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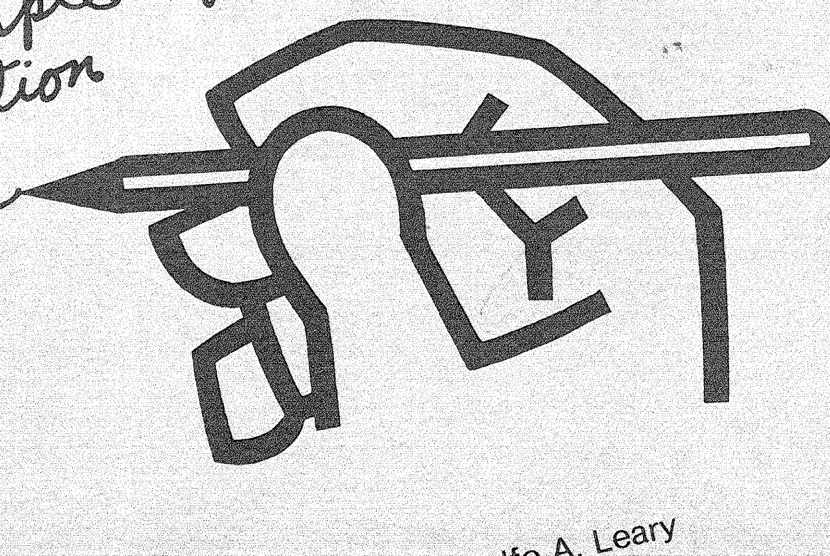

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How To Use Two Computational Strategies To Solve Simple System Identification Problems

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System identification has been defined as the process of determing a difference or differential equation such that it describes a physical process in accordance with some predetermined criterion (Sage and Melsa 1971). One should not, however, restrict it to physical processes, as many of the same principles apply to biological and social processes as well. In fact, system identification is an integral part of systems analysis, and solving the system identification problem may be viewed as one step toward the construction, testing, and analysis of mathematical characterizations of system processes. For an application of system identification concepts to the study of forest dynamics see Leary (1970).

One approach to the system identification problem (others are discussed by Sage and Melsa 1971) is to treat observations on a system as the boundary conditions governing the solution of a functional equation. Often-used functional equation types are first-order ordinary differential or difference equations. We can limit ourselves to first-order equations because any higher-order linear or nonlinear differential or difference equation may be converted to a system of simultaneous first-order equations. In most instances the only practical method of solving the resultant boundary-value problem is to use digital or hybrid (for continuous systems) computers. Generalized digital computer programs have been written to solve nonlinear multipoint boundary-value problems (Childs et al. 1969, Leary and Skog);[1/] however, the uninitiated are often unable to grasp the sequence of operations involved in the programs, and therefore unable to make efficient use of them. The purpose of this paper is to provide the reader with a better understanding of the methods involved so he can make better use of the available programs. We do this by presenting two step-by-step hand-computed solutions to a simplified problem using a computational strategy developed by the author and the particular solutions perturbation method (Childs et al. 1969).

We use these two methods for solving the same problem because they are implemented by available computer programs; also, we want to make it clear that one method is not necessarily better than the other but that it is the

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1/ Leary, Rolfe A., and Kenneth E. Skog. A preliminary user's guide to QUASI. North Cent. For. Exp. Stn., St. Paul, Minn. 1970. (Unpublished report.)

formulation of the problem as a multipoint boundary-value problem that is important. Exactly how one solves the boundary-value problem is not crucial, because the results are identical. The problem that follows is, in fact, a simple two-point boundary-value problem. Meaningful problems will ordinarily be higher dimensional (>2) with several boundary conditions.

THE PROBLEM

The problem is hypothetical, chosen to minimize the hand computations. Briefly it is as follows: Assume the process of concern is thought to be governed by the set of first-order difference equations:

$$\Delta Y1/\Delta t = a_{11}\ Y1 + a_{12}\ Y2$$

$$\Delta Y2/\Delta t = a_{21}\ Y1 + a_{22}\ Y2 \quad \text{, where}$$

$$\Delta Y1/\Delta t = (Y1\ (t + k) - Y1\ (t))/((t + k) - t),$$

Yi are system state variables, and a_{ij} are numerical constants. Assume also that $a_{12} = .75$, $a_{22} = .25$, the initial conditions at $t=0$ are $Y1 = 1$, $Y2 = 4$, a_{11} and a_{21} are unknown, and we have an observation on the system at $t=1$, where $Y1 = 4.4$ and $Y2 = 5.6$. The question to be answered is, what values should a_{11} and a_{21} have so that $Y1 = 4.4$ and $Y2 = 5.6$ when $t=1$?

To solve this problem we formulate it as a two-point boundary-value problem as follows:

EQUATION 1.

$$\Delta Y1/\Delta t = Y3\ Y1 + (.75)\ Y2$$

$$\Delta Y2/\Delta t = Y4\ Y1 + (.25)\ Y2$$

$$\Delta Y3/\Delta t = 0$$

$$\Delta Y4/\Delta t = 0$$

with boundary conditions:

EQUATION 2.

$$Y1(t=0) = 1,\ Y1(t=1) = 4.4$$

$$Y2(t=0) = 4,\ Y2(t=1) = 5.6\ .$$

Clearly, the problem is to find initial conditions for Y3 and Y4 such that the boundary conditions are satisfied. Equation 1 with the conditions in Equation 2 is a nonlinear boundary-value problem.

TWO STRATEGIES FOR SOLUTION

Not all nonlinear two-point or multipoint boundary-value problems are solvable. But many meaningful ones can be solved using the strategies outlined in this section. Strategy One is a variation of the method known as the method of complementary functions, and Strategy Two is the method of particular solutions (Childs *et al*. 1969). The essential difference between the two methods concerns the form of the linear equation for which solutions are computed. The method of complementary functions utilizes solutions of both the nonhomogeneous and homogeneous forms of the linear equation. The method of particular solutions uses solutions of the nonhomogeneous equation only. The two strategies are therefore related and have several common operations, as follows:

Step	Operation
1	Linearize the functional equations analytically.
2	Set initial conditions for solution of functional equations.
3	Solve functional equations by numerical integration or iteration.
4	Solve algebraic equations for integration constants so the boundary conditions are satisfied.
5	Form new initial conditions, Y_{L_S} (t_o) via superposition[2/] and/or check for convergence by (Strategy One) comparing $Y_{L_S}^{K}$ (t_o) and $Y_{L_S}^{K-1}$ (t_o) or (Strategy Two) checking the smallness of the integration constants.
6	If not convergent, set new initial conditions governing solution of functional equations and go to Step 3.

2/ For a discussion of superposition of homogeneous and nonhomogeneous solutions of differential equations see any basic textbook; e.g., Martin and Reissner (1961), and Bellman (1968). For difference equations consult Goldberg (1961, page 123).

Strategy One

Strategy One is patterned after Baird (1969) and may be described by the following recurrence relations:

EQUATION 3.

$$\overset{\Delta}{Y}_N{}^K(t) = f(Y_N{}^K,t) \qquad , \; Y_N{}^K(t_o) = Y_{L_S}{}^{K-1}(t_o)$$

EQUATION 4.

$$\overset{\Delta}{Y}_L{}^K(t) = f(Y_N{}^K,t) + J[Y_N{}^K,t](Y_L{}^K-Y_N{}^K) \; , \; Y_L{}^K(t_o) = Y_{ij}$$

where

$\overset{\Delta}{Y} = \Delta Y/\Delta t$,

K is the iteration count,

N denotes the nonlinear equations,

L denotes the linear equations,

J is the Jacobian matrix at time t,

$Y_N{}^o(t_o) = Y_{i1}$,

Y_{ij} is a constant matrix of initial conditions, the best estimates of unknown i.c., exact values of known i.c., and specified i.c. for solution of the homogeneous equations, and

$Y_{L_S}(t_o)$ is the initial condition vector formed by superposition of one particular solution and one or more homogeneous solutions.

Column 1 of Y_{ij} contains the initial conditions governing the solution of the nonhomogeneous form of Equation 4. Y_{i2} contains the initial conditions governing the first solution of the homogeneous form of Equation 4. Subsequent columns govern subsequent homogeneous solutions.

This strategy is different from the one usually employed in the method of complementary functions in that the initial conditions for the particular solution of Equation 4 do not change from one iteration (value of K) to the next. The primary advantage of this approach is that it minimizes relative error growth during execution of the algorithm.

The integration constants used in superposition are determined so that the boundary conditions are satisfied; i.e., by solving the system of algebraic equations:

$$y1_c(t_i) = y1_p(t_i) + c_1\, y1_{1h}(t_i) + c_2\, y1_{2h}(t_i) = BC1(t_i) \text{ and}$$

$$y2_c(t_i) = y2_p(t_i) + c_1\, y2_{1h}(t_i) + c_2\, y2_{2h}(t_i) = BC2(t_i) \;,$$

where c denotes the complete solution of the linearized equation.

The operation at Step 1 for the method of complementary functions requires that we prepare the basic nonlinear system, the nonhomogeneous form of the linearized equation, and as many homogeneous forms of the linearized equations as there are unknown initial conditions. Using the notational convention $\Delta Y1 = \Delta Y1/\Delta t$ (the first forward difference) we have:

$$\Delta Y1 = Y3\ Y1 + (.75)Y2 \qquad = f_1$$

$$\Delta Y2 = Y4\ Y1 + (.25)Y2 \qquad = f_2$$

$$\Delta Y3 = 0 \qquad = f_3$$

$$\Delta Y4 = 0 \qquad = f_4$$

$$\begin{matrix} \Delta Y5 = \\ \Delta Y6 = \\ \Delta Y7 = \\ \Delta Y8 = \end{matrix} \begin{bmatrix} Y3\ Y1 + (.75)Y2 \\ Y4\ Y1 + (.25)Y2 \\ 0 \\ 0 \end{bmatrix}_t + \begin{bmatrix} \partial f_1/\partial Y1 & \partial f_1/\partial Y2 & \partial f_1/\partial Y3 & \partial f_1/\partial Y4 \\ \partial f_2/\partial Y1 & \partial f_2/\partial Y2 & \partial f_2/\partial Y3 & \partial f_2/\partial Y4 \\ \partial f_3/\partial Y1 & \partial f_3/\partial Y2 & \partial f_3/\partial Y3 & \partial f_3/\partial Y4 \\ \partial f_4/\partial Y1 & \partial f_4/\partial Y2 & \partial f_4/\partial Y3 & \partial f_4/\partial Y4 \end{bmatrix}_t \cdot \begin{bmatrix} Y5 - Y1 \\ Y6 - Y2 \\ Y7 - Y3 \\ Y8 - Y4 \end{bmatrix}_t$$

$$\begin{matrix} \Delta Y9 = \\ \Delta Y10 = \\ \Delta Y11 = \\ \Delta Y12 = \end{matrix} \qquad \begin{bmatrix} \\ \dfrac{\partial f_i}{\partial Y_j}\ i=j=1,2,3,4 \\ \\ \end{bmatrix}_t \cdot \begin{bmatrix} Y9 \\ Y10 \\ Y11 \\ Y12 \end{bmatrix}_t$$

$$\begin{matrix} \Delta Y13 = \\ \Delta Y14 = \\ \Delta Y15 = \\ \Delta Y16 = \end{matrix} \qquad \begin{bmatrix} \\ \dfrac{\partial f_i}{\partial Y_j}\ i=j=1,2,3,4 \\ \\ \end{bmatrix}_t \cdot \begin{bmatrix} Y13 \\ Y14 \\ Y15 \\ Y16 \end{bmatrix}_t$$

In the above system of 16 equations, the first four constitute the nonlinear system, the next four the nonhomogeneous form of the linearized equation, and the last eight the homogeneous form of the linearized equation. As will be clear later, the solutions of 9 to 12 and 13 to 16 above will differ because of different initial conditions. They should in fact be linearly independent to prevent ill-conditioning in the system of linear algebraic equations that is solved for the integration constants.

We are ready to begin Step 2. Let us use as initial estimates for the initial conditions on Y3 and Y4 the values 0.5 and 0.5; to ensure independence of homogeneous solutions we purposely choose Y11 = 1, Y12 = 0, and Y15 = 0, Y16 = 1. Iteration 1 follows:

Step 2: Set i.c. for functional equations	Step 3: Solve functional equations; i.e., evaluate equations 1-16 and add to i.c. to give Yi(t=1)	
Y1(t=0) = 1.0	ΔY1 = 3.5	Y1(t=1) = 4.5
Y2(0) = 4.0	ΔY2 = 1.5	Y2(1) = 5.5
Y3(0) = .5 estimated	ΔY3 = 0	Y3(1) = .5
Y4(0) = .5 estimated	ΔY4 = 0	Y4(1) = .5
Y5(0) = 1.0	ΔY5 = 3.5	Y5(1) = 4.5
Y6(0) = 4.0	ΔY6 = 1.5	Y6(1) = 5.5
Y7(0) = .5 estimated	ΔY7 = 0	Y7(1) = .5
Y8(0) = .5 estimated	ΔY8 = 0	Y8(1) = .5
Y9(0) = 0	ΔY9 = 1.0	Y9(1) = 1.0
Y10(0) = 0	ΔY10 = 0	Y10(1) = 0
Y11(0) = 1.0	ΔY11 = 0	Y11(1) = 1.0
Y12(0) = 0	ΔY12 = 0	Y12(1) = 0
Y13(0) = 0	ΔY13 = 0	Y13(1) = 0
Y14(0) = 0	ΔY14 = 1.0	Y14(1) = 1.0
Y15(0) = 0	ΔY15 = 0	Y15(1) = 0
Y16(0) = 1.0	ΔY16 = 0	Y16(1) = 1.0

Clearly,

$$\Delta 14 = (\partial f2/\partial Y1)Y13 + (\partial f2/\partial Y2)Y14 + (\partial f2/\partial Y3)Y15 + (\partial f2/\partial Y4)Y16$$

$$= Y4\ Y13 + .25\ Y14 + 0\ Y15 + Y1\ Y16$$

$$= .5(0) + .25(0) + 0(0) + 1(1) = 1$$

Step 4: Solve for integration constants so boundary conditions are satisfied,

Time	Particular		First homogeneous		Second homogeneous		Boundary conditions (observation)
0	$\begin{bmatrix} Y5 = 1 \\ Y6 = 4 \end{bmatrix}$	$+c_1$	$\begin{bmatrix} Y9 = 0 \\ Y10 = 0 \end{bmatrix}$	$+c_2$	$\begin{bmatrix} Y13 = 0 \\ Y14 = 0 \end{bmatrix}$	$\begin{matrix} = \\ = \end{matrix}$	$\begin{matrix} 1.0 \\ 4.0 \end{matrix}$
1	$\begin{bmatrix} Y5 = 4.5 \\ Y6 = 5.5 \end{bmatrix}$	$+c_1$	$\begin{bmatrix} Y9 = 1 \\ Y10 = 0 \end{bmatrix}$	$+c_2$	$\begin{bmatrix} Y13 = 0 \\ Y14 = 1 \end{bmatrix}$	$\begin{matrix} = \\ = \end{matrix}$	$\begin{matrix} 4.4 \\ 5.6 \end{matrix}$

This simplifies to

$$c_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -.1 \\ +.1 \end{bmatrix} \quad \text{or} \quad \begin{matrix} c_1 = -.1 \\ c_2 = +.1 \end{matrix} \quad .$$

Step 5: Form new initial conditions for the linearized equations using the superposition principle; i.e.,

$$\begin{bmatrix} Y5(t{=}0) = 1.0 \\ Y6(t{=}0) = 4.0 \\ Y7(t{=}0) = \ \ .5 \\ Y8(t{=}0) = \ \ .5 \end{bmatrix} + (c_1{=}-.1) \begin{bmatrix} Y9(t{=}0) = 0 \\ Y10(t{=}0) = 0 \\ Y11(t{=}0) = 1 \\ Y12(t{=}0) = 0 \end{bmatrix} + (c_2{=}.1) \begin{bmatrix} Y13(t{=}0) = 0 \\ Y14(t{=}0) = 0 \\ Y15(t{=}0) = 0 \\ Y16(t{=}0) = 1 \end{bmatrix} = \begin{bmatrix} 1.0 \\ 4.0 \\ .4 \\ .6 \end{bmatrix}$$

Then check for convergence by comparing initial conditions at successive iterations; i.e., compare $Y_{L_S}^K(t_o)$ and $Y_{L_S}^{K-1}(t_o)$. When $K=1$, the comparison is between $Y_{L_S}^K(t_o)$ and Yi. Thus, our comparison is:

$$Y_{L_S}^{1}(t_o) \text{ and } Yi$$

$$\begin{bmatrix} 1.0 \\ 4.0 \\ .4 \\ .6 \end{bmatrix} \quad \begin{bmatrix} 1.0 \\ 4.0 \\ .5 \\ .5 \end{bmatrix}$$

Clearly, convergence has not occurred.

Step 6: According to Equation 3, the initial conditions governing the solution of the nonlinear equations at the start of iteration 2 are given by $Y_{L_S}^{1}(t_o)$, the result from Step 5. Thus, in iteration 2 which follows, the initial conditions for Y3 and Y4 are 0.4 and 0.6, respectively.

Step 2: Set i.c., for functional equations	Step 3: Solve functional equations, i.e., evaluate equations 1-16 and add to i.c.	
Y1(0) = 1.0	ΔY1 = 3.4	Y1(1) = 4.4
Y2(0) = 4.0	ΔY2 = 1.6	Y2(1) = 5.6
Y3(0) = .4 from Iteration 1	ΔY3 = 0	Y3(1) = .4
Y4(0) = .6 from Iteration 1	ΔY4 = 0	Y4(1) = .6
Y5(0) = 1.0	ΔY5 = 3.5	Y5(1) = 4.5
Y6(0) = 4.0	ΔY6 = 1.5	Y6(1) = 5.5
Y7(0) = .5 initial estimates	ΔY7 = 0	Y7(1) = .5
Y8(0) = .5 initial estimates	ΔY8 = 0	Y8(1) = .5
Y9(0) = 0	ΔY9 = 1.0	Y9(1) = 1.0
Y10(0) = 0	ΔY10 = 0	Y10(1) = 0
Y11(0) = 1.0	ΔY11 = 0	Y11(1) = 1.0
Y12(0) = 0	ΔY12 = 0	Y12(1) = 0
Y13(0) = 0	ΔY13 = 0	Y13(1) = 0
Y14(0) = 0	ΔY14 = 1.0	Y14(1) = 1.0
Y15(0) = 0	ΔY15 = 0	Y15(1) = 0
Y16(0) = 1.0	ΔY16 = 0	Y16(1) = 1.0

Step 4: Solve for integration constants so boundary conditions are satisfied (from Iteration 1, Step 4, we see that conditions at t=0 do not affect the solution),

Time	Particular		First homogeneous		Second homogeneous		Boundary conditions (observation)
1	$\begin{bmatrix} Y5 = 4.5 \\ Y6 = 5.5 \end{bmatrix}$	$+ c_1$	$\begin{bmatrix} Y9 = 1 \\ Y10 = 0 \end{bmatrix}$	$+ c_2$	$\begin{bmatrix} Y13 = 0 \\ Y14 = 1 \end{bmatrix}$	=	$\begin{bmatrix} 4.4 \\ 5.6 \end{bmatrix}$

$$c_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -.1 \\ +.1 \end{bmatrix}, \text{ thus } \begin{array}{l} c_1 = -.1 \\ c_2 = +.1 \end{array}$$

Step 5: Form new initial conditions via superposition,

$$\begin{bmatrix} Y5(t=0) = 1.0 \\ Y6(\ \ 0) = 4.0 \\ Y7(\ \ 0) = \ .5 \\ Y8(\ \ 0) = \ .5 \end{bmatrix} + (c_1=-.1) \begin{bmatrix} Y9(t=0) = 0 \\ Y10(\ \ 0) = 0 \\ Y11(\ \ 0) = 1 \\ Y12(\ \ 0) = 0 \end{bmatrix} + (c_2=.1) \begin{bmatrix} Y13(t=0) = 0 \\ Y14(\ \ 0) = 0 \\ Y15(\ \ 0) = 0 \\ Y16(\ \ 0) = 1 \end{bmatrix} = \begin{bmatrix} 1.0 \\ 4.0 \\ .4 \\ .6 \end{bmatrix}$$

and check for convergence by comparing results with those produced at Step 5 of the previous iteration; i.e., compare $YL_S^2(t_o)$ and $YL_S^1(t_o)$. Thus:

$$\begin{bmatrix} 1.0 \\ 4.0 \\ .4 \\ .6 \end{bmatrix} \text{ vs. } \begin{bmatrix} 1.0 \\ 4.0 \\ .4 \\ .6 \end{bmatrix},$$ and we see that convergence has occurred.

We may conclude, therefore, that the following equations may be used to approximate the true equations governing the observed process:

$$\frac{\Delta Y1}{\Delta t} = (.4)Y1 + (.75)Y2$$

$$\frac{\Delta Y2}{\Delta t} = (.6)Y1 + (.25)Y2 .$$

Strategy Two

This computational strategy is taken from Childs et al. (1969), and may be described by the recurrence relations:

EQUATION 5.

$$\overset{\Delta}{Y}_N^{K}(t) = f(Y_N^{K},t) \qquad , \; Y_N^{K}(t_o) = Y_{L_S}^{K-1}(t_o)$$

EQUATION 6.

$$\overset{\Delta}{Y}_L^{K}(t) = f(Y_N^{K},t) + J[Y_N^{K},t](Y_L^{K}-Y_N^{K}), \; Y_L^{K}(t_o) = Y_{ij} \; ,$$

where all notation is identical to that in Equations 3 and 4, and where

Y_{ij} is a nonconstant matrix of initial conditions, the first column of which contains the unperturbed initial conditions. Other columns contain initial conditions that have been perturbed, and

$Y_{L_S}(t_o)$ is the initial condition vector formed by superimposing (in this case) two perturbed particular solutions on one unperturbed solution.

The integration constants are determined by solving the system of algebraic equations:

$$a_1\, y1_{1p}(t_i) + a_2\, y1_{2p}(t_i) + a_3\, y1_{3p}(t_i) = BC1(t_i)$$

$$a_1\, y2_{1p}(t_i) + a_2\, y2_{2p}(t_i) + a_3\, y2_{3p}(t_i) = BC2(t_i)$$

$$a_1 \qquad + a_2 \qquad + a_3 \qquad = 1$$

for a_i, i=1,2,3. The third equation is a supplementary condition that the a_i must meet for this method.

The operation at Step 1 for the particular solutions perturbation method requires that we prepare the basic nonlinear system and (for this problem) three sets of the nonhomogeneous form of Equation 6. Thus we have:

$$\begin{aligned}
\Delta Y1 &= \\
\Delta Y2 &= \\
\Delta Y3 &= \\
\Delta Y4 &=
\end{aligned}
\begin{bmatrix} Y3\ Y1 + (.75)Y2 \\ Y4\ Y1 + (.25)Y2 \\ 0 \\ 0 \end{bmatrix}
\quad
\begin{aligned}
&= f_1 \\
&= f_2 \\
&= f_3 \\
&= f_4
\end{aligned}$$

$$\begin{aligned}
\Delta Y5 &= \\
\Delta Y6 &= \\
\Delta Y7 &= \\
\Delta Y8 &=
\end{aligned}
\begin{bmatrix} Y3\ Y1 + (.75)Y2 \\ Y4\ Y1 + (.25)Y2 \\ 0 \\ 0 \end{bmatrix}_t
+
\begin{bmatrix} \partial f_1/\partial Y1 & . & . & \partial f_1/\partial Y4 \\ \partial f_2/\partial Y1 & . & . & \partial f_2/\partial Y4 \\ \partial f_3/\partial Y1 & . & . & \partial f_3/\partial Y4 \\ \partial f_4/\partial Y1 & . & . & \partial f_4/\partial Y4 \end{bmatrix}_t
\cdot
\begin{bmatrix} Y5-Y1 \\ Y6-Y2 \\ Y7-Y3 \\ Y8-Y4 \end{bmatrix}_t$$

$$\begin{aligned}
\Delta Y9 &= \\
\Delta Y10 &= \\
\Delta Y11 &= \\
\Delta Y12 &=
\end{aligned}
\begin{bmatrix} Y3\ Y1 + (.75)Y2 \\ Y4\ Y1 + (.25)Y2 \\ 0 \\ 0 \end{bmatrix}_t
+
\begin{bmatrix} \\ \dfrac{\partial f_i}{\partial Y_j} \quad i=j=1,2,3,4 \\ \\ \end{bmatrix}_t
\cdot
\begin{bmatrix} Y9-Y1 \\ Y10-Y2 \\ Y11-Y3 \\ Y12-Y4 \end{bmatrix}_t$$

$$\begin{aligned}
\Delta Y13 &= \\
\Delta Y14 &= \\
\Delta Y15 &= \\
\Delta Y16 &=
\end{aligned}
\begin{bmatrix} Y3\ Y1 + (.75)Y2 \\ Y4\ Y1 + (.25)Y2 \\ 0 \\ 0 \end{bmatrix}_t
+
\begin{bmatrix} \\ \dfrac{\partial f_i}{\partial Y_j} \quad i=j=1,2,3,4 \\ \\ \end{bmatrix}_t
\cdot
\begin{bmatrix} Y13-Y1 \\ Y14-Y2 \\ Y15-Y3 \\ Y16-Y4 \end{bmatrix}_t .$$

The first four equations above constitute the nonlinear system. Equations 5 to 8, 9 to 12, and 13 to 16 are nonhomogeneous forms of the linearized equations. The solution of these three sets may be expected to differ because of different initial conditions and should be linearly independent.

We are now ready to begin Step 2. Notice that we again estimate the unknown initial conditions on Y3 and Y4 as 0.5. Notice also that the initial conditions for Y11 and Y16 are a constant multiple (in this case 1.2) of those for Y3 and Y4, respectively. Iteration 1 follows:

Step 2: Set i.c. for functional equations	Step 3: Solve functional equations; i.e., evaluate equations 1-16 and add to i.c. to give Yi(t=1)	
Y1(t=0) = 1.0	ΔY1 = 3.5	Y1(t=1) = 4.5
Y2(0) = 4.0	ΔY2 = 1.5	Y2(1) = 5.5
Y3(0) = .5	ΔY3 = 0	Y3(1) = .5
Y4(0) = .5	ΔY4 = 0	Y4(1) = .5
Y5(0) = 1.0	ΔY5 = 3.5	Y5(1) = 4.5
Y6(0) = 4.0	ΔY6 = 1.5	Y6(1) = 5.5
Y7(0) = .5	ΔY7 = 0	Y7(1) = .5
Y8(0) = .5	ΔY8 = 0	Y8(1) = .5
Y9(0) = 1.0	ΔY9 = 3.5+.1 = 3.6	Y9(1) = 4.6
Y10(0) = 4.0	ΔY10 = 1.5+ 0 = 1.5	Y10(1) = 5.5
Y11(0) = (.5)(1.2)=.6	ΔY11 = 0	Y11(1) = .6
Y12(0) = .5	ΔY12 = 0	Y12(1) = .5
Y13(0) = 1.0	ΔY13 = 3.5+ 0 = 3.5	Y13(1) = 4.5
Y14(0) = 4.0	ΔY14 = 1.5+.1 = 1.6	Y14(1) = 5.6
Y15(0) = .5	ΔY15 = 0	Y15(1) = .5
Y16(0) = (.5)(1.2)=.6	ΔY16 = 0	Y16(1) = .6

Step 4: Solve for integration constants so boundary conditions are satisfied.

$$\underset{\text{Unperturbed}}{a_1 \begin{bmatrix} Y5 = 4.5 \\ Y6 = 5.5 \\ 1.0 \end{bmatrix}} + \underset{\text{First perturbed}}{a_2 \begin{bmatrix} Y9 = 4.6 \\ Y10 = 5.5 \\ 1.0 \end{bmatrix}} + \underset{\text{Second perturbed}}{a_3 \begin{bmatrix} Y13 = 4.5 \\ Y14 = 5.6 \\ 1.0 \end{bmatrix}} = \underset{\text{Boundary conditions (observation)}}{\begin{bmatrix} 4.4 \\ 5.6 \\ 1.0 \end{bmatrix}}$$

The solution is $a_3 = 1$, $a_2 = -1$, $a_1 = 1$.

Step 5: Check for convergence; i.e., are a_2 and a_3 very near zero? Because they are not we form new initial conditions via superposition; i.e.,

$$(a_1=1)\begin{bmatrix} Y5(t=0)=1 \\ Y6(\ \ 0)=4 \\ Y7(\ \ 0)=.5 \\ Y8(\ \ 0)=.5 \end{bmatrix} +(a_2=-1)\begin{bmatrix} Y9(t=0)=1 \\ Y10(\ \ 0)=4 \\ Y11(\ \ 0)=.6 \\ Y12(\ \ 0)=.5 \end{bmatrix} +(a_3=1)\begin{bmatrix} Y13(t=0)=1 \\ Y14(\ \ 0)=4 \\ Y15(\ \ 0)=.5 \\ Y16(\ \ 0)=.6 \end{bmatrix} = \begin{bmatrix} 1 \\ 4 \\ .4 \\ .6 \end{bmatrix}$$

and start Iteration 2.

Step 2: Set i.c. for functional equations		Step 3: Solve functional equations			
Y1(t=0) =	1.0	ΔY1 =	3.4	Y1(t=1) =	4.4
Y2(0) =	4.0	ΔY2 =	1.6	Y2(1) =	5.6
Y3(0) =	.4	ΔY3 =	0	Y3(1) =	.4
Y4(0) =	.6	ΔY4 =	0	Y4(1) =	.6
Y5(0) =	1.0	ΔY5 =	3.4	Y5(1) =	4.4
Y6(0) =	4.0	ΔY6 =	1.6	Y6(1) =	5.6
Y7(0) =	.4	ΔY7 =	0	Y7(1) =	.4
Y8(0) =	.6	ΔY8 =	0	Y8(1) =	.6
Y9(0) =	1.0	ΔY9 =	3.4+.08=3.48	Y9(1) =	4.48
Y10(0) =	4.0	ΔY10 =	1.6+ 0 =1.6	Y10(1) =	5.6
Y11(0) =	(.4)(1.2)=.48	ΔY11 =	0	Y11(1) =	.48
Y12(0) =	.6	ΔY12 =	0	Y12(1) =	.6
Y13(0) =	1.0	ΔY13 =	3.4+ 0 =3.4	Y13(1) =	4.4
Y14(0) =	4.0	ΔY14 =	1.6+.12=1.72	Y14(1) =	5.72
Y15(0) =	.4	ΔY15 =	0	Y15(1) =	.4
Y16(0) =	(.6)(1.2)=.72	ΔY16 =	0	Y16(1) =	.72

Step 4: Solve for integration constants so boundary conditions are satisfied

$$a_1 \begin{bmatrix} 4.4 \\ 5.6 \\ 1.0 \end{bmatrix} + a_2 \begin{bmatrix} 4.48 \\ 5.6 \\ 1.0 \end{bmatrix} + a_3 \begin{bmatrix} 4.4 \\ 5.72 \\ 1.0 \end{bmatrix} = \begin{bmatrix} 4.4 \\ 5.6 \\ 1.0 \end{bmatrix}$$

The solution is $a_1 = 1$, $a_2 = 0$, and $a_3 = 0$.

Step 5: Check for convergence. Clearly, since $a_2=a_3=0$, convergence has occurred.

In this case also we conclude that the following equations may be used to approximate the process that was observed:

$$\Delta Y1/\Delta t = (.4)\ Y1 + (.75)\ Y2$$

$$\Delta Y2/\Delta t = (.6)\ Y1 + (.25)\ Y2\ .$$

DISCUSSION

It is clear that by following the steps outlined previously we have obtained convergence in both cases in 2 iterations. Furthermore, convergence is to the same values, as we asserted earlier. Thus both computational strategies, for which there are available digital computer programs, are suited to solve this two-point boundary-value problem.

There are at least two points that appear to warrant some discussion. First, the example selected was extremely simple and probably could be solved by other means. We would like to emphasize that the procedures used for this example apply virtually unaltered to problems that are orders of magnitude more complex; e.g., for time-dependent coefficients (nonstationary processes), "missing observation" situations, unobservable state variables, etc. For several worked examples see Leary and Skog (1972).

Second, the computational strategies employed here compare favorably with other methods of solving nonlinear boundary-value problems such as quasi-linearization, and provide an efficient method of solving a variety of meaningful problems.

SOME RECENT PUBLICATIONS OF THE NORTH CENTRAL FOREST EXPERIMENT STATION

Weights and Centers of Gravity for Red Pine, White Spruce, and Balsam Fir, by H. M. Steinhilb and John R. Erickson. USDA For. Serv. Res. Pap. NC-75, 7 p., illus. 1972.

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Size of Oak Advance Reproduction: Key to Growth Following Harvest Cutting, by Ivan L. Sander. USDA For. Serv. Res. Pap. NC-79, 6 p., illus. 1972.

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- Participating with all State forestry agencies in cooperative programs to protect, improve, and wisely use our Country's 395 million acres of State, local, and private forest lands.
- Managing and protecting the 187-million acre National Forest System.

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