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MATHEMATICAL FUNCTIONS FOR PREDICTING GROWTH AND YIELD OF BLACK WALNUT PLANTATIONS IN THE CENTRAL STATES

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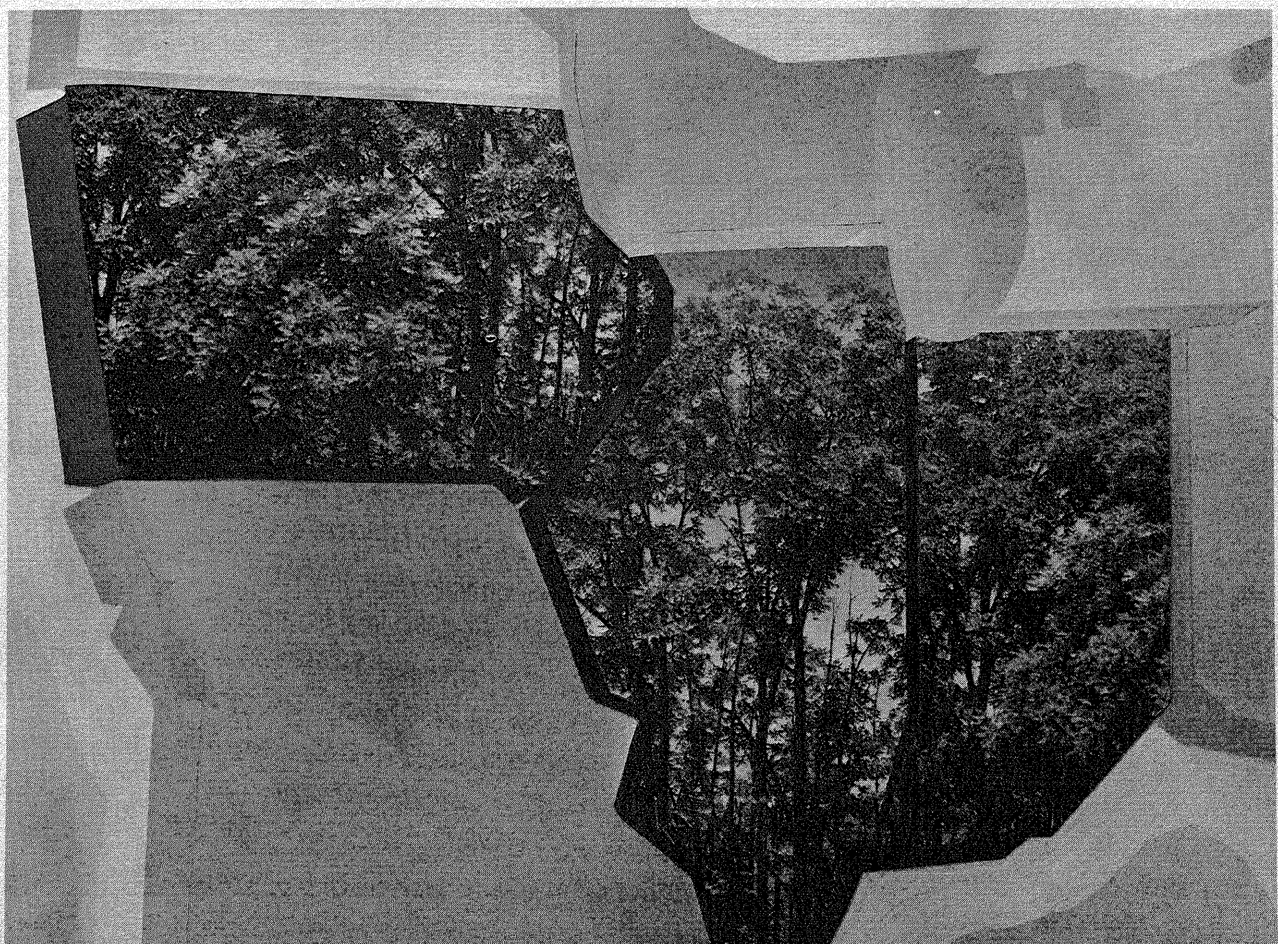

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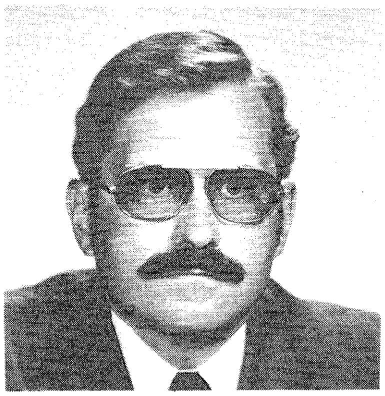

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MATHEMATICAL FUNCTIONS FOR PREDICTING GROWTH AND YIELD OF BLACK WALNUT PLANTATIONS IN THE CENTRAL STATES

Raymond S. Ferell and Allen L. Lundgren

Probably no other single species has stimulated more interest in growing trees for profit than has black walnut. The expectation of high returns on investment, the sheer beauty of the wood, and the mystique that seems to be associated with products from the wood have generated great interest in planting, growing, and selling the species. At the same time, however, not enough is known about the costs and returns from growing and selling walnut to provide the potential investor with the information necessary to make sound decisions.

As a minimum, the landowner needs to be able to estimate what costs will be incurred, what quantity of products he can get and when, and at what price he will be able to sell his product in order to evaluate return to his investment. To do this he will need a model that will predict the yield of walnut at different ages and under several different conditions. Such a model should be adaptable to computer simulation.

Several studies estimating the potential returns from a plantation investment in growing black walnut have been made (Ferell and Bentley 1969, Callahan and Smith 1974). These have been generally limited to a few alternatives and conditions. Growth and yield data for black walnut are also available from many sources (Funk 1966, Dillow and Hawker 1971). This information is generally descriptive in nature and also applicable only to a narrow set of alternatives.

L. F. Kellogg (1940), in an unpublished report of the Central States Forest Experiment Station of the USDA Forest Service, summarized an extensive analysis of data collected in 1928, 1929, and 1930 from more than 200 black walnut plantations, located primarily in Iowa, Illinois, and Indiana. The plantations were well stocked and essentially undisturbed. These data, although old, are the most comprehensive available, covering a wide range of site and stand conditions. In Kellogg's data base, site index (at 50 years) ranged from 40 to 80; stand age from 10 to 75 years; initial stocking from 436 to 4,480 trees per acre (square spacing from 10 feet by 10 feet to 3 feet by 3 feet); surviving number of trees per acre from 119 to 966; total height from 18 to 82 feet; and d.b.h. from 1 to 23 inches. Unfortunately, the original data were not available--only the graphed and tabularized summaries that had already been smoothed and harmonized.

Kellogg derived regression equations for basal area, number of trees, and cubic foot volume per acre as a function of original spacing, site index, and age (Appendix). Tree heights were estimated from site index curves prepared for the plots. Board foot volumes were derived from board foot-cubic foot ratios for average stand diameters. These and other data are presented in tables in his report.

In order to develop a more generalized model better suited for use in computer simulation, new equations were fitted to the tabularized data in the report. This paper presents the results of this work.

THE MODEL

The volume per acre at any plantation age can be determined by multiplying the average volume per tree of average diameter at that age times the number of trees per acre at that age. Volumes are expressed in both cubic feet and in board feet.

Two general equation forms (Lundgren and Dolid 1970) were used to represent tree volumes as a function of average stand height and diameter:

(1) a monomolecular equation of the form

$$V=H(b_1+b_2e^{b_3D}), \text{ and}$$

(2) an exponential-monomolecular equation of the form

$$V=b_1H(1.0-e^{b_2D})^{b_3}$$

where:

V = volume per tree in cubic feet or board feet

H = total height in feet of dominant and codominant trees

D = average diameter breast height outside bark, in inches, of a tree

b_1, b_2, b_3 = parameters to be estimated

e = base of natural logarithms.

The monomolecular equation is a simpler form that has no point of inflection, whereas the exponential-monomolecular function has the more general S-shape associated with many biological growth functions. To use either function it is necessary to know or to estimate tree height and diameter.

The same two equation forms were used to estimate total height:

$$(1)\quad H=S(b_1+b_2 e^{b_3 A})$$

$$(2)\quad H=b_1 S(1.0-e^{b_2 A})^{b_3}$$

where:

S = site index (total height in feet at tree age 50)

A = age of tree in years.

The following equation form was fitted to the tree diameter data:

$$D=b_1 S(1.0-b_2 e^{b_3 A+b_4 S})e^{b_5 T+b_6 S}$$

where:

T = number of trees per acre at age A

$b_1 \ldots b_6$ = parameters to be estimated.

The number of trees surviving at any age was estimated by the following general form:

$$T=(b_1 No/(1.0-e^{b_2 S}))e^{b_3 A\ No}$$

where:

T = number of surviving trees per acre at specified age

No = initial number of trees per acre at time of plantation establishment.

With the above equations and knowing the site index and initial number of trees per acre, one can estimate the surviving number of trees per acre. Using this estimated number of trees per acre, the site index, and age, one can estimate the average stand diameter and height for any age. From this diameter and height one can estimate stand timber volumes per acre at that same age. This, then, is the basic timber yield model derived from Kellogg's data.

PROCEDURE

The specified mathematical functions were fitted to the data given by Kellogg (1940) by the NONLIN program, a modification of the nonlinear regression program of Marquardt (1964). The form of the function and an initial estimate of the parameters is specified. The program, through successive approximations, estimates the parameters using a least squares approach. The program also gives the value of the standard error.

RESULTS

Volume

Most black walnut is sold by board-foot volume measured by the Doyle log rule, but parameters for other board-foot rules and also cubic-foot volume were also estimated (tables 1-3).

For board-foot volume by the Doyle rule, the equation based on the monomolecular function (table 2) is:

$$V=(H)(-1.1705+0.5241\ e^{0.1037\ D}).$$

A tree 80 feet in height and 20 inches in diameter would have a volume of:

$$V=(80)(-1.1705+0.5241\ e^{0.1037(20)})$$

$$V=240.0 \text{ board feet.}$$

Using the exponential-monomolecular function (table 3), we would estimate the volume of the same tree to be:

$$V=26.5436(80)(1.0-e^{-0.05153(20)})^{4.8977}$$

$$V=244.6 \text{ board feet (Doyle).}$$

Both functions give reasonably close estimates within the range of the data. For extrapolations, the exponential-monomolecular function is more reasonable.

Total Height

The total height of the dominant and codominant trees on a site is one of the more commonly used measures of site potential, and is required to estimate volumes. The

Table 1.--*Major assumptions used for determining volume by different rules for black walnut timber*

Volume measurement	Stump height	Top diameter	Log length	Trim allowance	Notes
	Feet	*Inches*	*Feet*	*Feet*	
Total cubic feet	1.0	--	--	--	Volume measured by planimeter
Merchantable cubic feet	1.0	4.0	4.0	--	Volume measured by planimeter
Board-foot volume:					
International ¼ inch	1.0	5.0	12.0	0.3	
Scribner (10 inch)	1.0	10.0	12.0	.3	
Scribner (8 inch)	1.0	8.0	12.0	.3	
Doyle[1]	1.0	5.0	12.0	.3	

[1]Converted from International ¼ inch using factors from Forestry Handbook, page 1.59 (Forbes 1956).

Table 2.--*Parameters of the monomolecular functions describing the volume of black walnut trees reported by Kellogg (1940)*[1]

Predicted variable	Standard error	Maximum error[2]	Parameters		
			b_1	b_2	b_3
Merchantable cubic foot volume	0.319	1.0	-0.41937	0.31843	0.07272
Total cubic foot volume	.889	3.6	-.31916	.26255	.07727
Board-foot volume:					
Scribner (10 inch)	3.53	9.9	12.7368	-19.4986	-.03600
Scribner (8 inch)	2.61	7.4	-4.6526	3.0437	.04784
International ¼ inch (5 inch)	8.47	22.1	-2.4654	1.6250	.06935
Doyle	6.13	18.6	-1.1705	.5241	.10370

[1]$V = H(b_1 + b_2 e^{b_3 D})$.

[2]Maximum difference between observed and predicted values over the range of the data.

Table 3.--*Parameters of the exponential-monomolecular functions describing the volume of black walnut trees reported by Kellogg (1940)*[1]

Predicted variable	Standard error	Maximum error[2]	Parameters		
			b_1	b_2	b_3
Merchantable cubic foot volume	0.459	1.9	162.9683	-0.004488	2.0966
Total cubic foot volume	.845	2.7	192.3086	- .003543	1.9939
Board-foot volume:					
Scribner (10 inch)	5.16	13.5	5.4778	- .20283	29.1136
Scribner (8 inch)	4.24	10.9	11.0799	- .08552	6.0640
International ¼ inch (5 inch)	8.62	23.9	47.1558	- .02869	2.9549
Doyle	5.71	16.8	26.5436	- .05153	4.8977

[1]$V = b_1 H(1.0 - e^{b_2 D})^{b_3}$.

[2]Maximum difference between observed and predicted values over the range of the data.

relation of height (H) to age (A), for a known site index (S) was determined using the two equation forms discussed earlier.

For the simpler monomolecular function, the height equation is:

$$H=S(1.2582-1.2027e^{-0.0308A}).$$

For this function the standard error = 0.5284 feet; and the maximum error (the maximum difference between observed and predicted data over the range of data) = 1.1 feet.

For the exponential-monomolecular function, the height equation is:

$$H=1.2756(S)(1.0-e^{-0.02836A})^{0.87915}.$$

For this function the standard error = 0.5102 feet; and the maximum error = 1.3 feet.

On a site index of 60, a 30-year-old black walnut tree would be expected to be 46.8 feet high by the first equation and 46.9 feet high by the second. Clearly, within the range of the original data, both functions give essentially the same results. For extrapolations beyond the range of the data, the exponential-monomolecular model is preferred.

Diameter

The diameter of black walnut trees on a particular site is a function of site productivity, average age of trees, and the number of trees growing on the site. A generalized function that would apply to all sites, ages, and stocking was fitted for a six-parameter function that recognized the interactions of site and age, and site and stocking. The equation for this function is:

$$D=0.50(\text{Site Index})(1.0-(0.548)e^{-0.01009(\text{Age})+0.00361(\text{SI})})e^{-0.00136(\#\text{ trees})-0.00537(\text{SI})}.$$

For this function the standard error = 0.3014 inches; and the maximum error (the maximum difference between observed and predicted values over the range of the data) = 1.060 inches.

Using this formula, the diameter of a 20-year-old tree on site 60 growing in a stand of 200 trees per acre is predicted to be:

$$D=0.510(60)(1.0-(0.548)e^{-0.01009(20)+0.00361(60)})e^{-0.00136(200)-(0.00537)(60)}$$

$$D=7.5\text{ inches.}$$

Mortality and Residual Trees

In order to determine the total trees per acre remaining at any time, a function was fitted to describe the residual trees. This can be used to determine mortality in a plantation before thinning. The number of trees remaining will depend upon how many were planted, the site, and how long the trees have been growing.

The general equation is:

$$T=\frac{0.8757(\text{Initial Stocking})}{(1.0-e^{-0.04559(\text{Site Index})})}e^{-0.000024(\text{Age})(\text{Initial stocking})}.$$

For this equation, the standard error = 158.93 trees; and the maximum error = 524 trees.

A plantation planted at a 10-foot square spacing (435 trees per acre) on site 60 and allowed to grow for 20 years would have:

$$T=\frac{0.8757(435)}{(1.0-e^{-0.04559(60)})}e^{-0.000024(20)(435)}$$

$$T=331\text{ trees per acre.}$$

As expected, this equation for forecasting residual trees showed the poorest fit of all the equations reported here. Nevertheless, it is fairly reliable at earlier ages on medium sites and can be used in simulating black walnut plantation growth until the first thinning.

CONCLUSIONS

The functions presented here describe the tables reported by Kellogg reasonably well. Because these functions cover a rather wide range of conditions, they should be useful for simulating growth and yield in black walnut stands until a better data base can be developed. Used with caution, these functions provide the best expression of black walnut growth and yield available at the present time.

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APPENDIX

Several extensive tables in Kellogg's 1940 report were derived from the following three equations reported by Kellogg:

1. Total basal area:

$$\log Y = 0.0930 - 0.0824 \log X_1$$
$$+0.7082 \log X_2 + 0.4324 \log X_3$$

where:

Y=total basal area (sq. ft./acre)
X_1=original spacing in feet (square root of the area per tree)
X_2=site index (height in feet at 50 years)
X_3=stand age in years.

2. Number of trees:

$$\log Y = 5.7714 - 0.4979 \log X_1$$
$$-0.8916 \log X_2 - 0.8609 \log X_3$$

where:

Y=number of surviving trees per acre
X_1,X_2,X_3=as above.

3. Volume in cubic feet:

$$\log Y = -0.9047 + 1.5480 \log X_2 + 0.8704 \log X_3$$

where:

Y=volume in cubic feet per acre.
X_2,X_3= as above.

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1976. Mathematical functions for predicting growth and yield of black walnut plantations in the Central States. USDA For. Serv. Gen. Tech. Rep. NC-24, 5 p. North Cent. For. Exp. Stn., St. Paul, Minn.

Mathematical functions were fitted to unpublished summaries of yield data collected and reported by L. F. Kellogg in 1940 for unmanaged black walnut plantations in the Central States. They cover a wide range of conditions and provide the best data base available for simulating growth and yield in plantations.

OXFORD: 564:561:562:176.1 *Juglans nigra*. KEY WORDS: height, diameter, mortality, board-foot-volume, cubic-foot-volume.

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