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How to formulate and solve **optimal stand density over time** problems for even-aged stands using dynamic programming

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HOW TO FORMULATE AND SOLVE "OPTIMAL STAND DENSITY OVER TIME" PROBLEMS FOR EVEN-AGED STANDS USING DYNAMIC PROGRAMMING[1]

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Any intermediate cutting in a forest stand has implications for the growth and yield of the stand following cutting. For example, a severe thinning in a young plantation may significantly reduce the range of possible residual densities as the stand gets older. In general, each cutting decision in a stand affects all future growth, cutting decisions, and returns. Stated another way, forest managers face sequential or interdependent decision-making problems when planning the intermediate harvests in forest stands. An optimal sequence of such interrelated decisions can be derived using the procedure called dynamic programming (DP).

Dynamic programming has been extensively applied in many other areas: inventory and production decisions, allocation and control problems, and in systems design (Bellman 1957, Nemhauser 1966, Wagner 1975). In forestry, however, DP has been used sparingly. Arimizu (1958) used it to regulate intermediate cutting with the objective of producing a maximum harvest volume. Hool (1965), using simply a "cut" or "do not cut" strategy, applied a discrete DP model. Later he introduced a Markov chain approach to production control using a DP model (Hool 1966). Amidon and Akin (1968) compared traditional marginal analysis with DP for determining optimal growing stock and found the latter to be more flexible and convenient. Other authors have illustrated the feasibility of DP for deriving optimal cutting schedules for timber stands (Risvand 1969, Kilkki and Vaisanen 1969, Schreuder 1971).

Unfortunately many of the above papers are difficult to follow because explicit derivation of the solution procedures is lacking. An additional shortcoming of several of the papers is the absence of suitable forest growth models—ones directly related to the decision variable. These two factors, plus the unfamiliarity of most readers with the special conditions which must be met for a problem to be solved as a DP problem, account for the limited application of DP in forestry.

The purpose of this paper is, therefore, not only to derive a set of optimal stand densities over time (an optimum thinning schedule), but also to introduce DP to the reader in a comprehensive, easy-to-understand, way.

CONVERTING TRADITIONAL PROBLEM STATEMENTS INTO THE FORM NEEDED FOR A DP SOLUTION

Traditional Description

A general description of a forest stand may be given in a number of ways: by specifying an individual attribute of the stand, such as its age, species composition, or standing crop. Or, one may give a description by specifying an ordered pair of attributes of the stand (age, standing crop), (standing crop, height), etc. The number and nature of the attributes used to characterize a stand are somewhat arbitrary, but an often-used stand attribute, when just one will suffice, is basal area per acre. We shall refer to basal area per acre as the stand state variable, and associate with it an age variable, although the latter will not be considered a state variable. Other variables that characterize stand attributes of interest are net basal area

[1] *Based on a paper presented by the authors at the Midwest Forest Mensurationists' Meeting at Pingree Park, Colorado, August 14-17, 1978.*

growth per unit area, basal area harvested per unit area, and some measure of return from the material harvested. Although these variables change in value every year, we will consider their values at periodic intervals. We can summarize the variables used to characterize our forest stand at any particular time, t, as follows:

Stand attribute	Symbol used to designate the attribute	Name given to the variable
standing crop basal area per unit area (pua)	B_{t-1}	state variable
net periodic basal area growth pua	ΔB_t	net growth
basal area harvested pua	Y_t	decision variable
return from basal area harvested	R_t	return

Forest managers frequently want either to maximize physical yield or returns from forest stands. A mathematical statement of this relation for maximum physical yield is:

$$\underset{\text{(over all harvest cuts)}}{\text{Maximize}} \quad \sum_{t=1}^{N} \alpha H_{t-1} Y_t$$

where N = number of the final stage, and

αH_{t-1} = numerical constant times mean height of trees removed at the beginning of stage t.

A statement of this relation for maximum return is:

$$\underset{\text{(over all harvest cuts)}}{\text{Maximize}} \quad \sum_{t=1}^{n} R_t, \text{ where } R_t \text{ is related to } Y_t.$$

Of course, the amount cut at any time t, Y_t, cannot be greater than the amount present at that time, B_{t-1}, nor less than 0.

A geometric portrayal of the above problem statement shows the traditional saw-toothed pattern of stand development following cutting (fig. 1). Maximum physical yield comes from thinning

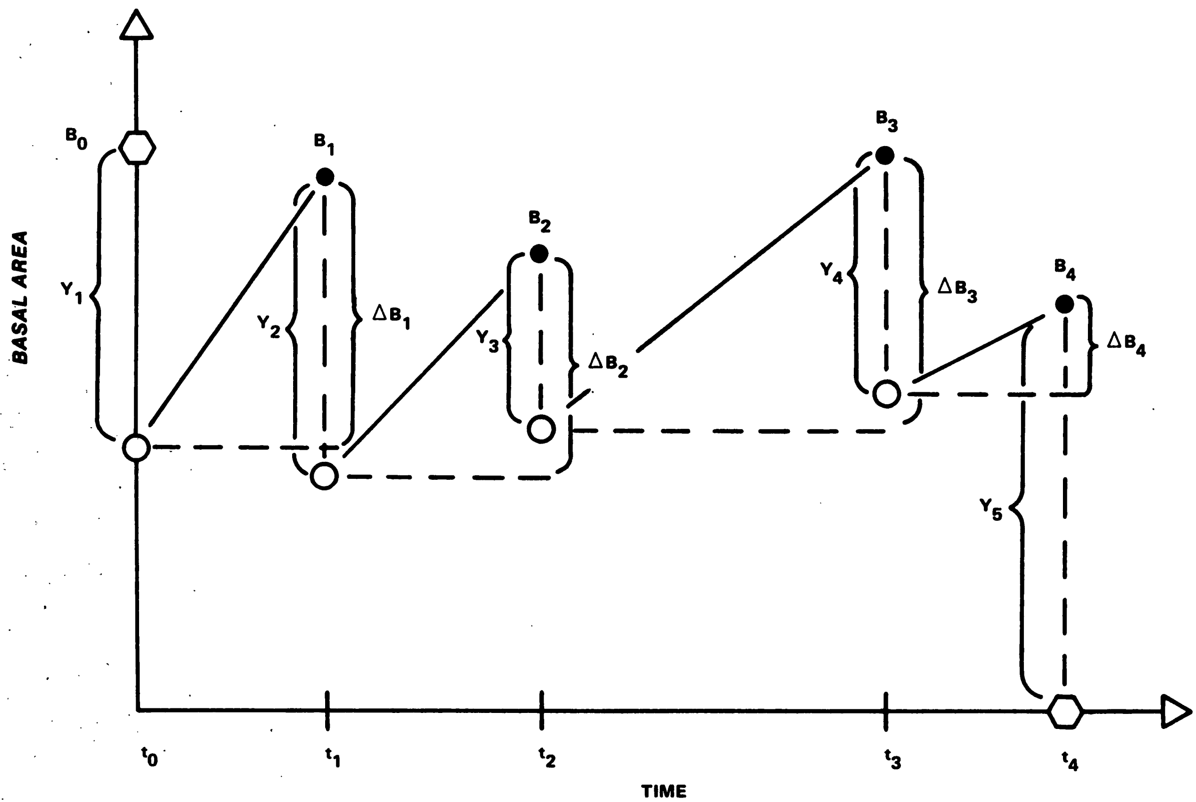


Figure 1. — *Traditional formulation of the "optimal stand density over time" problem expressed in terms of growth periods (t), initial and final stand basal areas (B), and amounts harvested at each cutting (Y).*

intensities Y_t, t = 1,2,3,4, in such a way that their sum, plus the final harvest cut, Y_5, is a maximum, or that the return from the four intermediate and one final harvest cut is a maximum.

Let us now convert the familiar problem into a form that lends itself to dynamic programming. To do this we must convert it into a description for a multistage decision problem.

Multistage Decision Process Description

The limited use of DP in forestry has come about not because forestry has no suitable problems, but rather because many potential users have not known how to formulate problems so that DP can be used to solve them. Part of the difficulty may stem from the DP symbolism—the shorthand used in expressing problems in a multistage decision process. Another source of confusion is the variety of decision process frameworks: deterministic, stochastic, finite time, infinite time, discrete state, continuous time, etc. We will treat two types of deterministic, finite time problems in this paper. They are both discrete time, one being discrete state and one continuous state (fig. 2).

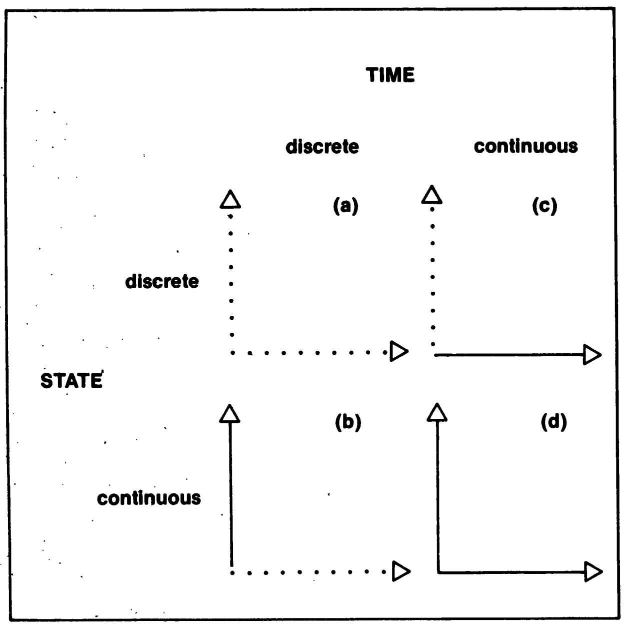


Figure 2. — *Kinds of deterministic multistage decision problems. Problems of type (a) and (b) are treated in this paper.*

Let us return briefly to the symbolism used in expressing multistage decision processes. In addition to the normal "let B designate basal area per unit area" type of symbolism, we have the following:

(1) Small "n" designates one of the discrete times called a stage. "Stage" can also be read as "growth period." Thus, B_n designates basal area at the end of stage (growth period) n.

(2) "(,)" signifies the arguments in a traditional mathematical function. Normally the arguments are related in some arithmetic manner, such as
$y=f(x,z)$, where $f=x+z$ or x/z, etc.

(3) "[,]" signifies a more general situation than "(,)". The arguments in the square brackets can be simple variables, such as x, as well as functional relationships between x and, say, z. Thus, [x,z] may be shorthand for $x+z-f(x,z)$, where $f = x/z$. Normally "[ ]" designates a transformation.

An additional source of confusion in DP stems from using symbols for a functional relation and its numerical value. Normally in mathematics one does not write $y= y(x,z)$, because the y to the left of the equality designates the numerical value of the function on the right side. Normally one would say $y= f(x,z)$, with f designating the manner in which x and z are combined. But in DP problem formulation it is normal to use expressions such as $T= T[x(1),u(1),\ldots,u(N)]$. It is usually safe to associate the letter "T" with the transformation implicit in the brackets, but sometimes "T" stands for the numerical value of the quantity on the righthand side of the equality. In the latter case, it should have a stage number attached—for example, T_n—to indicate over how many stages the transformation applies.

(4) "{ }" is typically used as

$$\max_{u(1)\ldots u(N)} \left\{T[x(1),u(1),\ldots,u(N)]\right\}$$

subject to any restrictions

to designate "the maximum value of T[x(1), u(1), . . . ,u(N)]that can be obtained by adjusting u(1), . . . ,u(N) without infringing on any of the restrictions".

Now let's convert the traditional expression of the problem into the form required for multistage decision problems. First, we need to identify the old variables in their new garb:

- n — instead of the continuous time variable, t, we shall use the discrete variable n to designate a stage in time. Ordinarily, "n" takes on values from 1 to N. The value of N is often used to classify the problem; e.g., if $N=4$, we refer to a four-stage process. In this paper a stage is treated as a growth period of unspecified length.
- B_{n-1} — basal area per unit area at the beginning of stage n.
- Y_n — the amount of basal area removed per unit area at the beginning of stage n.
- ΔB_n — net periodic basal area increment per unit area during stage n.
- R_n — the return at the beginning of stage n. It is related to the amount cut, Y_n—i.e., the decision variable.
- $(B_{n-1}-Y_n)$ — the residual basal area per unit area at the beginning of growth period n.

The traditional and multistage formulations of the optimum stand density problem are actually very similar. The traditional form was given in figure 1, and the multistage form in figure 3. The time axis in (1) has been divided into stages of uniform length in (3), each with a beginning and an end. The standing crop, $B_0, B_1, \ldots, B_{N-1}$, is reduced at the beginning of each stage by an amount $Y_1, Y_2, \ldots Y_N$ to give residual standing crop at the beginning of each stage, (B_0-Y_1), $(B_1-Y_2), \ldots, (B_{N-1}-Y_N)$. The latter standing crop values form the base on which growth is based during the stage, e.g., ΔB_1 occurs during stage one, begins with (B_0-Y_1), and ends with B_1. Similarly, ΔB_2 occurs during stage two, begins with (B_1-Y_2), and ends with B_2, etc.

With these old ideas in new garb, let us re-express our objective using

$$\max_{Y_1, Y_2, \ldots, Y_N} \sum_{n=1}^{N} R_n$$

subject to: $B_n = B_{n-1} - Y_n + \Delta B_n$,
$0 \leq Y_n \leq B_{n-1}$, with B_0 given, and
$R_n = R_n(B_{n-1}, Y_n)$.

The objective is to determine how much to thin at stage 1, Y_1, stage 2, $Y_2, \ldots$, and stage N, Y_N, such

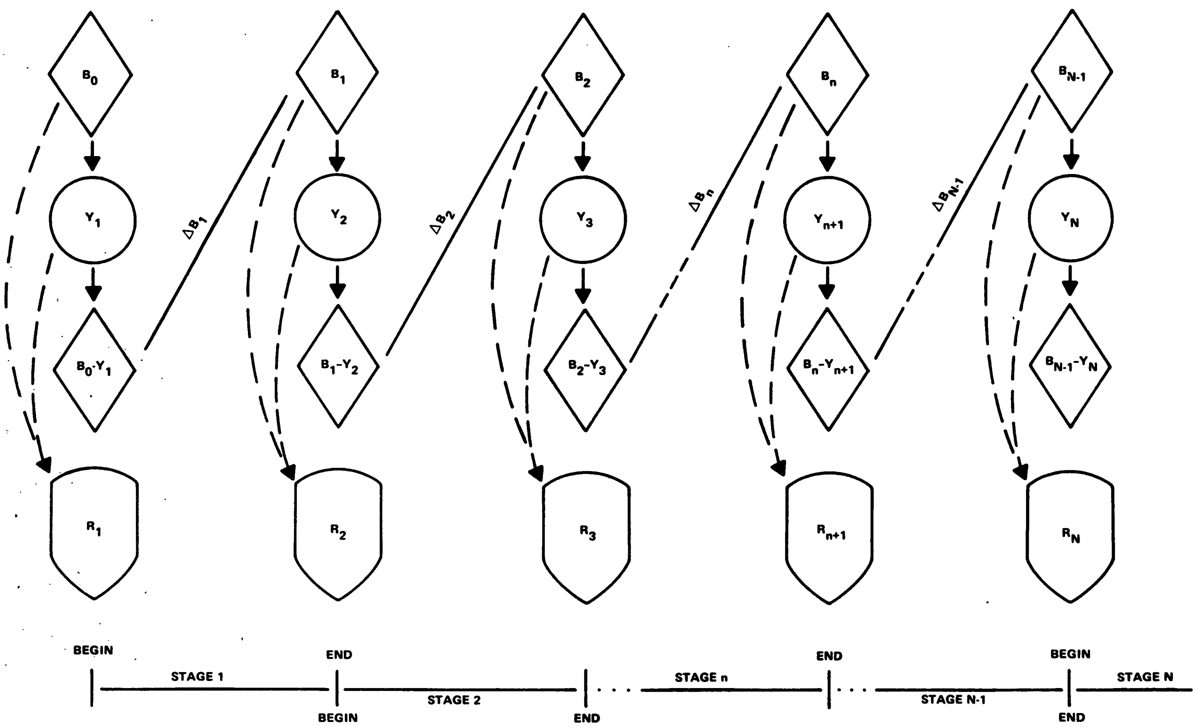


Figure 3. — *Multistage decision problem formulation of the "optimal stand density over time" problem. The decision variable, Y, is contained in the circle. Note the numbering of the stages, and that a point in time may, simultaneously, be the end of stage n-1 and the beginning of stage n.*

that the sum of the returns from the intermediate cuts plus the final clearcut is maximized when the forest standing crop basal area, B_n, changes as indicated.

Looking at the change in more detail we have

$$\begin{pmatrix}\text{Basal area}\\ \text{at } \textit{end} \text{ of}\\ \text{stage } n\end{pmatrix} = \begin{pmatrix}\text{Basal area}\\ \text{at } \textit{end} \text{ of}\\ \text{stage } n\text{-}1\end{pmatrix} - \begin{pmatrix}\text{Basal area}\\ \text{removed at}\\ \textit{beginning} \text{ of}\\ \text{stage } n\end{pmatrix} + \begin{pmatrix}\text{Net basal area}\\ \text{growth } \textit{during}\\ \text{stage } n\end{pmatrix}, \text{ or}$$

$$B_n = (B_{n-1}-Y_n) + \Delta B_n \qquad (1)$$

Equation (1) is called the recursive relationship for stage dependence.

Looking further at the growth component of (1), we have

$$\Delta B_n = g(B_{n-1}-Y_n, S, \text{age}), \qquad (2)$$

where g is an unspecified (for now) mathematical combination of what is available to grow during stage n, $(B_{n-1}-Y_n)$, site (S), and stand age.

This, of course, is a stand growth formulation as opposed to an individual tree growth formulation. Equation (2) can be simplified by removing age from the argument list and letting the mathematical function itself carry age's proxy, stage:

$$\Delta B_n = g_n(B_{n-1}-Y_n, S) \qquad (3)$$

Substituting equation (3) into (1) gives a more complete statement of the "recursive relationship for stage dependence":

$$\begin{aligned} B_n &= (B_{n-1}-Y_n) + g_n (B_{n-1}-Y_n, S) \qquad (4)\\ &= T_n[B_{n-1}, Y_n] \text{ for a given site.}\end{aligned}$$

A fundamental stratagem in formulating multistage problems is to show that the state of the system, our B_{n-1}, at any stage is a function of the initial system state, B_0, and the intervening decision variables $Y_1 \ldots Y_{n-1}$. This can be shown by repeated substitution into the stage transformation relation T[ ],

$$B_n = T_n[B_{n-1}, Y_n].$$

Recall $\quad B_{n-1} = T_{n-1}[B_{n-2}, Y_{n-1}]$, so

$$B_n = T_n[T_{n-1}[B_{n-2}, Y_{n-1}], Y_n].$$

Again, $\quad B_{n-2} = T_{n-2}[B_{n-3}, Y_{n-2}]$, so

$$B_n = T_n\big[T_{n-1}\big[T_{n-2}[B_{n-3}, Y_{n-2}], Y_{n-1}\big], Y_n\big].$$

If n=3, and we look only at the variables in brackets, we see the only state variable used is B_0 — the initial system state — but all the decision variables are used, i.e.,

$$B_3 = T_3'[B_0, Y_1, Y_2, Y_3]. \qquad (5)$$

This says simply that the basal area at the end of stage 3 is dependent on the basal area present initially, B_0, the amount cut (Y_n, n = 1,2,3), and any relationships between B_n and Y_n that are implicit in the stage transformation T[,]. The "′" on T in equation (5) simply means that the substitution of previous stage transformations has been carried to completion— completion being when B_0 is the only state variable in the list of stage transformation arguments.

A parallel stratagem is employed for the stage return, R_n. Figure 3 shows that the return at the beginning of stage n, R_n, is affected by the decision variable at the beginning of nth stage, Y_n, and the state variable at the end of the previous stage, B_{n-1}, i.e., R_2 is affected by Y_2 and B_1. Thus

$$R_n = R_n(B_{n-1}, Y_n), \qquad (6)$$

and, as before, $B_{n-1} = T_{n-1}[B_{n-2}, Y_{n-1}]$.

Carried to completion we have the "recursive relationship for stage dependence," i.e.,

$$R_n = R_n'(B_0, Y_1, Y_2, Y_3, \ldots, Y_{n-1}, Y_n). \qquad (7)$$

This simply states that the return from managing a forest to the beginning of the nth stage is related to how much has been harvested, Y_n (n=1, . . . , n), how much standing crop was present initially, B_0, and the relationship (invisible in this notation) between growing stock and cut given in

$$B_n = (B_{n-1}-Y_n) + g_n(B_{n-1}-Y_n, S).$$

SOLVING MULTISTAGE DECISION PROBLEMS WITH DP

Up to this point we have concentrated on converting the concepts and terminology used in expressing traditional forest management problems to those used in expressing multistage decision problems (table 1). Now let's solve our problem using DP.

A universal limitation on using DP to solve multistage decision problems is called the decomposability constraint, and it applies to the total return function

$$TR_N = TR(R_1, R_2, \ldots, R_N)$$

Table 1. — *Traditional and multistage decision descriptions of the problem of determining an optimal cutting regime for one rotation of an even-aged stand*

Traditional description	Multistage decision description
Cutting will be spread out over the rotation.	A series of decisions is required.
Standing crop cannot be more than previous standing crop plus growth. Maximum cut is a clearcut.	There are restrictions on the possible values of the state variable B_{n-1} and the decision variable Y_n.
Standing crop in a stand depends on how much was available to grow. Amount cut at any given time depends on amount cut earlier in the rotation.	The state at the end of stage n, B_n, depends on the state at the previous stage and the decision, (Y_n) which depends on previous decisions $Y_{n-1}, Y_{n-2}, \ldots, Y_1$.
Cutting opportunities occur at X-year intervals of which there are N.	The process runs for N stages.
Each cutting opportunity results in some (possibly zero) level of cutting.	Each stage of the process requires a decision, Y_n, that can be converted to a return, R_n.
The total return from managing a forest depends on the returns from each cut.	The total return from N stages of the process, TR_N, depends on the returns from each stage according to some functional relationship, i.e., $TR_N = TR(R_N, \ldots, R_n, R_{n-1}, \ldots, R_1)$.

for an N-stage process. According to Nemhauser (1966), the objective of decomposing the total return function into N equivalent subproblems is to have a formulation where:

1) Each subproblem contains only one state variable B_{n-1} and one decision variable Y_n, $n=1,2,\ldots,N$.
2) Each of the subproblems will be roughly equivalent to a one-stage optimization problem, but all the decisions are interdependent.
3) The optimal decision Y_n^* (for $n=1$, $n=2, \ldots n=N$) will be derived one at a time.
4) The optimal solutions from the subproblems are combined to derive the optimal solution to the whole problem.
5) The decomposition, according to the principle of DP, insures that the number of feasible solutions does not change from the original number, and the value of the objective function associated with each feasible solution also does not change.

Two mathematical properties of the return function are sufficient for decomposition to be valid: separability and monotonicity. They are discussed in Nemhauser (1966), pages 34 to 39. The simple return functions used in this paper are known to be decomposable.

We now need to develop our return function. The total return (TR_N) from the first stage through the Nth stage is some as-yet-unspecified function of the individual stage returns

$$TR_N = TR(R_N, \ldots, R_n, \ldots, R_1) \tag{8}$$

where $R_n = R_n(B_{n-1}, Y_n) = R_n'(B_0, Y_1, \ldots, Y_n)$ for $n=1, \ldots, N$.

Let us introduce a new variable $f_N(B_0)$ as follows:

$f_N(B_0)$ = the optimal value of TR_N using the optimal cutting policy over N stages (growth periods) starting from the state B_0.

$$= \max_{Y_N, Y_{N-1}, \ldots, Y_1} \left\{ TR(R_N, \ldots, R_1) \right\} \tag{9}$$

subject to constraints

The problem is to choose Y_n, $(n=1, \ldots, N)$ so that equation (9) is true.

A simple and reasonable optimal return function that can be decomposed is

$$f_N(B_0) = \max_{Y_N, Y_{N-1}, \ldots, Y_1} \left\{ \sum_{n=1}^{N} R_n \right\} \tag{10}$$

subject to constraints

Recall the return at the beginning of stage n is

$R_n = R_n(B_{n-1}, Y_n)$ and that the stage transformation is

$$B_{n-1} = T_{n-1}[B_{n-2}, Y_{n-1}] = T_{n-1}'[B_0, Y_1, Y_2, \ldots, Y_{n-2}, Y_{n-1}]$$

as well as

$$R_n = R_n'(B_0, Y_1, Y_2, \ldots, Y_{n-1}, Y_n).$$

So, in expanded terms, equation (10) is:

$$f_N(B_0) = \max_{Y_1, \ldots, Y_N} \left\{ R_1(B_0, Y_1) + R_2(B_1, Y_2) \ldots + \ldots R_n(B_{n-1}, Y_n) \ldots + \ldots R_N(B_{N-1}, Y_N) \right\}$$

subject to constraints

or

$$f_N(B_0) = \max_{Y_1, \ldots, Y_N} \left\{ R_1(B_0, Y_1) + R_2'(B_0, Y_1, Y_2) \ldots + \ldots R_n'(B_0, Y_1, \ldots, Y_n) \ldots + \ldots R_N'(B_0, Y_1, \ldots, Y_{N-1}, Y_N) \right\}.$$

subject to constraints

When the total return function is decomposable, we can modify the above equation as follows and not change its value:

$$f_N(B_0) = \max_{0 \leq Y_1 \leq B_0} \left\{ R_1 + \max_{Y_2, \ldots, Y_N} \left\{ R_2'(B_0, Y_1, Y_2) \ldots + \ldots R_n'(B_0, Y_1, \ldots, Y_n) \ldots + \ldots R_N'(B_0, Y_1, \ldots, Y_N) \right\} \right\}.$$

Done once legitimately, it may be repeated:

$$f_N(B_0) = \max_{0 \leq Y_1 \leq B_0} \left\{ R_1 + \max_{0 \leq Y_2 \leq B_1} \left\{ R_2'(B_0, Y_1, Y_2) + \max_{Y_3, \ldots, Y_N} \left\{ R_3'(B_0, Y_1, Y_2, Y_3) \ldots + \ldots R_N'(B_0, Y_1, Y_2, \ldots, Y_N) \right\} \right\} \right\}.$$

Expressing this in a recursive manner gives

$$f_N(B_0) = \max_{0 \leq Y_1 \leq B_0} \left\{ R_1 + f_{N-1}(B_1) \right\}, \quad (11)$$

where $R_1 = R_1[B_0, Y_1]$, and $B_1 = T_1[B_0, Y_1]$.

Decomposing $f_{N-1}(B_1)$ in equation (11) further, we have, generally

$$f_{N-(n-1)}(B_{n-1}) = \max_{0 \leq Y_n \leq B_{n-1}} \left\{ R_n + f_{N-n}(B_n) \right\} \quad (12)$$

for $n = N, N-1, \ldots, 2, 1$, when we use the backward solution approach. Note that equation (12) is identical to equation (11) when $n=1$ — i.e., at the final solution stage. Describing the variables in equation (12), we have

$f_{N-(n-1)}(B_{n-1})$ = the cumulative return over $N-(n-1)$ stages when the backward solution reaches the beginning of the nth stage, with state variable B_{n-1}, having used optimal decision variables $Y_n, Y_{n+1}, \ldots, Y_N$.

R_n = the return at the beginning of nth stage $= R_n(B_{n-1}, Y_n)$, with Y_n the basal area thinned at the beginning of the nth stage, and

$f_{N-n}(B_n)$ = the cumulative return over $N-n$ stages when the backward solution reaches the beginning of the $(n+1)$ stage, with state variable B_n, having used optimal decision variables $Y_{n+1}, Y_{n+2}, \ldots, Y_N$, and

$B_n = T_n[B_{n-1}, Y_n]$.

Figure 4 summarizes how equation (12) works in the backward solution approach.

Equation (12) expresses what is called the functional recurrence relation.

If we have a four-stage decision problem such as the one portrayed traditionally in figure 5a and in the multistage form in 5b, the equation embodying the functional recurrence relation is given in 5c. From 5a we see that we know two states of the system: B_0, the initial state, and B_4 the final state. Since we are clearcutting the stand at the beginning of the Nth stage, $B_N = 0$.

Now let's work the equation in 5c backwards for four stages.

Start: $n=4$

$$f_1(B_3) = \max_{Y_4} \left\{ R_4 + f_0(B_4) \right\}$$

a) From figure 5a we see that $B_4 = 0$, therefore $f_0(B_4) = 0$, and

$$f_1(B_3) = \max_{Y_4^* = B_3} \left\{ R_4 \right\}$$

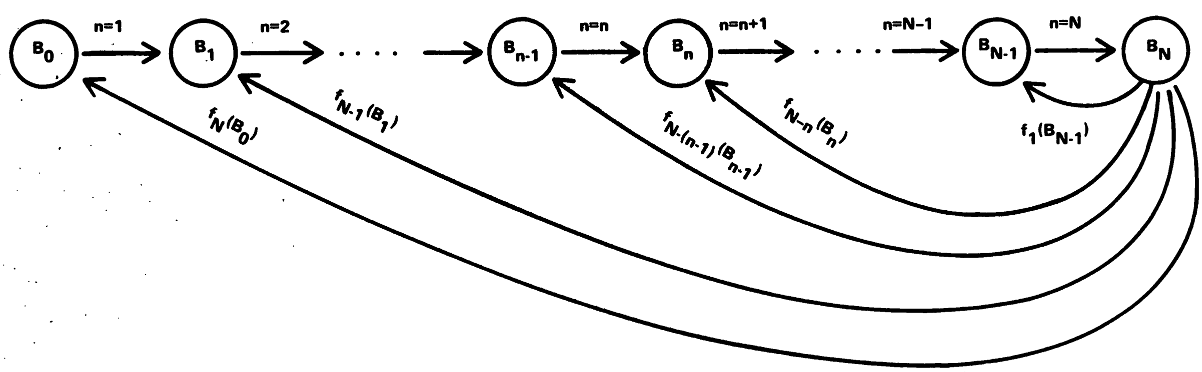
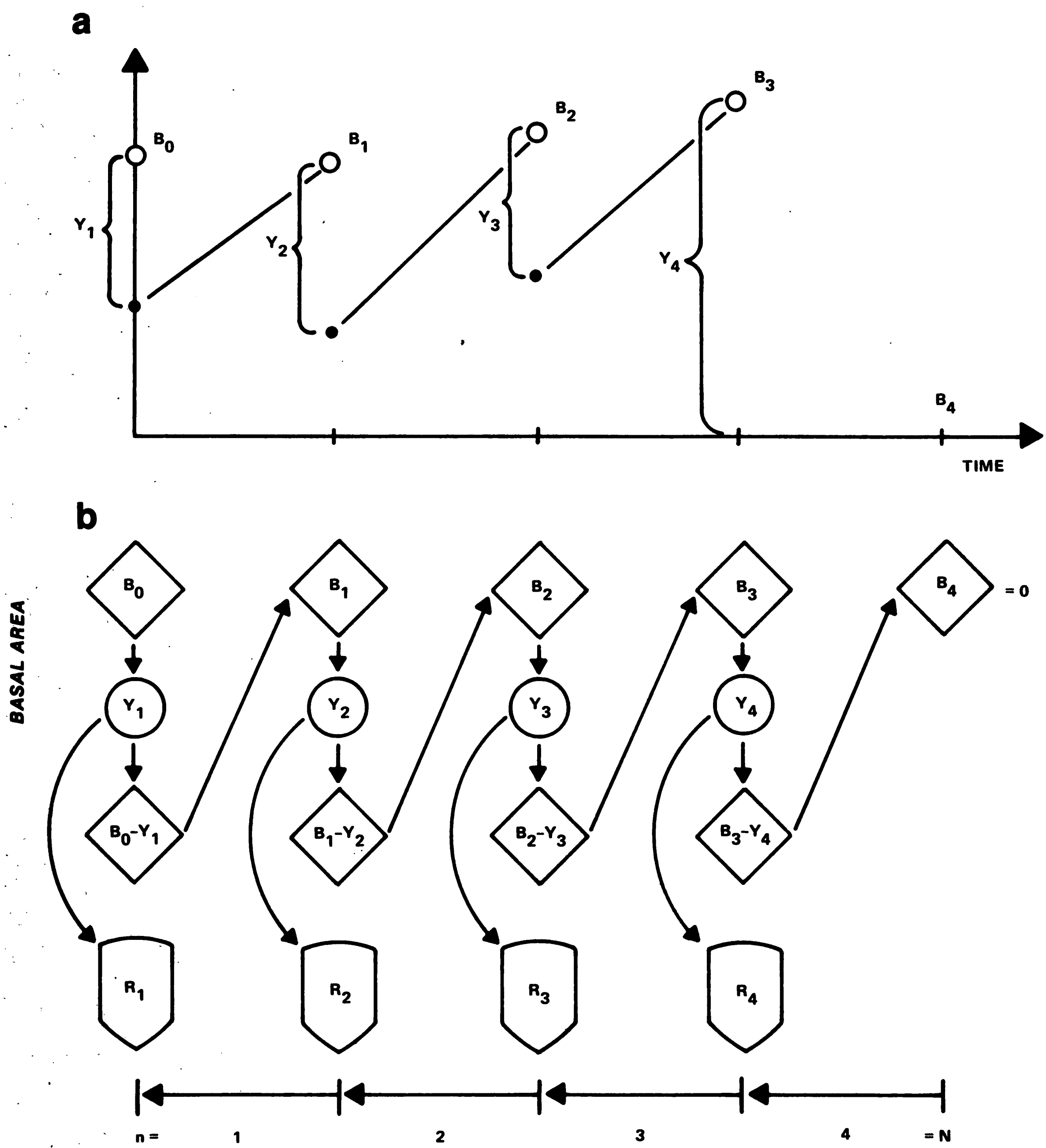


Figure 4. — *Schematic summary of relations given in equation (12).*



c

$$f_{N-(n-1)}(B_{n-1}) = \underset{Y_n}{\mathrm{Max}} \left\{ R_n + f_{N-n}(B_n) \right\} \text{, for } n = 4,3,2,1.$$

Figure 5. — *Traditional geometric (a), multistage decision (b), and DP formulations (c) of the "optimal stand density over time" problem.*

b) The optimal decision is to clearcut at beginning of stage 4, at which time there is B_3 basal area.

Let n=3

$$f_2(B_2) = \max_{Y_3} \left\{ R_3 + f_1(B_3) \right\},$$

with $f_1(B_3)$ coming from the above equation.

When n=2

$$f_3(B_1) = \max_{Y_2} \left\{ R_2 + f_2(B_2) \right\}, \quad \text{and}$$

when n=1, the final solution stage,

$$f_4(B_0) = \max_{Y_1} \left\{ R_1 + f_3(B_1) \right\}. \quad (13)$$

Repeatedly substituting what is known into equation (13) gives

$$f_4(B_0) = \max_{Y_1} \left\{ R_1 + \max_{Y_2} \left\{ R_2 + f_2(B_2) \right\} \right\}, \text{ and}$$

$$f_4(B_0) = \max_{Y_1} \left\{ R_1 + \max_{Y_2} \left\{ R_2 + \max_{Y_3} \left\{ R_3 + f_1(B_3) \right\} \right\} \right\}, \text{ and finally}$$

$$f_4(B_0) = \max_{Y_1} \left\{ R_1 + \max_{Y_2} \left\{ R_2 + \max_{Y_3} \left\{ R_3 + \max_{Y_4} \left\{ R_4 + f_0(B_4) \right\} \right\} \right\} \right\}, \text{ or}$$

$$= \max_{Y_1,Y_2,Y_3,Y_4} \left\{ \sum_{n=1}^{N=4} R_n \right\}.$$

To solve the DP problem above, we need to know the numerical values of two variables: B_0 and $f_0(B_4)$. B_0 is the initial stand basal area and $f_0(B_4)$ is the cumulative return over 0 stages. Because we are clearcutting at the beginning of stage four we know $f_0(B_4) = 0$. This allows us to determine the innermost maximum, and progressively work our way to the outer maximum.

NONCALCULUS SEARCH OF A NETWORK USING DP

Suppose we possess the ability to simulate the development of forest stands with and without thinning. This ability may be based on, say, a stand or individual tree growth projection system. Then, for a species on a given site class, an initial stand density (B_0) branches according to different levels of basal area removed at the beginning of the stage (growth period). The basal area at the end of the first growth period is given by $B_n = B_{n-1} - Y_n + \Delta B_n$, with n=1. The process is repeated for as many values of Y_1 (thinning levels) as we wish to try. Let us say we try three different values of Y_1, giving rise to three values of B_1 at the end of the first growth period. For each of the three resulting values of B_1, there can be several values of Y_2 tested, giving a set of B_2 values at the end of the second growth period. Carried to completion, the result is a network such as that shown in figure 6. This figure shows that from one to three cutting intensities were tried at the beginning of each stage for each state at that stage. Thus, at the beginning of stage three there are four possible states (BA = 185, ^{1}145, 175, and ^{2}145), and each was tested with two thinning intensities. Of course, many more thinning intensities can be tested, but the resultant computational load is best handled by a computer. Since we are dealing with a number of stages and a discrete number of states at each stage, the problem can be classified as discrete stage-discrete state (fig. 2).

To facilitate handling the functional recurrence equation of DP we attempt to find the thinning schedule that will maximize total cords harvested from this hypothetical stand by applying the formulation

$$f_{N-(n-1)}(B_{n-1}) = \max_{Y_n} \left\{ V_n + f_{N-n}(B_n) \right\} \quad (14)$$

to EACH NODE OF EACH STAGE. (We have substituted V_n for R_n because the objective is maximum physical volume yield.)

Of course, N=4, so

$$\text{for } n=4 \;\; f_1(B_3) = \max_{Y_4} \left\{ V_4 + f_0(B_4) \right\}.$$

As before, $f_0(B_4) = 0$, because of the pre-established decision to clearcut at the beginning of stage N(=4). Thus,

$$f_1(B_3) = \max_{Y_4^*=B_3} \left\{ V_4 \right\}$$

This equation applies for every value of B_3, so we list them here along with the cordwood yield that would result from clearcutting the stand with B_3 square feet of basal area:

$f_1(176) = 60$ $\quad$ $f_1(175) = 75$
$f_1(140) = 55$ $\quad$ $f_1(148) = 70$
$f_1(160) = 60$ $\quad$ $f_1(165) = 75$
$f_1(146) = 65$ $\quad$ $f_1(152) = 70$

Having evaluated equation (14) for every node under B_3 in figure 6 we proceed backward to the

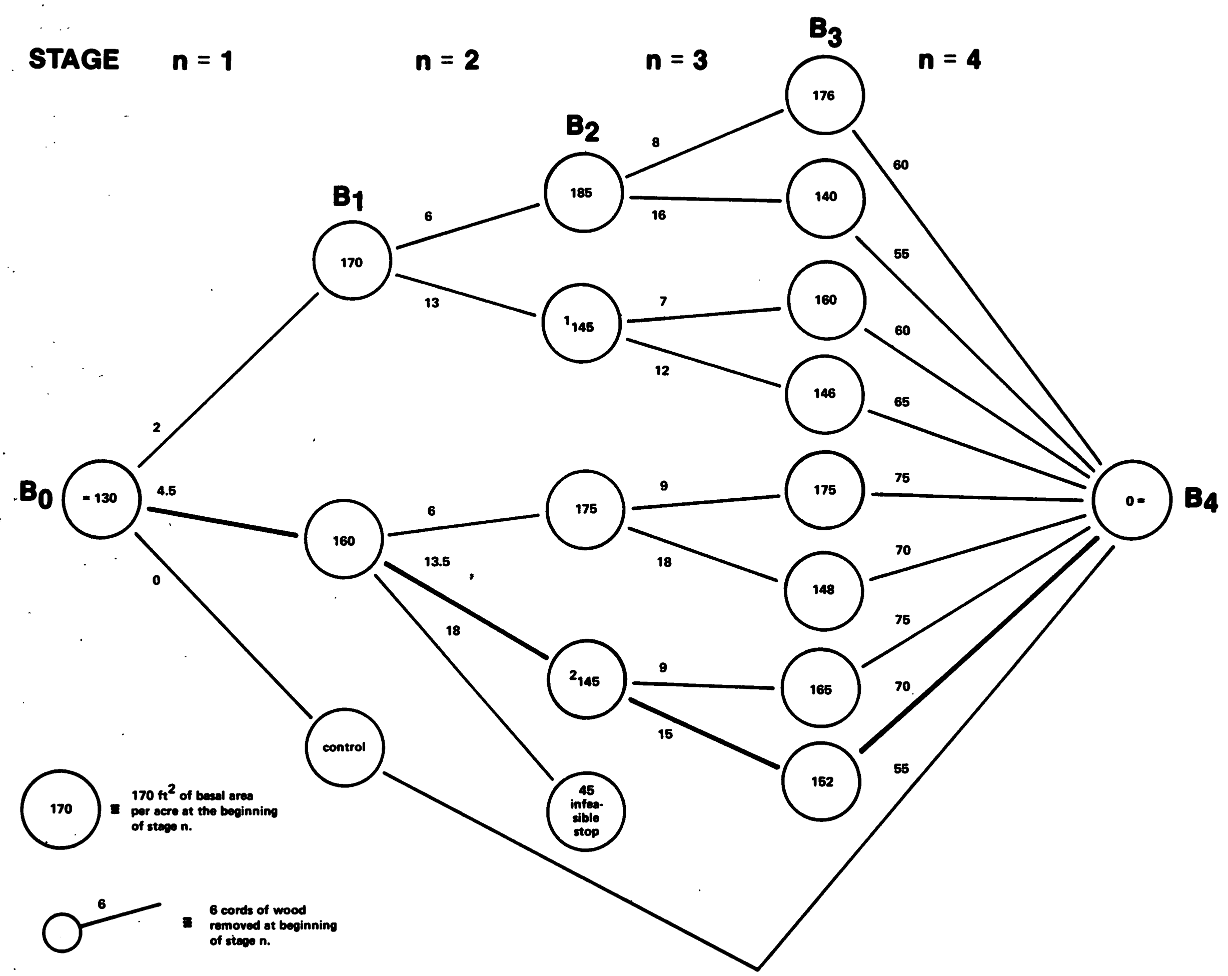


Figure 6. —*Network of simulated stand densities and harvest cuts for a four-stage process. Darkened line is the optimal cutting policy.*

nodes under B_2 that represent the system state at the beginning of stage three.

$$\text{For } n=3 \quad f_2(B_2) = \max_{Y_3} \left\{ V_3 + f_1(B_3) \right\}. \quad (15)$$

Again we evaluate equation (15) for every value of B_2 and, for each value of B_2, every value of Y_3. Thus,

$$f_2(185) = \max_{Y_3} \begin{Bmatrix} 8 + f_1(176) \\ 16 + f_1(140)^* \end{Bmatrix} = \max_{Y_3} \begin{Bmatrix} 8 + 60 \\ 16 + 55^* \end{Bmatrix} = 71$$

Comparing the sums (8 + 60) and (16 + 55), we choose the larger, and place a "*" to indicate that the 71 came from the route $f_2(185) \rightarrow f_1(140)$. Continuing for the remaining three nodes under B_2 gives

$$f_2({}^1145) = \max_{Y_3} \begin{Bmatrix} 7 + f_1(160) \\ 12 + f_1(146)^* \end{Bmatrix} = \max_{Y_3} \begin{Bmatrix} 7 + 60 \\ 12 + 65^* \end{Bmatrix} = 77$$

$$f_2(175) = \max_{Y_3} \begin{Bmatrix} 9 + f_1(175) \\ 18 + f_1(148)^* \end{Bmatrix} = \max_{Y_3} \begin{Bmatrix} 9 + 75 \\ 18 + 70^* \end{Bmatrix} = 88$$

$$f_2({}^2145) = \max_{Y_3} \begin{Bmatrix} 9 + f_1(165) \\ 15 + f_1(152)^* \end{Bmatrix} = \max_{Y_3} \begin{Bmatrix} 9 + 75 \\ 15 + 70^* \end{Bmatrix} = 85$$

For n=2

$$f_3(B_1) = \max_{Y_2} \left\{ V_2 + f_2(B_2) \right\}$$

$$f_3(170) = \max_{Y_2} \begin{Bmatrix} 6 + f_2(185) \\ 13 + f_2({}^1145)^* \end{Bmatrix} = \max_{Y_2} \begin{Bmatrix} 6 + 71 \\ 13 + 77^* \end{Bmatrix} = 90$$

$$f_3(160) = \max_{Y_2} \begin{Bmatrix} 6 + f_2(175) \\ 13.5 + f_2({}^2145)^* \end{Bmatrix} = \max_{Y_2} \begin{Bmatrix} 6 + 88 \\ 13.5 + 85^* \end{Bmatrix} = 98.5$$

For n=1 (the final solution stage)

$$f_4(B_0) = \max_{Y_1} \left\{ V_1 + f_3(B_1) \right\}$$

$$f_4(130) = \max_{Y_1} \begin{Bmatrix} 2 + f_3(170) \\ 4.5 + f_3(160)^* \\ 0 + \text{control} \end{Bmatrix} = \max_{Y_1} \begin{Bmatrix} 2 + 90 \\ 4.5 + 98.5^* \\ 0 + 55 \end{Bmatrix} = 103$$

Then, the optimal thinning schedule can be traced back through the nodes by locating the starred return functions as follows (see heavy line in fig. 6):

$$\text{Start } (130) \xrightarrow{n=1} (160) \xrightarrow{n=2} (145) \xrightarrow{n=3} (152) \xrightarrow{n=N=4} (0) \text{ End}$$

The associated returns (cord volume yields) are
$4.5 + 13.5 + 15 + 70 = 103$ cords/acre.

$$\text{Note that } f_4(B_0) \equiv \sum_{n=1}^{N=4} V_n^* = 103 \text{ cords/acre.}$$

The basal areas that must be removed at the beginning of each stage to give these cord yields were omitted from figure 6 to avoid clutter. They are as follows: At the beginning of $n=1$, thin from 130 to 90 sq. ft., at the beginning of $n=2$, thin from 160 to 80 sq. ft., at the beginning of $n=3$, thin from 145 to 90 sq. ft., and at the beginning of $n=N=4$, clearcut (remove 152 sq. ft.).

Thus, total basal area harvested is $40 + 80 + 55 + 152 = 327$ sq. ft./acre, to give 103 cords/acre. The reader is invited to find a thinning policy from among those shown in figure 6 that gives a greater total yield in cords per acre. If the problem involved more than four stages and more than two or three thinning intensities at the start of each stage, the number of alternative policies would increase rapidly, making the digital computer a necessity. Of course, the optimal path through the network is not necessarily **the** optimal way of thinning the stand; rather, it is the best from among the ways simulated. Now let's look at an approach that can be used to compute the exact optimal thinning intensity for each stage of a four-stage process.

CALCULUS SEARCH USING DP

In the previous example we dealt with a problem formulated in terms of discrete stages as well as discrete states. Now let's consider a problem formulated so there is a continuum of possible states at each stage (see fig. 2a,b, for comparison). The basic unknowns are the same as in the previous example: How much do we thin a stand, on a particular site, at each of four times so that we maximize total cordwood volume production? Figure 5a provides a geometric picture of what we know and what we need to determine.

We know:

B_0 = the initial basal area stocking of the stand,

B_4 = 0 because we want to clearcut at the beginning of stage four, so

Y_4^* = B_3 because the optimal decision is to cut as much as is present at the beginning of the last stage.

We assume:

V_n = the cordwood volume harvested at the beginning of stage n, or
= $\alpha H_{n-1} Y_n$, where

H_{n-1} = average height of trees removed at the beginning of stage n,

α = a model parameter, and

Y_n = basal area cut at the beginning of stage n.

An acceptable periodic basal area growth equation is given by

$$\Delta B_n = aS(B_{n-1} - Y_n) - b(B_{n-1} - Y_n)^m, \quad (16)$$

$(B_{n-1} - Y_n)$ = residual stand basal area/acre at the beginning of stage n,

S = $(Si)^r$, where Si is site index, and

a,b,m = positive numerical constants.

We do not know:

$Y_n, (n=1,2,3)$ = the basal area to be removed in thinnings at the beginning of stages one, two, and three so that total yield is maximized.

Again we employ the functional recurrence equation for DP that implements a backward solution:

$$f_{N-(n-1)}(B_{n-1}) = \max_{0 \leq Y_n \leq B_{n-1}} \left\{ V_n + f_{N-n}(B_n) \right\} \quad (17)$$

with $n = N, N-1, \ldots, 2, 1,$
$N = 4$, and
$V_n = R_n$ in equation (12).

Starting with $n=4$, we have

$$f_1(B_3) = \max_{Y_4} \left\{ \alpha H_3 Y_4 + f_0(B_4) \right\},$$

where $f_0(B_4) = 0$ because $B_4 = 0$. (Recall we clearcut at the beginning of stage 4, so in the equation

$B_4 = (B_3 - Y_4) + \Delta B_4$,

$B_4 = (B_3 - Y_4) + g_4(B_3 - Y_4, \text{Site})$).

But this is

$$B_4 = 0 + g_4(0, \text{Site}) = 0,$$

since $Y_4^* = B_3$, therefore

$$f_1(B_3) = \alpha H_3 B_3 \quad .$$

We see this result characterized in figure 7a. The interpretation of this is, although we do not know the standing crop basal area at the beginning of stage four, we should cut all of it.

Now, with n=3,

$$f_2(B_2) = \max_{0 \leq Y_3 \leq B_2} \left\{ V_3 + f_1(B_3) \right\} \text{ ,or, from above,}$$

$$= \max_{0 \leq Y_3 \leq B_2} \left\{ \alpha H_2 Y_3 + \alpha H_3 B_3 \right\}. \qquad (18)$$

Our strategy is to express the quantity in brackets (equation 18) in terms of Y_3, or quantities that determine Y_3. Since $B_3 = B_2 - Y_3 + \Delta B_3$ (see fig. 3), we substitute as follows:

$$f_2(B_2) = \max_{Y_3} \left\{ \alpha H_2 Y_3 + \alpha H_3 (B_2 - Y_3 + \Delta B_3) \right\}, \text{ but}$$

$\Delta B_3 = aS(B_2 - Y_3) - b(B_2 - Y_3)^m$, giving

$$f_2(B_2) = \max_{Y_3} \left\{ \alpha H_2 Y_3 + \alpha H_3 (B_2 - Y_3 + aS(B_2 - Y_3) - b(B_2 - Y_3)^m) \right\}. \qquad (19)$$

$$= \max_{Y_3} \left\{ \alpha \Theta_3 (Y_3) \right\}.$$

Now, the quantity in brackets, the quantity we want to maximize by selecting an appropriate value of Y_3, is expressed in terms of Y_3 and B_2 (ignoring the H_2 and H_3 that are determined outside the system). We can use ordinary calculus methods to obtain the value of Y_3 that maximizes

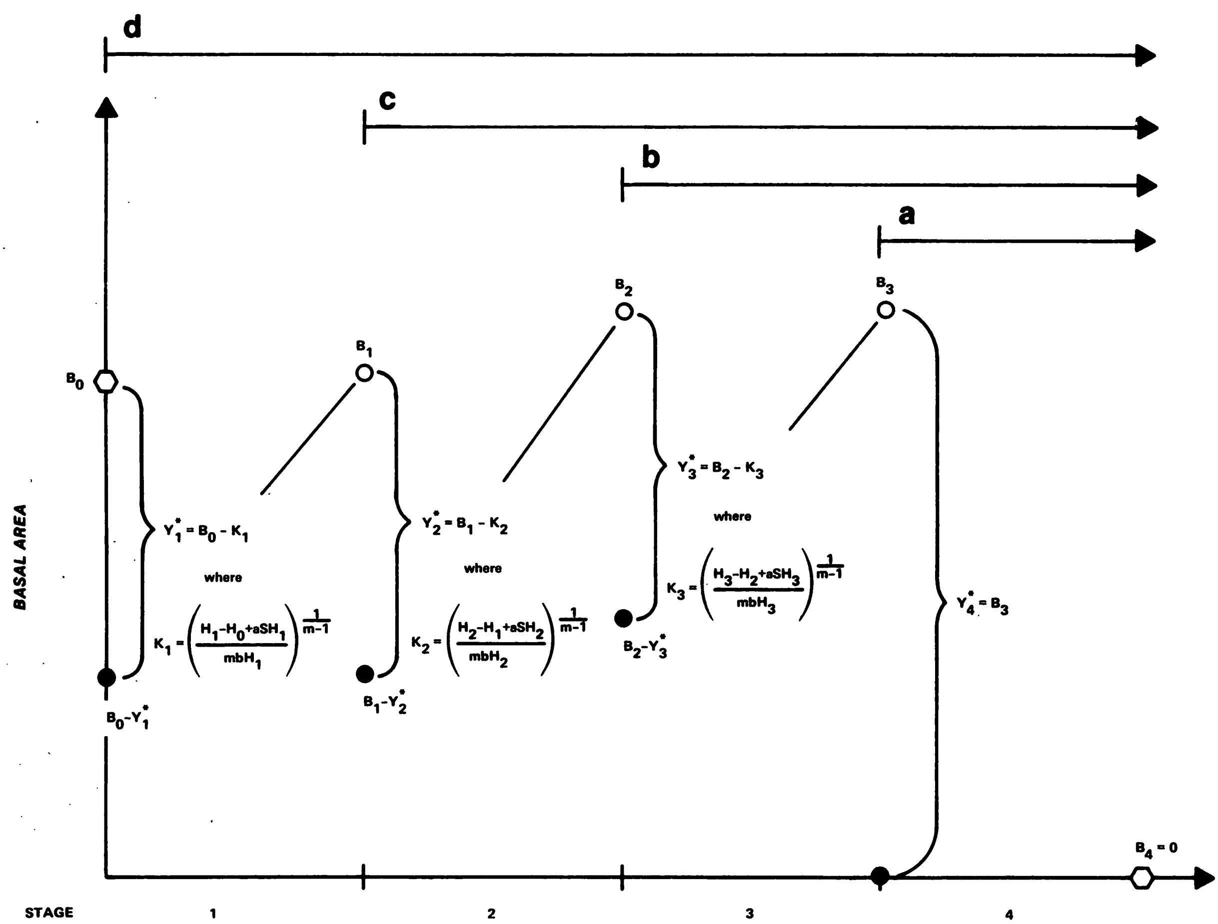


Figure 7. — *Optimal amounts to thin, Y_n^*, $n = 1 \ldots N$, at the beginning of each stage. The amounts to thin are derived analytically at stage four first, stage three next, etc. Because of the repeating pattern of variables in the equation for K_n, the actual application of the formulas can proceed in a forward direction.*

Θ_3 by taking the partial derivative of Θ_3 with respect to Y_3, equating to zero and solving for Y_3:

$$\frac{\partial\Theta_3(Y_3)}{\partial Y_3} = 0.$$

This gives the value of Y_3 that maximizes Θ_3,

$$Y_3^* = B_2 - K_3,$$

$$\text{where } K_3 = \left(\frac{H_3 - H_2 - aSH_3}{mbH_3}\right)^{\frac{1}{m-1}} > 0, \text{ and}$$

$$K_3 \leqslant B_2 \text{ to be biologically meaningful.}$$

This condition does not necessarily occur when B_2 is small. The problem generated if $K_3 > B_2$ implying a negative harvest, Y_3^*, will be discussed after the final solution to the dynamic programming problem has been derived.

A sufficient condition for a maximum rather than a minimum is that the second partial derivative of Θ_3 with respect to Y_3 is negative. It can be shown that this requires $a > 0$, $b > 0$, $Y_3 < B_2$ and $m > 1$. All these conditions generally hold. Since B_2 is the stand basal area at the end of the second stage, the thinning at the beginning of the third stage cannot affect B_2 (see fig. 3), so we assume $\partial B_2/\partial Y_3 = 0$. What the optimum value of Y_3, Y_3^*, does affect is the residual basal area at the beginning of stage 3, ΔB_3^*, V_3^*, Y_4^*, and V_4^*.

Substituting the optimal value of Y_3,

$$Y_3^* = B_2 - K_3,$$

back into equation (19) gives

$$\begin{aligned} f_2(B_2) &= \alpha(H_2(B_2 - K_3) + H_3(K_3 + aSK_3 - bK_3^m)), \text{ or} \\ &= \alpha(H_2(B_2 - K_3) + H_3P_3), \text{ where} \\ P_3 &= K_3 + aSK_3 - bK_3^m. \end{aligned}$$

What is the optimal amount of basal area to thin at the beginning of stage three? It depends on the amount of standing crop present at the beginning of stage three (B_2) and K_3. And B_2 we do not yet know.

Getting on with the search, we let n=2,

$$\begin{aligned} f_3(B_1) &= \max_{0\leqslant Y_2\leqslant B_1} \left\{V_2 + f_2(B_2)\right\} \\ &= \alpha \max_{Y_2} \left\{H_1Y_2 + H_2(B_2 - K_3) + H_3P_3\right\} \\ &= \alpha \max \; \Theta_2(Y_2), \end{aligned}$$

where $B_2 = B_1 - Y_2 + \Delta B_2 = T[B_1, Y_2]$.

$$\frac{\partial\Theta_2}{\partial Y_2} = 0 \text{ results in}$$

$$Y_2^* = B_1 - K_2,$$

$$\text{where } K_2 = \left(\frac{H_2 - H_1 + aSH_2}{mbH_2}\right)^{\frac{1}{m-1}} \geqslant 0.$$

Note that P_3 is not a function of Y_2 because of the volume return function and the growth model we used, so $\partial P_3/\partial Y_2 = 0$.

Then

$$f_3(B_1) = \alpha(H_1(B_1 - K_2) + H_2(P_2 - K_3) + H_3P_3),$$

where

$$P_2 = K_2 + aSK_2 - bK_2^m.$$

What is the optimal amount of basal area to thin at the beginning of stage two? It depends on how much is present to thin (B_1), and K_2; they depend on things we do not yet know. But, they are determined as follows, when

n=1

$$\begin{aligned} f_4(B_0) &= \max_{Y_1} \left\{V_1 + f_3(B_1)\right\}, \text{ and} \\ &= \max_{Y_1} \left\{\alpha H_0Y_1 + \alpha(H_1(B_1 - K_2) + H_2(P_2 - K_3) + H_3P_3)\right\}. \end{aligned} \tag{20}$$

where $B_1 = B_0 - Y_1 + \Delta B_1 = T_1[B_0, Y_1]$.

This results in the following optimal amount to thin at the beginning of stage one:

$$Y_1^* = B_0 - K_1,$$

$$\text{where } K_1 = \left(\frac{H_1 - H_0 + aSH_1}{mbH_1}\right)^{\frac{1}{m-1}} \geqslant 0.$$

Substituting Y_1^* into equation (20) gives the cumulative optimal return over four stages:

$$\begin{aligned} f_4(B_0) = \alpha\, (H_0(B_0 - K_1) + H_1(P_1 - K_2) + \\ H_2(P_2 - K_3) + H_3P_3), \end{aligned} \tag{21}$$

or

$$= \alpha(H_0(B_0 - K_1) + \sum_{n=1}^{N-1} H_n(P_n - K_{n+1})), \text{ with } K_N = K_4 = 0$$

$$= \sum_{n=1}^{N=4} \alpha H_{n-1}Y_n^* = \sum_{n=1}^{N=4} V_n^*,$$

since

$$P_n = K_n + aSK_n - bK_n^m, \text{ and}$$

$$K_n = B_{n-1} - Y_n^*.$$

At last we know the optimal amount to thin at the beginning of the first period. This puts us in the position of being able to compute the optimal standing crop present at the start of stage two:

$$\begin{aligned}
B_1^* &= B_0 - Y_1^* + \Delta B_1^*, \text{ or} \\
B_1^* &= B_0 - (B_0 - K_1) + \Delta B_1^*, \text{ where} \\
\Delta B_1^* &= aS(B_0 - Y_1^*) - b(B_0 - Y_1^*)^m, \text{ or} \\
\Delta B_1^* &= aS(B_0 - (B_0 - K_1)) - b(B_0 - (B_0 - K_1))^m \\
&= aSK_1 - bK_1^m, \text{ therefore} \\
B_1^* &= K_1 + aSK_1 - bK_1^m.
\end{aligned}$$

The optimal amount to thin at the beginning of stage two is

$$\begin{aligned}
Y_2^* &= B_1^* - K_2, \text{ or} \\
&= K_1 + aSK_1 - bK_1^m - K_2.
\end{aligned}$$

The optimal standing crop at the beginning of stage three is given by

$$\begin{aligned}
B_2^* &= B_1^* - Y_2^* + \Delta B_2^* \\
&= B_1^* - (B_1^* - K_2) + aSK_2 - bK_2^m \\
&= K_2 + aSK_2 - bK_2^m.
\end{aligned}$$

The optimal amount to thin at the beginning of stage three is, then,

$$\begin{aligned}
Y_3^* &= B_2^* - K_3, \\
&= K_2 + aSK_2 - bK_2^m - K_3, \text{ and}
\end{aligned}$$

the optimal standing crop at the end of stage three is

$$\begin{aligned}
B_3^* &= B_2^* - Y_3^* + \Delta B_3^* \\
&= B_2^* - (B_2^* - K_3) + aSK_3 - bK_3^m \\
&= K_3 + aSK_3 - bK_3^m.
\end{aligned}$$

The final harvest cut, then, is

$$\begin{aligned}
Y_4^* &= B_3^*, \\
&= K_3 + aSK_3 - bK_3^m.
\end{aligned}$$

We can check our computations by summing the optimal harvest amounts and comparing this figure with the value of the functional recurrence relation $f_4(B_0)$. We note, however, that the latter is expressed in cords and the former in square feet.

Total basal area cut (thinnings + final harvest cut)

$$\begin{aligned}
&= (B_0 - K_1) \\
&+ K_1 + aSK_1 - bK_1^m - K_2 \\
&+ K_2 + aSK_2 - bK_2^m - K_3 \\
&+ K_3 + aSK_3 - bK_3^m \\
&= (B_0 - K_1) + \sum_{n=1}^{3} (K_n + aSK_n - bK_n^m - K_{n+1})
\end{aligned}$$

with $K_4 = 0$.

Cordwood volume of total cut

$$= \alpha\ H_0(B_0 - K_1) + \sum_{n=1}^{3} \alpha\, H_n(K_n + aSK_n - bK_n^m - K_{n+1})$$

with $K_4 = 0$.

This agrees with the value given by equation (21).

DISCUSSION

Two conditions determine whether DP will yield an optimal policy. One condition, which applies to the total return function, must be met for every kind of multistage DP problem. Called the decomposition condition (see Nemhauser 1966), it was used in the previous two examples. The second condition states that DP is applicable only to multistage decision processes where: (1) the state at the end of stage n depends only on the state at the beginning of stage n, on the decision at the beginning of stage n, and on the stage; (2) the calculus-based search procedure is used. If our concern is with so-called stationary (nontime-dependent) processes, stage does not explicitly enter the determination of state. An example of a system where the state at the end of stage n depends on more than the immediately previous state is as follows

$$\begin{aligned}
B_n &= B_{n-1} - Y_n + \Delta B_n \text{ where} \\
\Delta B_n &= g(B_{n-1}, B_{n-2}, B_{n-3}, Y_n, \text{Site}).
\end{aligned}$$

In this case the stage transformation equation is

$$B_n = T_n[B_{n-1}, B_{n-2}, B_{n-3}, Y_n, \text{Site}], \text{ instead of}$$

$$B_n = T_n[B_{n-1}, Y_n, \text{Site}].$$

Although growth functions may be made more realistic by having historical components, adequate functions have been developed using only the previous state.

The severity of the restriction on stage-dependence is not what it first appears, because it applies only when one uses a calculus-based search procedure. We gave examples of a noncalculus-based search procedure and a calculus-based search. In the first example the stage dependence constraint does not apply, but in the second it does.

Although not addressed directly, the question of determining the optimal rotation age can be examined using the following equation from the previous section:

$$f_N(B_0) = \sum_{n=1}^{N} \alpha H_{n-1} Y_n^*. \qquad (22)$$

By determining the numerical values of this function for several values of N it should be possible to estimate the number of stages that will maximize the optimal yield provided that the growth model possesses a biological limit as N becomes large. In a sense, we do a sensitivity analysis of equation (22) on N.

Conducting a sensitivity analysis on N is particularly easy in this case because of the way the same algebraic form is repeated in the equation for the optimal amount to thin, Y_n. In fact, we can, by simple induction, write the equation for any value of N, as follows:

for n=N, B_N=0, and $f_0(B_N)$=0 (assuming a clearcut at start of stage N)

$$V_N^* = \alpha H_{N-1} B_{N-1}, \text{ and}$$

for n=N−1, N−2,..., 3, 2, 1,

$$Y_n^* = B_{n-1} - \left(\frac{H_n - H_{n-1} + aSH_n}{mbH_n}\right)^{\frac{1}{m-1}}, \text{ or}$$

$$= B_{n-1} - K_n \leqslant B_{n-1}, \text{ with } K_n \geqslant 0.$$

The equation for the cumulative optimal return over N−(n−1) stages is

$$f_{N-(n-1)}(B_{n-1}) = \alpha\,(H_{n-1}(B_{n-1} - K_n) + \sum_{n=n}^{N-1} H_n(P_n - K_{n+1}))\ ,$$

with $K_N = 0$, and
$P_n = K_n + aSK_n - bK_n^{\,m}$ for n=n, n+1, . . . ,N−1.
Recall

$$K_n = \left(\frac{H_n - H_{n-1} + aSH_n}{mbH_n}\right)^{\frac{1}{m-1}}.$$

At the final solution stage, when n=1,

$$f_N(B_0) = \sum_{n=1}^{N} \alpha H_{n-1} Y_n^*.$$

By letting N=4, 5, 6, and 7, for example, and finding $f_4(B_0)$, $f_5(B_0)$, . . . , $f_7(B_0)$, we can plot their respective values over the years corresponding to a rotation of four stages, five stages, etc. The optimal biological rotation age is reached in the forward solution when the average total return is maximized (maximum mean annual increment).

Let us return now to the problem of a negative harvest indicated when

$$Y_n^* = B_{n-1} - K_n < 0 \text{ because } K_n > B_{n-1}.$$

We can interpret K_n as the optimal residual basal area at the beginning of period n right after the thinning. If K_n is greater than the basal area at the end of stage n−1, this would indicate that the stand has not yet reached the optimal basal area for a biologically meaningful thinning. The manager has to wait until the stand has enough basal area to warrant a thinning, i.e., should not cut anything. Only at that time, derivation of an optimal dynamic programming solution becomes meaningful and the condition $K_n > B_{n-1}$ will not occur thereafter as long as net growth is nonnegative. To show this, consider any stage n at which

$$B_{n-1} > K_n$$

i.e., a thinning is biologically reasonable. The optimal residual basal area, K_n, becomes the base for additional basal area growth in the next stage. For the next thinning to be meaningful

$$B_n = K_n + \Delta B_n > K_{n+1} \text{ or } \Delta B_n > K_{n+1} - K_n\ .$$

Our experience with numerous derivation of optimal thinning schedules for varying initial conditions is that this condition holds after $B_{n-1} > K_n$.

The significance of the generality of the calculus approach made possible by using growth equation (16) is increased by the fact that the growth equation form is that of a modified Richards function (Richards 1959) that has been shown applicable to a variety of tree species (Pienaar and Turnbull 1973, Moser and Hall 1969).

The calculus search for an optimal cutting policy is likely to give a more precise estimate of cutting intensities than the network search. The latter finds the optimal combination of those cutting intensities simulated and entered in the network. A calculus-based search finds the exact optimal amount to cut at each cutting cycle, and is not dependent on what was simulated.

Comparing the recurrence equation indexing as used in this paper with what could be called the standard method of indexing (Bellman 1957) shows that the standard indexing is

$$f_n(B_n) = \max_{Y_n} \left\{ R_n + f_{n-1}(B_{n-1}) \right\}, \qquad (23)$$

where n = 1, 2, 3, . . . , N (backward approach), and

$$B_{n-1} = B_n - Y_n + \Delta B_n.$$

The indexing introduced in this paper is

$$f_{N-(n-1)}(B_{n-1}) = \max_{Y_n} \left\{ R_n + f_{N-n}(B_n) \right\}, \qquad (24)$$

where $n = N, N-1, \ldots, 3,2,1$ (backward approach), and

$$B_n = B_{n-1} - Y_n + \Delta B_n.$$

The importance of the new indexing method can be appreciated if we examine typical cases of backward solutions to several problems with different numbers of stages (fig. 8). The traditional approach (fig. 8a) begins by relabeling the periods backwards in time. Thus, the final time period is represented as $n=1$, no matter what the value of N. A continuing source of confusion with the standard indexing can be seen by examining the indexes above, say, growth period two. They are $f_4(B_4)$, $f_3(B_3)$, $f_2(B_2)$, and $f_1(B_1)$ for five-, four-, three-, and two-stage processes, respectively. The standard indexing only tells us how many growth periods from the end we have progressed.

The new indexing approach for the same set of problems is shown in figure 8b. No relabeling of periods is necessary, even though we are doing a backward solution. Let us again examine the indexes above growth period two, i.e., $f_4(B_1)$, $f_3(B_1)$, $f_2(B_1)$, and $f_1(B_1)$.

It can be seen that this method of indexing does the following:

(1) It preserves one's sense that states of the system, B_n, should have indexes that increase from left to right (see fig. 8b).

(2) The index, $n-1$, on, B_{n-1}, the state variable, is the same for the nth growth period no matter how many stages the problem involves.

(3) The index on f tells how many stages we have come from the Nth stage.

(4) The sum of the indexes on f and B gives the value of N.

(5) The index $(n-1)$ on B_{n-1} in equation (24) indicates that the nth growth period starts with the state variable B_{n-1}, and this corresponds to the traditional stand basal area identity:

$$B_n = B_{n-1} - Y_n + \Delta B_n.$$

If, for economic or other reasons the basal area per acre removed must exceed a minimum amount, this new condition can be eliminated from

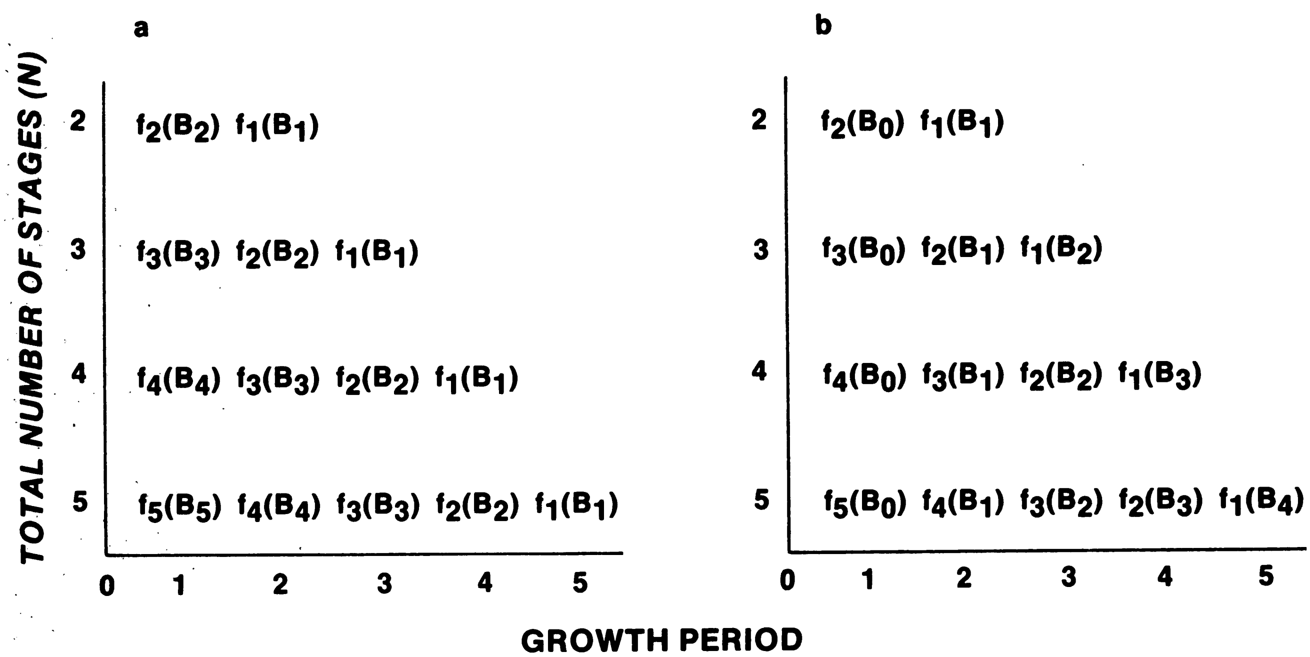


Figure 8. — *Comparative indexes on the functional recurrence equation's lefthand side when using the standard DP approach (a) and the new approach (b) for a backward solution.*

the constraint equation by applying the Lagrangian multiplier method (see Nemhauser 1966).

CONCLUSIONS

Dynamic programming can be used to solve "optimal stand density over time" problems expressed in either of two forms: a network of simulated yields or a mathematical stand growth equation. With the former, one can use a noncalculus-based search procedure; with the latter, one can use a calculus-based search. When the stand growth model has the biologically reasonable form of the one suggested here, it is simple a matter to determine optimal stand density over any number of growth periods. This, in turn, makes it easy to determine the optimal rotation age by doing a sensitivity analysis of the total return on the number of stages in the decision process.

A new method is presented for indexing the functional recurrence equation of DP that makes it much easier to follow the solution procedure stage by stage.

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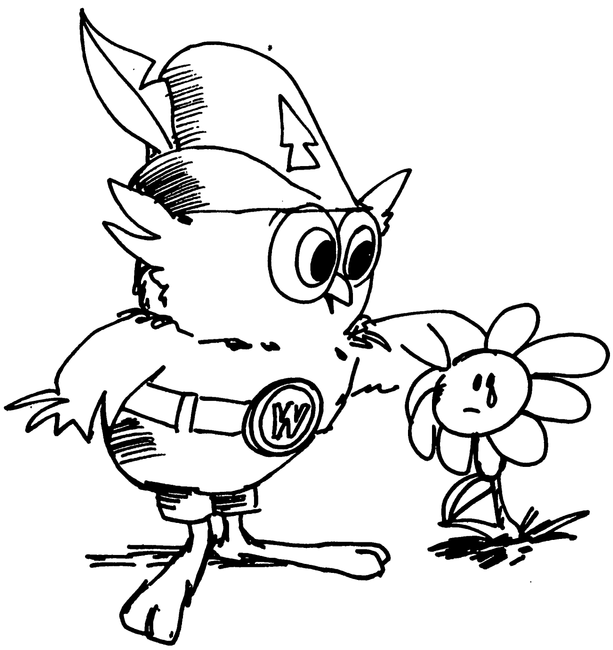

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