Self-referencing Taper Curves for Loblolly Pine

Mike Strub[1], Chris Cieszewski[2], and David Hyink[3]

Abstract.—We compare the traditional fitting of relative diameter over relative height with methods based on self-referencing functions and stochastic parameter estimation using data collected by the Virginia Polytechnic Institute and State University Growth and Yield Cooperative. Two sets of self-referencing equations assume known diameter at 4.5 feet inside (dib) and outside (dob) bark; and one set assumes known only dob at 4.5 feet. A fourth degree polynomial in one minus relative height describes taper. The proposed methods improved the error sum of squares by up to 47 percent over the traditional method.

We demonstrate here the application of self-referencing curves to taper equations. Individual tree taper is commonly modeled by predicting relative diameter (diameter at a reference height divided by diameter at breast height) as a function of relative height (reference height divided by total height). Many of the models proposed in the past have not resulted in estimated diameter outside bark at 4.5 feet equal to diameter at breast height. Valentine and Gregoire (2001) proposed a model that forces the estimated diameter outside bark to be equal to diameter at breast height. Goelz and Burk (1996) and Cieszewski *et al.* (2000) argue that such constraints result in parameter estimates describing biased curve shapes. Our study proposes parameter estimation techniques that avoid such a bias while using equations that estimate diameter outside bark at 4.5 feet equal to diameter at breast height.

Data

Data for this study were collected by the Virginia Polytechnic Institute and State University Growth and Yield Cooperative. Two trees were felled at initial (1980-1982 dormant seasons) and second thinning and in buffers of unthinned control plots (1992-1994 dormant seasons) in a thinning study established at 186 locations across the natural range of loblolly pine. Disks were cut from felled trees beginning at the stump and every 4 feet to approximately a 2-inch top. Diameter inside and outside bark were measured for each disk. Diameter at breast height and total height were also measured on each tree.

Methods

To model tree taper we used a fourth degree polynomial in relative diameter versus relative height having two inflection points, which is a desirable characteristic of taper equations. We further assumed that taper curves are anamorphic in nature.

This results in the following model:

$$\frac{d}{dbh} = a_1(1-h/ht) + a_2(1-h/ht)^2 + a_3(1-h/ht)^3 + a_4(1-h/ht)^4 ,$$

where h = reference height, d = diameter inside or outside bark at the reference height, d.b.h. = diameter at breast height, ht = total height, d/d.b.h. = relative diameter, and h/ht = relative height.

Common practice has been to fit this equation to all the data pairs, or to multiply both sides of the equation by d.b.h. to obtain an equation in diameter inside or outside bark that can be fit to the data:

$$d = dbh \cdot a_1(1-h/ht) + a_2(1-h/ht)^2 + a_3(1-h/ht)^3 + a_4(1-h/ht) \tag{1}$$

[1] Forest Biometrician, Weyerhaeuser Company, P.O. Box 1060, Hot Springs, AR 71902. Phone: 501–624–8504; fax: 501–624–8505; e-mail: mike.strub@weyerhaeuser.com.

[2] Assistant Professor of Fiber Supply Assessment, The University of Georgia, Warnell School of Forest Resources, Athens, GA 30602–2152. Phone: 706–542–8169; fax: 706–542–8356; e-mail: biomat@smokey.forestry.uga.edu.

[3] Senior Scientific Advisor, Weyerhaeuser Company, P.O. Box 9777, Federal Way, WA 98063–9777. Phone: 253–924–6315; fax: 253–924–6736; e-mail: dave.hyink@weyerhaeuser.com.

This is analogous to fitting a guide curve to dominant height and age pairs when developing site index curves. Hereafter we refer to fitting equation (1) to the data as fitting a guide curve. An alternate approach is to obtain a self-referencing curve according to the generalized algebraic difference approach developed by Cieszewski and Bailey (2000). We introduce into the equation an unobservable variable, *X*, that varies from tree to tree.

$$\frac{d}{dbh} = X\left[a_1(1-h/ht) + a_2(1-h/ht)^2 + a_3(1-h/ht)^3 + a_4(1-h/ht)^4\right]$$

Multiplying both sides of the equation by d.b.h. results in:

$$d = dbh \bullet X[a_1(1-h/ht) + a_2(1-h/ht)^2 + a_3(1-h/ht)^3 + a_4(1-h/ht)^4]$$

However, this model is overparameterized. The substitution of Z = a_4 • dbh • x results in:

$$d = Z\left[b_1(1-h/ht) + b_2(1-h/ht)^2 + b_3(1-h/ht)^3 + (1-h/ht)^4\right] \quad (2)$$

where $b_1=a_1/a_4$, $b_2=a_2/a_4$, and $b_3=a_3/a_4$, and Z is estimated for each tree by minimizing the error sum of squares for each tree:

$$SSe_j = \sum_{i=1}^{Nj}\left(d_{ij} - Y_j f_{ij}\right)^2 \quad (3)$$

where:

SSe_j is the error sum of squares for the jth tree

N_j is the number of diameter measurements for the jth tree

d_{ij} is the ith diameter measurement for the jth tree

Z_j is a level parameter for the jth tree

$$f_{ij} = b_1(1-h_{ij}/ht_j) + b_2(1-h_{ij}/ht_j)^2 + b_3(1-h_{ij}/ht_j)^3 + (1-h_{ij}/ht_j)^4$$

h_{ij} is the height of the ith diameter measurement for the jth tree

ht_j is the total height of the jth tree

Taking the first derivative of SSe_j with respect to Z_j, setting it equal to zero and solving for Z_j results in the least squares estimate of Z_j:

$$Z_j = \frac{\sum_{i=1}^{N_j} d_{ij} f_{ij}}{\sum_{i=1}^{N_j} f_{ij}^2} \quad (4)$$

These estimates can be used in the SAS NLIN procedure to obtain least squares estimates of the global parameters b_2, b_3, and b_4. The value of Z_j can be calculated given the global parameters and retained for subsequent data points from the same tree. Details of this method are described in Strub and Cieszewski (2002) for the more general case where Z_j cannot be analytically solved for in a closed form. Note that this method is not valid if a_4 is zero. However, if a_4 is zero, the model would have only one inflection point and would not be appropriate for a taper function. Substituting equation (3) into equation (2) results in minimizing the error sum of squares.

$$SSe_j = \sum_{i=1}^{Nj}\left(d_{ij} - \frac{\sum_{i=1}^{N_j} d_{ij} f_{ij}}{\sum_{i=1}^{N_j} f_{ij}^2} f_{ij}\right)^2 \text{; and } SSe = \sum_{j=1}^{M}\sum_{i=1}^{Nj}\left(d_{ij} - \frac{\sum_{i=1}^{N_j} d_{ij} f_{ij}}{\sum_{i=1}^{N_j} f_{ij}^2} f_{ij}\right)^2 \quad (5)$$

Numerical techniques must be used to estimate the global parameters b_2, b_3, and b_4 by minimizing this error sum of squares. Separate parameters can be obtained for diameter inside and diameter outside bark equations. Application of these equations requires an estimate of both diameter inside and outside bark at some index height, usually 4.5 feet, just as application of site index curves requires an estimate of dominant height at an index age (Cieszewski *et al*. 2000). Given d.b.h., the following equation describes the outside bark taper profile.

$$d = \frac{dbh\left[b_1(1-h/ht) + b_2(1-h/ht)^2 + b_3(1-h/ht)^3 + (1-h/ht)^4\right]}{b_1(1-4.5/ht) + b_2(1-4.5/ht)^2 + b_3(1-4.5/ht)^3 + (1-4.5/ht)^4} \quad (6)$$

If bark thickness is measured at d.b.h., diameter inside bark at 4.5 feet can be substituted for d.b.h. in equation (6) to obtain the inside bark taper profile. Often only diameter outside bark at 4.5 feet (diameter at breast height) is known. A system of equations that predicts both diameter inside and outside bark from only diameter at breast height can be developed by modeling Z in the inside bark equation as a linear function of Z in the outside bark equation (see justification for this assertion in the Results section). Implementing this maneuver results in the following definition of the error sum of squares for the jth tree when both inside and outside bark measurements are given the same weight.

$$SSe_j = \sum_{i=1}^{Nj} \left(dib_{ij} - Z_j f_{ij}\right)^2 + \sum_{i=1}^{Nj} \left(dob_{ij} - (a_0 + a_1 Z_j) g_{ij}\right)^2 \qquad (7)$$

Where:

$f_{ij} = bib_1 \, (1 - h_{ij} / ht_j) + bib_2 \, (1 - h_{ij} / ht_j)^2 + bib_3 \, (1 - h_{ij} / ht_j)^3 + (1 - h_{ij} / ht_j)^4$

$g_{ij} = bob_1 \, (1 - h_{ij} / ht_j) + bob_2 \, (1 - h_{ij} / ht_j)^2 + bob_3 \, (h_{ij} / ht_j)^3 + (1 - h_{ij} / ht_j)^4$ bib_1, bib_2, bib_3, bob_1, bob_2, bob_3, a_0, a_1 are global parameters.

Minimizing the error sum of squares with respect to Z_j

$$Z_j = \frac{\sum_{i=1}^{N_j} dob_{ij} g_{ij} + a_1 \sum_{i=1}^{N_j} dib_{ij} f_{ij} + a_0 a_1 \sum_{i=1}^{N_j} f_{ij}^2}{\sum_{i=1}^{N_j} g_{ij}^2 + a_1^2 \sum_{i=1}^{N_j} f_{ij}^2}$$

The remaining eight global parameters (bib_1, bib_2, bib_3, bob_1, bob_2, bob_3, a_0, a_1) can be estimated with non-linear regression. Once the parameters are estimated, a single Z value can be calculated from d.b.h. by using equation (8):

$$Z = \frac{dbh}{bob_1(1-4.5/ht) + bob_2(1-4.5/ht)^2 + bob_3(1-4.5/ht)^3 + (1-4.5/ht)^4}$$

The outside-bark taper profile can be determined from equation (9) and the inside-bark equation can be estimated from equation (10):

$$dob = Z\left[bob_1(1-h/ht) + bob_2(1-h/ht)^2 + bob_3(1-h/ht)^3 + (1-h/ht)^4\right]$$

$$dib = (a_0 + a_1 Z)\left[bib_1(1-h/ht) + bib_2(1-h/ht)^2 + bib_3(1-h/ht)^3 + (1-h/ht)^4\right]$$

Results

Equation (1) was fit to the data resulting in a guide curve that is similar to the traditional methods of modeling tree taper. The model is shown for both diameter outside bark in black and diameter inside bark in gray in figure 1a. Parameter estimates are given in table 1. Given d.b.h. and total height, diameter outside bark at 4.5 feet or d.b.h. can be estimated from the outside bark guide curve. The difference between the estimated d.b.h. and the input d.b.h. is shown in figure 1b for a range of input d.b.h.'s and total heights. Note the large discrepancy between input d.b.h. and estimated d.b.h. for many reasonable d.b.h. and total height pairs. This difference suggests that methodology for ensuring the consistency of input d.b.h. and estimated d.b.h. is well grounded.

Figure 1c shows relative diameter outside bark versus relative height for each data observation. Most of the data are plotted with gray circles representing each data point. Four selected trees were plotted as black diamonds, crosses, triangles, or squares. Note that these four trees tend to lie at the top, middle, or bottom of the data points. This suggests that a system of curves analogous to site index curves could better approximate tree taper. A single guide curve fit through the middle of the data range will not adequately describe trees with taper profiles like the black diamonds and squares.

Equation (2) was fit to both inside and outside bark diameter. Parameter estimates are given in table 2. The taper profile for selected total heights is shown for diameter outside bark in

Table 1.—*Parameter estimates for the guide curve, equation 1*

Model	Parameter	Estimate
Diameter outside bark	a_1	1.4195
	a_2	2.1537
	a_3	-6.9315
	a_4	4.5998
Diameter inside bark	a_1	1.2879
	a_2	1.9968
	a_3	-6.1580
	a_4	3.9148

Table 2.—*Parameter estimates for the self-referencing curve, equation 2*

Model	Parameter	Estimate
Diameter outside bark	b_1	0.314256
	b_2	0.462600
	b_3	-1.504593
Diameter inside bark	b_1	0.330808
	b_2	0.507229
	b_3	-1.571667

Figure 1.—(a) *Relative diameter inside and outside bark versus relative height for the guide curve, equation (1); (b) error in d.b.h. estimate obtained from the outside bark guide curve; (c) observed relative diameter outside bark versus relative height is shown for the majority of the data in gray circles (four selected trees were plotted as black diamonds, crosses, triangles, or squares); (d) relative diameter outside bark versus relative height for selected total heights and the self-referencing curve, equation (2); (e) relative diameter inside bark versus relative height for selected total heights and the self-referencing curve, equation (2); and (f) Zdob versus Zdib and the linear fit (the Zdob = Zdib line is shown in gray).*

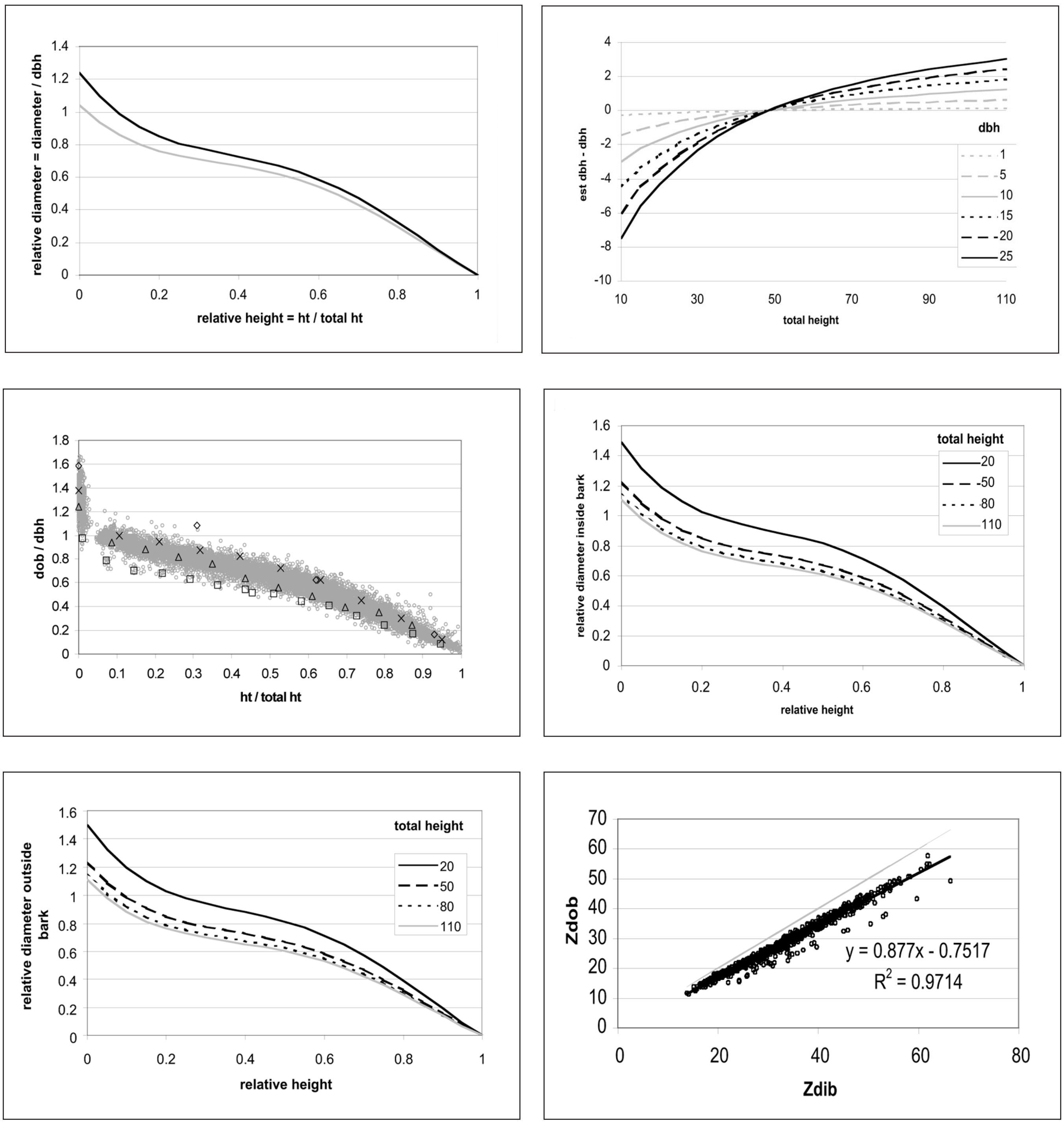

figure 1d and diameter inside bark in figure 1e. Note that the taper profile changes dramatically as total height changes. Fitting the outside bark equation resulted in an estimate of Z_j, Zdob, for each tree, and fitting the inside bark equation resulted in an estimate of Z_j, Zdib, for each tree. Applying these taper equations (eq. (6)) requires estimating both Zdob and Zdib. Obtaining these estimates requires knowledge of both diameter inside and outside bark at some reference point usually breast height or 4.5 feet. An alternate approach was developed that requires only d.b.h. Figure 1f shows the strong linear relationship between Zdob and Zdib. This suggests that error sum of squares described in equation (7) could be minimized to obtain inside and outside bark taper estimates based on equation (9) and inside bark estimates based on equation (10). A common Z is estimated from d.b.h. using equation (8). Table 3 gives parameter estimates for these equations.

Discussion and Conclusion

The base-age-invariant parameter estimation (Bailey and Clutter 1974) used in this research can be successfully fit with sufficient data and fairly straightforward programming using SAS (SAS Institute 1990). The data must consist of repeated measures, which are pooled cross-sectional and longitudinal data. The number of measurements in each series has to be at least two, and the number of series must be greater than the number of global parameters considered in the model. The proposed method does not violate regression principles.

Table 3.—*Parameter estimates for the self-referencing curves, equations 8, 9, and 10*

Model	Parameter	Estimate
Diameter outside bark	bob_1	0.311990
	bob_2	0.464576
	bob_3	-1.505517
Diameter inside bark	a_0	-0.009822
	a_1	0.850341
	bib_1	0.333494
	bib_2	0.505446
	bib_3	-1.571255

Table 4.—*Sum of squared errors for the guide curve approach and self-referencing equations*

Model	Guide curve	Self-referencing equations
Diameter inside bark	2372.8	1098.0
Diameter outside bark	2436.7	1456.8
Total inside bark and outside bark	4809.5	2554.8
Zib=ao+a1*Zob		2912.3

Error sum of squares are listed in table 4 for the guide curve approach, the self-referencing curves when both diameter inside and outside bark are known at breast height, and the self referencing curves when only d.b.h. is known. The total error sum of squares for both inside and outside bark guide curves was reduced by 47 percent when self-referencing curves based on separate Zdob and Zdib were used to describe tree taper. The total error sum of squares for both inside and outside bark guide curves was reduced by 39 percent when self-referencing curves based on a common Z were used to describe tree taper. These curves have the additional advantage of returning estimated diameter outside bark at 4.5 feet equal to d.b.h. The senior author has applied the same techniques to the more complex equations of Max and Burkhart (1976) with equal success, which demonstrates the broad utility of self-referencing curves.

The proposed methods are suitable for fitting applications with other dependent variables such as per acre basal area, survival, and yield.

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Literature Cited

Bailey, R.L.; Clutter, J.L. 1974. Base-age invariant polymorphic site curves. Forest Science. 20(2): 155–159.

Cieszewski, C.J.; Bailey, R.L. 2000. Generalized algebraic difference approach: theory based derivation of dynamic site equations with polymorphism and variable asymptotes. Forest Science. 46(1): 116–126.

Cieszewski, C.J.; Harrison, M.W.; Martin, S.W. 2000. Examples of practical methods for unbiased parameter estimation in self-referencing functions. In: Cieszewski, C.J., ed. Proceedings, 1st International conference on measurements and quantitative methods and management; 1999 November 17–18; Jekyll Island, GA.

Goelz, J.C.G.; Burk, T.E. 1996. Measurement error causes bias in site index equations. Canadian Journal of Forest Research. 26: 1586–1593.

Max, T.A.; Burkhart, H.E. 1976. Segmented polynomial regression applied to taper equations. Forest Science. 22(3): 283–289.

SAS Institute. 1990. SAS guide to macro processing, version 6, 2d ed. Cary, NC: SAS Institute, Inc.

Strub, M.R.; Cieszewski, C.J. 2002. Fitting global site index parameters when plot or tree site index is treated as a local nuisance parameter. In: Proceedings, symposium on statistics and information technology in forestry; 2002 September 8–12; Blacksburg, VA.

Valentine, H.T.; Gregoire, T.G. 2001. A switching model of bole taper. Canadian Journal of Forest Research. 31: 1400–1409.