Monte Carlo approaches to sampling forested tracts with lines or points

Harry T. Valentine, Jeffrey H. Gove, and Timothy G. Gregoire

Abstract: Several line- and point-based sampling methods can be employed to estimate the aggregate dimensions of trees standing on a forested tract or pieces of coarse woody debris lying on the forest floor. Line methods include line intersect sampling, horizontal line sampling, and transect relascope sampling; point methods include variable- and fixed-radius plot sampling, and point relascope sampling. We demonstrate that the line methods can be interpreted as applications of importance sampling and that point methods can be interpreted as two-stage applications of importance sampling and crude Monte Carlo. Interestingly, each of the line methods also can be implemented as a point method. Operationally, the two stages of a point method effectively reduce to a single stage. Estimators of target parameters are derived from the Monte Carlo approach for all six methods. Two new methods of estimating cross-sectional areas of slanted or tilted log-shaped objects are suggested for use in line intersect sampling. Boundary problems also are discussed.

Résumé : Plusieurs méthodes d'échantillonnage par ligne et par point peuvent être utilisées pour estimer les dimensions agrégées des arbres debout sur une surface forestière ou des débris ligneux grossiers étendus sur le sol forestier. Les méthodes linéaires comprennent l'échantillonnage par intersection linéaire, l'échantillonnage par ligne horizontale et l'échantillonnage par virée au relascope ; les méthodes par point incluent l'échantillonnage par placette à rayon variable ou fixe et l'échantillonnage par point au relascope. Nous démontrons que les méthodes par ligne peuvent être interprétées comme des applications de l'échantillonnage par importance et que les méthodes par point peuvent être interprétées comme les applications en deux phases de l'échantillonnage par importance et par Monte Carlo brut. Il est intéressant de constater que chacune des méthodes par ligne peut aussi être implantée comme une méthode par point. En pratique, les deux phases d'une méthode par point sont effectivement réduites à une seule phase. Les estimateurs des paramètres cibles sont dérivés de la méthode Monte Carlo pour les six méthodes. Deux nouvelles méthodes pour estimer la surface radiale des billes inclinées ou renversées sont suggérées en lien avec l'échantillonnage par intersection linéaire. Les problèmes de démarcation sont également discutés.

[Traduit par la Rédaction]

Introduction

Estimates of the masses and sizes of trees standing in a forest or pieces of coarse woody debris lying on the forest floor often are requisite to ecological analyses. Several specialized sampling methods employing randomly located points or lines have been developed to estimate these quantities. These methods include line intercept or intersect sampling, horizontal line sampling, transect relascope sampling, point relascope sampling, and fixed- and variable-radius plot sampling.

Variable- and fixed-radius plot sampling methods can be developed from at least four theories (Eriksson 1995). What may not be generally recognized is that both point and line methods can be interpreted as rather straightforward applications of Monte Carlo integration, namely crude Monte Carlo and importance sampling. Crude Monte Carlo is a continuous analog of simple random sampling with replacement. Importance sampling is a continuous analog of sampling with probability proportional to size. Both methods have been widely used in the field of stochastic simulation (e.g., Rubinstein 1981, Chap. 4). Monte Carlo methods also have been used to estimate attributes of physical objects, namely the dimensions of trees and logs (e.g., Valentine et al. 1984; Furnival et al. 1986; Gregoire et al. 1986, 1995; Van Deusen and Baldwin 1993). In this paper we present Monte Carlo approaches to the estimation of attributes of coarse woody debris and standing trees per unit land area of forested tract. We suspect that the Monte Carlo approach to the point and line methods may be easier to understand than many of the existing descriptions and, therefore, may facilitate more wide-spread application of these useful methods in ecological research. In any event, Monte Carlo provides a simple unifying theory for line and point methods.

We begin by showing that crude Monte Carlo is indeed a continuous analog of simple random sampling with replacement. We then move to Monte Carlo approaches to line and point methods for the estimation of attributes of coarse woody debris and standing trees per unit land area. A symbol table is provided in Appendix A.

Continuous analog of random sampling

Coarse woody debris on the forest floor may comprise fallen branches, fallen trees, and logging slash. To introduce the method of crude Monte Carlo, we shall consider the problem of estimating the volume of a large log-shaped piece of coarse woody debris. First we consider the estimation of volume by simple

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H.T. Valentine[1] **and J.H. Gove.** USDA Forest Service, Northeastern Research Station, P.O. Box 640, Durham, NH 03824-0640, U.S.A.
T.G. Gregoire. Yale University, School of Forestry and Environmental Studies, 360 Prospect Street, New Haven, CT 06511-2189, U.S.A.

[1]Corresponding author (e-mail: hvalentine@fs.fed.us).

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random sampling, then we move to the continuous analog.

The ends of the log of interest need not be square and its shape need not be straight; measurement of the length of the log (L) with a taut tape establishes a straight reference axis. Let us imagine that the log is divided along the length axis into N segments, each segment of length $\Delta l = L/N$. Let $l_i = i \times \Delta l$ denote the length from the butt (or most basal point) of the log to the distal end of the ith segment ($i = 1, 2, \ldots, N$). Therefore, the ith segment occurs between lengths l_{i-1} and l_i. The volume of the log from the butt to the distal end of the ith segment is denoted $V(l_i)$. Thus, the volume of the log is $V(l_N) = V(L)$. The volume of the ith segment (ΔV_i) is

$$\Delta V_i = V(l_i) - V(l_{i-1})$$

The N segments make up a discrete population from which we can select a simple random sample of n segments for the estimation of the total volume of the log, $V(L)$.

We shall select a segment from the log in a somewhat unorthodox way by drawing a length at random. The segment that occurs at a selected length from the butt enters the sample. The random length is uL, where u is a random number between 0 and 1 (from a uniform distribution). The kth segment in the log is selected as a "sample segment" if

$$l_{k-1} < uL \leq l_k \tag{1}$$

Because the method of selection is simple random sampling with replacement, the selection probability of the sample segment is $\Delta l/L = 1/N$. A design-unbiased estimator ($\widetilde{V}(L)$) of the volume of the log is

$$\begin{aligned} \widetilde{V}(L) &= L \times \Delta V_k \\ &= L \times \frac{\Delta V_k}{\Delta l} \\ &= L \times \left[\frac{V(l_k) - V(l_{k-1})}{l_k - l_{k-1}}\right] \end{aligned} \tag{2}$$

In other words, the estimate of log volume is provided by the volume per unit length of the sample segment × log length. Obtaining the requisite measurement of volume could be difficult because the sample segment is embedded in the log, so let us move to the less demanding, continuous analog of simple random sampling, i.e., crude Monte Carlo.

By definition, the volume of the log is the integral of cross-sectional area with respect to length. Let $A(l)$ denote the cross-sectional area of the log at point l along the axis of length ($0 \leq l \leq L$). Thus, the volume of the sample segment is

$$V(l_k) - V(l_{k-1}) = \int_{l_{k-1}}^{l_k} A(l)\,\mathrm{d}l$$

The mean value theorem for integrals indicates that a point, θ, exists along the axis of length between l_{k-1} and l_k such that

$$V(l_k) - V(l_{k-1}) = A(\theta) \times (l_k - l_{k-1}) \qquad l_{k-1} \leq \theta \leq l_k$$

It is evident that the mean cross-sectional area, $A(\theta)$, is equivalent to the volume per unit length of the sample segment, i.e.,

$$A(\theta) = \frac{V(l_k) - V(l_{k-1})}{l_k - l_{k-1}}$$

Substitution into [2] expresses the estimator of log volume in terms of the mean cross-sectional area of the sample segment, i.e.,

$$\widetilde{V}(L) = LA(\theta) \tag{3}$$

Although this estimator requires the measurement of a cross-sectional area instead of a volume, it appears to be useless because the measurement point, θ, is unknown. However, because θ and the random point, uL, fall between the points that define the sample segment, the distance between θ and uL can not exceed the length of the segment, i.e.,

$$|\theta - uL| \leq \Delta l$$

The length of any segment, Δl, obviously depends on the choice of N. If N approaches ∞, then Δl approaches 0 and the resultant population is continuous because the segments are infinitesimally short. Since Δl approaches 0, $|\theta - uL|$ also approaches 0, therefore,

$$\widetilde{V}(L) = L \times A(uL)$$

which provides the desired estimate of volume. If cross-sectional area is measured at several points $\theta_j = u_j L$ ($j = 1, 2, \ldots, n$), then

$$\widetilde{V}(L) = L \times \frac{\sum A(\theta_j)}{n} \tag{4}$$

Most samplers would choose to measure log diameters and convert them to cross-sectional areas for insertion into [4].

Other similar uses of crude Monte Carlo include the estimation of coverage areas and dry masses of logs. For example, to estimate coverage area, i.e., the amount of land area overlain by the horizontal projection of a log, measure log diameter at random points, calculate the average, and multiply by log length. The estimation of log mass requires a measurement (or estimate) of mass per unit length at θ_j. The mass per unit length can be estimated as the product of bulk density (dry mass per unit wet volume) × cross-sectional area. Sampling often can be made more efficient by generating measurement points as "antithetic pairs", i.e., uL and $(1 - u)L$ (e.g., Rubinstein and Samorodnitsky 1985).

Sampling a forested tract

In sampling a forested tract, our target parameter ordinarily would be the aggregate length, coverage area, volume, or dry mass of coarse woody debris per unit land area. Or, we may have an interest in the aggregate basal area or volume of the standing trees per unit land area. For the moment, however, let us assume that the target parameter is the volume of coarse woody debris per unit land area.

We consider a tract of land with area **A**, which can be tessellated into rectangular plots, not necessarily all the same size.

Let us assume that selection with probability proportional to size has yielded a plot of width X and length Y, so that opposite corners can be referenced with Cartesian coordinates (0, 0) and (X, Y). Let us further assume that all lengths, areas, and volumes have units m, m^2, and m^3, respectively.

Measurement of all the coarse woody debris on the plot is, of course, an option (after it is sawn loose from debris that extends off the plot). If the number of pieces of debris per unit land area is of interest, then some rule is needed to determine whether a piece that extends off the plot should be counted. For example, a piece may be counted if the basal (or big) end is on the plot. Subsampling, perhaps with the methods just described, is an alternative to the measurement of dimensions of the individual pieces.

Another alternative, and the one on which we shall concentrate, is to reduce the workload by subsampling the plot. For example, the plot can be divided along the x axis into N subplots, each with width $\Delta x = X/N$. The ith subplot falls between x_{i-1} and x_i $(i = 1, 2, \ldots, N)$. The kth subplot is selected by simple random sampling, with probability $1/N = \Delta x/X$, if

$$x_{k-1} < uX \leq x_k$$

Let $\mathcal{V}(x)$ denote the aggregate volume of coarse woody debris on a subplot, whose opposite corners have coordinates (0, 0) and (x, Y). Thus, $\mathcal{V}(x)$ can be considered as a continuous function over the domain $0 \leq x \leq X$. The volume of debris on the subplot defined by x_{k-1} and x_k is denoted by $\Delta\mathcal{V}_k = \mathcal{V}(x_k) - \mathcal{V}(x_{k-1})$. Thus, the volume of debris on the original "whole plot" $(\mathcal{V}(X))$ is unbiasedly estimated by

$$[5] \quad \begin{aligned} \widetilde{\mathcal{V}}(X) &= N \times \Delta\mathcal{V}_k \\ &= X \times \frac{\Delta\mathcal{V}_k}{\Delta x} \\ &= X \times \left[\frac{\mathcal{V}(x_k) - \mathcal{V}(x_{k-1})}{x_k - x_{k-1}}\right] \end{aligned}$$

Let $\mathcal{A}(x)$ be the aggregate cross-sectional area of coarse woody debris intersected by a line from $(x, 0)$ to (x, Y). In other words, $\mathcal{A}(x)$ would be the cross-sectional area of wood exposed if we were to cut along the line with a saw. By definition,

$$\mathcal{V}(x_k) - \mathcal{V}(x_{k-1}) = \int_{x_{k-1}}^{x_k} \mathcal{A}(x)\,\mathrm{d}\,x$$

The mean value theorem for integrals indicates the existence of a line from $(\phi, 0)$ to (ϕ, Y) where

$$[6] \quad \mathcal{V}(x_k) - \mathcal{V}(x_{k-1}) = \mathcal{A}(\phi) \times (x_k - x_{k-1}) \qquad x_{k-1} \leq \phi \leq x_k$$

Substituting [6] into [5], the estimator of the volume of debris on the whole plot becomes

$$[7] \quad \widetilde{\mathcal{V}}(X) = X\mathcal{A}(\phi)$$

The location of ϕ is, of course, unknown, though we do know that ϕ is between x_{k-1} and x_k and, therefore, $|\,\phi - uX| \leq \Delta x$.

If we abandon simple random sampling for crude Monte Carlo, the number of subplots, N, approaches ∞, Δx approaches 0, and, therefore, $\phi = uX$. Since Δx approaches 0, the subplot assumes the shape of a line. Let us refer to this subplot with vanishing width as a "line plot".

Whereas simple random sampling selects a subplot from the whole plot with probability $\Delta x/X = (1/X)\Delta x$, the continuous analog of simple random sampling selects the line plot from the whole plot with probability density $1/X$. Assuming the whole plot is selected from the forested tract with probability $XY/\mathbf{A}$, the line plot is selected from the tract in two stages with probability density $(XY/\mathbf{A})(1/X) = Y/\mathbf{A}$.

Equation [7] provides an estimate, $\widetilde{\mathcal{V}}(X)$, of the volume of coarse woody debris on the whole plot from a measurement of aggregate cross-sectional area along the line plot, i.e., $\mathcal{A}(\phi)$. The aggregate volume of coarse woody debris on the entire tract ($\mathbf{V}$) is estimated by

$$[8] \quad \begin{aligned} \widetilde{\mathbf{V}} &= \left(\frac{\mathbf{A}}{XY}\right)\widetilde{\mathcal{V}}(X) \\ &= \left(\frac{\mathbf{A}}{XY}\right)X\mathcal{A}(\phi) \\ &= \left(\frac{\mathbf{A}}{Y}\right)\mathcal{A}(\phi) \end{aligned}$$

The volume of coarse woody debris per unit land area, ρ_{V}, is estimated by

$$[9] \quad \widetilde{\rho}_{\mathrm{V}} = \frac{\mathcal{A}(\phi)}{Y}$$

$\mathbf{A}/XY$, in [8], is the reciprocal of the probability of the first-stage selection of the whole plot from the tract and $\mathbf{A}/Y$ is the reciprocal of the probability density associated with the two-stage selection of the line plot from the tract. This reciprocal density has a dimension of length. $\mathcal{A}(\phi)$, which is measured along the line plot, can be interpreted as a density with dimensions volume per unit length since $\mathcal{A}(\phi) = \mathrm{d}\,\mathcal{V}(\phi)/\mathrm{d}\,x$. Or, to put it more heuristically, the volume of coarse woody debris overlying the line plot is $\mathcal{A}(\phi)\Delta x$, and the volume per unit length over the width of the line plot is $(\mathcal{A}(\phi)\Delta x)/\Delta x$. Hence, in [9], $\mathcal{A}(\phi)/Y = (\mathcal{A}(\phi)\Delta x)/(Y\Delta x)$ has the requisite units of volume per unit area.

Importance sampling

The two-stage procedure, just described, afforded an intuitive derivation of the estimators [8] and [9]. However, as a selection mechanism for a line plot, this two-stage procedure is cumbrous. In general, the location of a line plot can be selected quite simply in one stage by importance sampling.

Let us define the contiguous forested tract in terms of an x axis that runs west to east and a y axis that runs south to north. Let the most western point of the tract define $x = 0$ and the most eastern point define $x = \mathbf{X}$, and let the most southern point define $y = 0$ and the most northern point define $y = \mathbf{Y}$. Thus, we assume that the tract of interest fits within a rectangle with dimensions $\mathbf{X} \times \mathbf{Y}$. Let $\mathcal{Y}(\phi)$ denote the length of a line from the southern boundary to the northern boundary at $x = \phi$ where

$0 \le \phi \le \mathbf{X}$. If the tract contains coves or concavities, $\mathcal{Y}(\phi)$ is the sum of lengths of the line segments contained within the tract. In a sense, we assume that the tract is divided into line plots, which run south to north. The dimensions of the line plot at $x = \phi$ are $\mathcal{Y}(\phi) \times \Delta x$, where $\Delta x \to 0$. By definition, the area of the tract, $\mathbf{A}$, is

$$\mathbf{A} = \int_0^{\mathbf{X}} \mathcal{Y}(x)\,\mathrm{d}\,x$$

The location of a line plot, ϕ, is selected with probability density proportional to line length, i.e., the distance between the southern and northern borders at ϕ. The requisite density function is

$$f(x) = \begin{cases} 0, & x < 0 \\ \mathcal{Y}(x)/\mathbf{A}, & 0 \le x \le \mathbf{X} \\ 0, & x > \mathbf{X} \end{cases}$$

The selection of ϕ is easily accomplished with von Neumann's acceptance–rejection method (see, e.g., Rubinstein 1981, section 3.4). Generate coordinates (ϕ, θ), where $\phi = u_1\mathbf{X}$ and $\theta = u_2\mathbf{Y}$. If (ϕ, θ) fall within the tract, then ϕ is selected with probability density $f(\phi) = \mathcal{Y}(\phi)/\mathbf{A}$. Otherwise, a new set of coordinates is generated with new random numbers. Alternatively, generate $\phi = u_1\mathbf{X}$ and u_2. If $u_2 \le \mathcal{Y}(\phi)/\mathbf{Y}$, then ϕ is selected. If $\mathcal{Y}(\phi)$ is a longer line plot than desired, it can be divided into segments $Y_1(\phi), Y_2(\phi), \ldots,$ one of which is selected with probability $Y_k(\phi)/\mathcal{Y}(\phi)$. The result is a line plot of length $Y_k(\phi)$ that is selected with probability density $[\,Y_k(\phi)/\mathcal{Y}(\phi)\,] \times [\,\mathcal{Y}(\phi)/\mathcal{A}\,] = Y_k(\phi)/\mathbf{A}$.

The estimators, [8] and [9], are unbiased for importance sampling (see Appendix C for proofs and variance formulas). Of course, $\mathbf{A}/\mathcal{Y}(\phi)$ or $\mathbf{A}/Y_k(\phi)$ substitutes for $\mathbf{A}/Y$, as appropriate.

Line methods

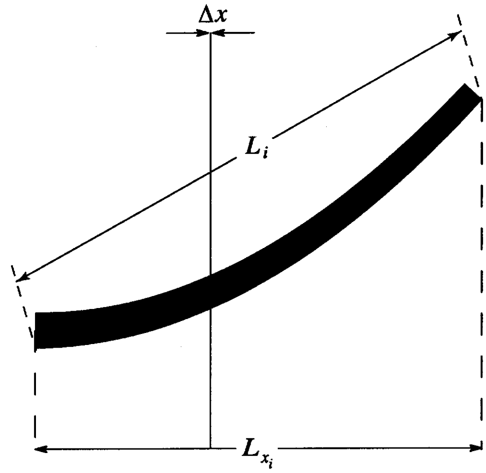
It is remarkable that [9] is identical to an estimator of aggregate volume per unit land area that is used in connection with line intersect (or line intercept) sampling (Kaiser 1983). As we shall see, this basic estimator, with qualification, also applies to other line methods, namely horizontal line sampling and transect relascope sampling. We now derive estimators for each of these methods.

Line intersect sampling

Line intersect sampling, which was developed to estimate logging slash, stems primarily from Warren and Olsen (1964). It seems to differ from line intercept sampling (Canfield 1941) primarily in terms of semantics: the latter is said to be a method of estimating coverage area (Warren 1990). Taking a Monte Carlo approach to line intersect sampling of coarse woody debris, we can estimate, in addition to aggregate volume, the aggregate projected length, true length, coverage area, and mass on a tract and per unit area. We also can estimate the number of pieces of coarse woody debris.

We assume that a line plot (or intersect line) is selected by importance sampling with probability density $Y/\mathbf{A}$. Each piece

Fig. 1. Projected length of a piece of debris (L_{x_i}) is measured perpendicularly to the line plot. True length (L_i) is measured end-to-end with a taut tape.



of debris that crosses the line plot has a true length and a projected length. Recall that the true length of a piece of debris is measured end-to-end with a taut tape. The "projected length" of a piece of coarse woody debris is measured perpendicular to the line plot (Fig. 1), i.e., the projected length is the true length projected onto the x axis. We already have established that $\mathbf{V}$ and ρ_{V} are estimated with measurements of cross-sectional area, which we interpret as volume per unit projected length, i.e., $(\mathcal{A}\Delta x)/\Delta x$. Other attributes are estimated analogously. For example, the aggregate coverage area of pieces of debris on the tract can be estimated with a measurement of coverage area per unit projected length on a line plot; aggregate true length is estimated by measurements of true length per unit projected length; and so on. Perhaps somewhat less intuitively, the number of pieces is estimated with reciprocals of projected lengths.

Let $\mathbf{L}_x$ and $\rho_{\mathbf{L}_x}$, respectively, denote the aggregate projected length of coarse woody debris on the tract and per unit area. If m pieces of coarse woody debris cross the line plot, the aggregate projected length of those portions of the pieces of the debris which cover the line plot is $m\Delta x$ and the aggregate projected length per unit projected length is $m\Delta x/\Delta x$. Since the Δx's cancel, the estimator of $\mathbf{L}_x$ is

$$\widetilde{\mathbf{L}}_x = \frac{\mathbf{A}}{Y} \times m$$

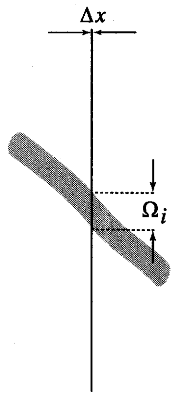
and the estimator of $\rho_{\mathbf{L}_x}$ is

$$\widetilde{\rho}_{\mathbf{L}_x} = \frac{m}{Y}$$

Thus, m, the count of the pieces of debris that cross the line plot, suffices to estimate $\mathbf{L}_x$ and $\rho_{\mathbf{L}_x}$.

Now let L_i and L_{x_i}, respectively, denote the true length and the projected length of the ith piece of coarse woody debris that crosses the line plot (Fig. 1). The true length per unit projected length of the ith piece of debris is L_i/L_{x_i}. The true length of that portion of the piece of debris which crosses the line plot is $(L_i/L_{x_i})\Delta x$ and the true length per unit projected length is $[(L_i/L_{x_i})\Delta x]/\Delta x$ or (L_i/L_{x_i}). Thus, the aggregate true length

Fig. 2. Measurement of the width of a piece of debris (Ω_i) where it crosses the line plot.



of coarse woody debris on the tract (**L**) is estimated by

$$\widetilde{\mathbf{L}} = \frac{\mathbf{A}}{Y}\sum_i^m \frac{L_i}{L_{x_i}}$$

and the aggregate true length per unit land area (ρ_L) by

$$\widetilde{\rho}_\mathrm{L} = \frac{1}{Y}\sum_i^m \frac{L_i}{L_{x_i}}$$

Let C_i denote the coverage area of the ith piece of debris that crosses the line plot. Let $\omega_i(x)$ denote the calipered width of the ith piece of debris at x. Hence,

$$C_i = \int_0^{L_{x_i}} \omega_i(x)\,\mathrm{d}\,x$$

Let Ω_i denote the calipered width (coverage area per unit projected length) of the ith piece of coarse woody debris where it crosses the line plot (Fig. 2). The area of the line plot covered by the ith piece of debris is $\Omega_i\,\Delta x$ and the coverage area per unit projected length is $(\Omega_i\,\Delta x)/\Delta x$. Thus, the coverage area of coarse woody debris on the tract (**C**) is estimated by

$$\widetilde{\mathbf{C}} = \frac{\mathbf{A}}{Y}\sum_i^m \Omega_i$$

and the aggregate coverage area per unit land area (ρ_C) by

$$\widetilde{\rho}_\mathrm{C} = \frac{1}{Y}\sum_i^m \Omega_i$$

Before we consider the estimation of mass, let us revisit the problem of estimating **V** and ρ_V, the aggregate volume of coarse woody debris on the tract and per unit land area, respectively. The aggregate cross-sectional area, $\mathcal{A}$, of [8] and [9] is, of course, the sum of the cross-sectional areas of the individual pieces of coarse woody debris over the line plot, i.e., $\mathcal{A} = \sum A_i$. The axis of true length of each piece of debris may be slanted with respect to the line plot and tilted with respect to (the horizontal projection of) the ground. Thus, the measurement of cross-sectional area can be problematic, at least without use of a saw. However, A_i can be estimated by at least five different methods (Appendix B).

To estimate the total dry mass on the tract or per unit area, substitute dry mass per unit projected length for cross-sectional area in the estimators. As was noted, dry mass per unit length can be calculated as cross-sectional area × bulk density. Unfortunately, bulk density varies within and among species so the conversion of cross-sectional area to dry mass per unit projected length may need to be undertaken piece by piece. Dry mass per unit length can be measured directly on a disk, provided length (i.e., the thickness of the disk) is measured before drying.

Finally, the number of pieces of coarse woody debris on the tract, **P**, is estimated by

$$\widetilde{\mathbf{P}} = \frac{\mathbf{A}}{Y}\sum_i^m \frac{1}{L_{x_i}}$$

and the pieces per unit land area, ρ_p, is estimated by

$$\widetilde{\rho}_P = \frac{1}{Y}\sum_i^m \frac{1}{L_{x_i}}$$

Horizontal line sampling

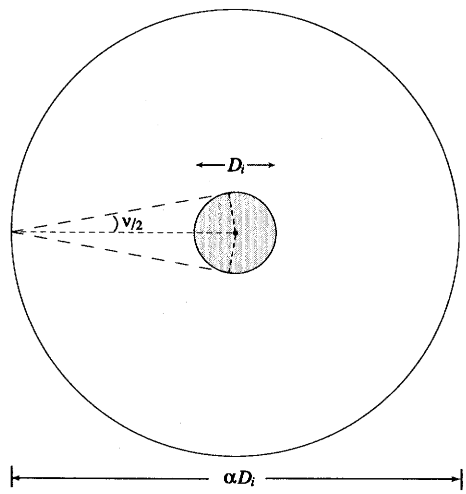
Forest mensurationists have devised clever devices to effectively stretch the coverage areas of small objects. For example, a proportional tree circle can be viewed as an attribute of a tree and, as such, the area of the tree circle is the effective coverage area of the tree. If a tree has diameter D at breast height (1.37 m), then the tree circle has diameter αD, where α has the same value for all trees (Fig. 3). An angle gauge, the angle of the gauge (ν) corresponding to the desired value of α, i.e., $\nu = 2\arcsin(1/\alpha)$ (e.g., Grosenbaugh and Stover 1957), ordinarily suffices to determine whether a sampler is within the perimeter of a tree circle, or whether a tree circle overlaps a line plot. In horizontal line sampling, a tree is measured or counted if its tree circle overlaps the transect line. If a tree circle overlaps a line plot, then so does a tree-circle diameter. We shall restrict our interest to the diameter that crosses the line plot perpendicularly. If we assume that the attributes of a tree are distributed uniformly along this diameter, then horizontal line sampling reduces to line intersect sampling of tree-circle diameters.

The projected length of a tree circle is equivalent to its diameter. Thus, using the estimators for projected length from line intersect sampling, the sum of tree-circle diameters on the tract and per unit area are estimated by $(\mathbf{A}/Y)m$ and m/Y, respectively, where m is the count of tree-circle diameters that cross the line plot perpendicularly. Division by α yields estimates of the sum of tree diameters on the tract (**D**) and per unit land area (ρ_D), i.e.,

$$\widetilde{\mathbf{D}} = \frac{\mathbf{A}}{Y}\times\frac{m}{\alpha}; \qquad \widetilde{\rho}_\mathrm{D} = \frac{1}{Y}\times\frac{m}{\alpha}$$

Aggregate tree volume per unit length, i.e., the volume of each tree divided by the diameter of its tree circle, yields estimates

Fig. 3. A tree with diameter D_i has a tree circle of diameter αD_i, where α is determined by the angle (ν) of an angle gauge, i.e., $\alpha = \csc(\nu/2)$.



of the aggregate tree volume on the tract (**T**) and per unit area (ρ_T), i.e.,

$$\tilde{\mathbf{T}} = \frac{\mathbf{A}}{Y}\sum_i^m \frac{T_i}{\alpha D_i}; \qquad \tilde{\rho}_T = \frac{1}{Y}\sum_i^m \frac{T_i}{\alpha D_i}$$

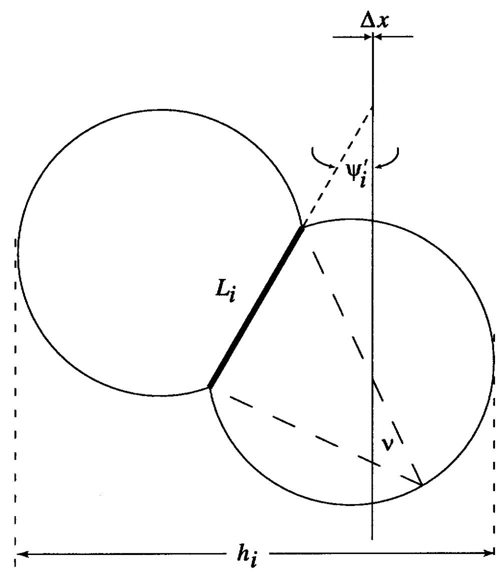
where D_i and T_i, respectively, are the measured diameter and volume of the ith tree whose tree-circle diameter crosses the line plot perpendicularly. An estimate of total basal area or basal area per unit land area obtains by substituting tree basal area for tree volume in the appropriate estimator. Substitution of 1 for T_i yields estimates of the total count of trees and tree density. This particular application of Monte Carlo is identical in operation and result to Grosenbaugh's (1958) description of horizontal line sampling, which stems from Strand (1957).

Transect relascope sampling

Transect relascope sampling (Ståhl 1998) is similar to horizontal line sampling, but the objects of interest are pieces of coarse woody debris lying on the forest floor, rather than standing trees. Figure 4 depicts the shape of the perimeter within which a line, defined by the two endpoints of a piece of coarse woody debris, fills the field of view of an angle gauge. The area defined by this perimeter assumes the shape of two identical intersecting circles with the line as the common chord (Ståhl 1998; also see Grosenbaugh 1958). For lack of a better term, we shall refer to the shape as a "dual circle" The area of the dual circle is proportional to the length-square of the common chord (Gove et al. 1999).

In the Monte Carlo approach to transect relascope sampling, we would ordinarily measure a piece of coarse woody debris if some portion of its dual circle overlapped some portion of the line plot. However, boundary issues arise, so we must establish some protocols. We require at least half of the length of a piece of coarse woody debris to fall within a region bounded by perpendicular lines at either end of the line plot (Fig. 5). A portion of the dual circle may overlap a portion of the line plot

Fig. 4. Dual circle of a piece of debris, the size of which is defined by the true length, L_i, of the piece of debris and the angle, ν, of the angle gauge. The axis of true length of the piece of debris forms and acute angle, ψ_i', with the line plot. The projected width of the dual circle, h_i, is calculated from ν and measurements of L_i and ψ_i' (see text).



even though more than half of the length of the piece of debris is out of bounds (Fig. 5). In this case, we should ignore the piece of debris. On the other hand, more than half of the length of a piece of debris may fall within bounds, though no portion of the dual circle overlaps the line plot; however, if some portion of the dual circle would overlap an extended or elongated line plot (Fig. 5), then we should accept the piece of debris. Other protocols could be defined, but these particular protocols accord with those which Ståhl (1998) established for transect relascope sampling.

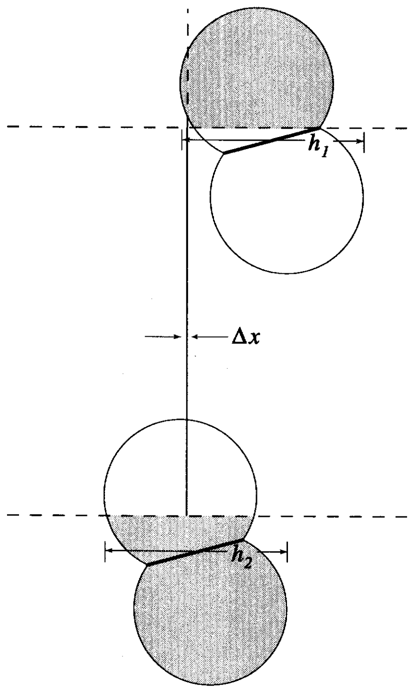
In the Monte Carlo approach to horizontal line sampling, we assumed that the attributes of a tree were distributed uniformly along the diameter of the tree circle that was perpendicular to the line plot. For the present problem, we assume that the attributes of the ith piece of coarse woody debris are distributed uniformly along a line whose length equals the projected width (h_i) of the dual circle (see Fig. 5). We assume that this "projection line" bisects (or is coincident with) the axis of true length of the piece of debris. Under this assumption and the caveats of the previous paragraph, transect relascope sampling reduces to line intersect sampling of the projection lines of dual circles.

Ståhl (1998) derived a formula for the length of the projection line, i.e.,

$$h_i = L_i\left(\frac{1}{\sin\nu} + \cot\nu\,\cos\psi_i'\right)$$

where L_i is the true length of the piece of debris, ν is the angle of the angle gauge (degrees), and ψ_i' is the acute angle formed by the central axis of the piece of debris and the line plot (Fig. 4). The angle ψ_i' can be calculated from the azimuth of the axis of true length of the piece of debris and the azimuth of the line plot.

Fig. 5. The projection line of the piece of debris of length h_1 crosses the line plot, though the line plot must be extended to intersect the dual circle. This piece of debris is measured. The piece of debris with a projection line of length h_2 is out of bounds, though the dual circle overlaps the line plot. This piece of debris is rejected.



The ratio of the true length of a piece of debris to the projected width of the dual circle is L_i/h_i. The true length of that portion of the piece of debris which crosses the line plot is $(L_i/h_i)\Delta x$ and the true length per unit projected length is $[(L_i/h_i)\Delta x]/\Delta x$ or (L_i/h_i). Thus, the total true length of coarse woody debris on the tract (**L**) and per unit land area (ρ_L) are estimated by

$$\tilde{\mathbf{L}} = \frac{\mathbf{A}}{Y}\sum_i^m \frac{L_i}{h_i}; \qquad \tilde{\rho}_L = \frac{1}{Y}\sum_i^m \frac{L_i}{h_i}$$

Substitution of coverage area, volume, or 1 for L_i yields estimates of aggregate coverage area, volume, or pieces of debris on the tract and per unit area. The coverage area or volume of the ith piece of coarse woody debris can be estimated unbiasedly by crude Monte Carlo, as was demonstrated above, or approximated by the trapezoidal rule.

Point methods

We now consider the subsampling of the line plot of length Y and width Δx, where Δx approaches 0. The line plot can be divided along its length into N smaller subplots, each with length $\Delta y = Y/N$. Of particular interest is the subplot that results when N approaches ∞ and Δy approaches 0. In this instance, we are left with a point plot with length Δy and width Δx, where both Δy and Δx approach 0. The resultant point plot is indistinguishable from a point, though we shall continue to regard it as rectangular in shape. This procedure selects the point plot from the line plot with probability density $1/Y$. Because the line plot was selected from the area of the tract with probability density $Y/\mathbf{A}$, the point plot is selected from the tract with probability density $(1/Y)(Y/\mathbf{A}) = 1/\mathbf{A}$. Thus, in general, the location of a point plot within a tract can be selected by the acceptance–rejection method: generate $\phi = u_1\mathbf{X}$ and $\theta = u_2\mathbf{Y}$. If (ϕ, θ) occur within the tract, they define the location of the point plot. Otherwise, generate new coordinates.

Once again, for illustrative purposes, let us assume that the target parameter is the aggregate volume of coarse woody debris per unit land area. Let $\mathcal{D}(x, y)$ denote the depth of coarse woody debris at (x, y). Let us assume that a corner of the point plot is located at (ϕ, θ), $0 \le \theta \le Y$. By definition, the aggregate cross-sectional area of coarse woody debris that crosses the line plot is

$$\mathcal{A}(\phi) = \int_0^Y \mathcal{D}(\phi, y)\,\mathrm{d}y$$

Substituting into [9] and replacing the tilde with a dot (to indicate that the estimate derives from measurements on a point plot),

$$\dot{\mathbf{V}} = \mathbf{A}\left(\frac{\int_0^Y \mathcal{D}(\phi, y)\,\mathrm{d}y}{Y}\right)$$

The "crude Monte Carlo estimator" of the integral is simply $Y \times \mathcal{D}(\phi, \theta)$, therefore,

$$\dot{\mathbf{V}} = \mathbf{A}\left(\frac{Y\mathcal{D}(\phi, \theta)}{Y}\right)$$

$$= \mathbf{A}\,\mathcal{D}(\phi, \theta)$$

or, dropping the location coordinates,

$$\dot{\mathbf{V}} = \mathbf{A}\,\mathcal{D}$$

The estimator of the volume of coarse woody debris per unit land area is simply the depth of wood covering the point plot, i.e.,

$$\dot{\rho}_V = \mathcal{D}$$

In this application, it may be useful to think of the depth of wood as having dimensions of volume ($\mathcal{D}\,\Delta x\,\Delta y$) per unit area ($\Delta x\,\Delta y$).

We can also estimate coverage area, **C**. Since the point plot is vanishingly small, a piece of woody debris (or any other object) either covers the land area of the point plot completely, or not at all. Assuming the point plot is covered by one piece of debris, the coverage area per unit land area of the point plot is $(\Delta x\Delta y)/(\Delta x\Delta y)$. Of course, more than one piece of debris may cover the point plot. If m pieces of debris cover the point plot, the estimator of **C** is

$$\dot{\mathbf{C}} = \mathbf{A}\sum_i^m \frac{\Delta x\Delta y}{\Delta x\Delta y}$$

$$= \mathbf{A}m$$

Thus, the coverage area per unit land is estimated by the count (m) of pieces of coarse woody debris covering the point plot, i.e.,

$$\dot{\rho}_C = m$$

It is very doubtful that the use of point plot, as just described, would be efficient for estimating coarse woody debris. Point-based sampling is most efficient when the variable of interest is distributed uniformly but most of the land surface is devoid of coarse woody debris. However, other variables that are distributed fairly uniformly per unit land area come to mind. For example, the volume of soil in the A horizon, per unit land area, can be estimated from the depth of this horizon at the point plot. Leaf-area index of a broad-leaved forest can be estimated, just after leaf fall, from the count of leaves covering the point plot. Root mass (or soil carbon) can be extracted with a core centered at the plot. That the core is larger than the plot is of no concern; root mass (or soil carbon) per unit cross-sectional area of the core provides the estimate of root mass (or soil carbon) per unit land area.

Variable-radius plot sampling

Variable-radius plot sampling, which stems from Bitterlich (1948), is also known as horizontal point sampling and Bitterlich sampling. The method generally provides very efficient estimates of the aggregate basal area and volume of standing trees per unit land area.

In a Monte Carlo approach to variable-radius plot sampling, the point plot is the only plot involved. We assume that each tree on the tract is endowed with a tree circle, the radius of which is αr_i, where r_i is bole radius at breast height. The coverage area of the ith tree circle is $\pi\,\alpha^2 r_i^2$ or $\alpha^2 B_i$, where B_i is the tree's basal area. Typically, α is assigned a value large enough to ensure that about 5 to 12 tree circles, on average, cover any point on the forest floor (Avery and Burkhart 1994). Whether a particular tree circle overlaps a point plot is determined with an angle gauge, the angle of the gauge corresponding to the value of α, as noted previously.

The count, m, of tree circles overlapping the point plot is an estimator of the coverage area of tree circles per unit land area. Accordingly, the estimators of total basal area on the tract (**B**) and per unit area ($\rho_{\mathbf{B}}$) are

$$\dot{\mathbf{B}} = \mathbf{A} \times \frac{m}{\alpha^2}; \qquad \dot{\rho}_{\mathbf{B}} = \frac{m}{\alpha^2}$$

where α^{-2} is the so-called "basal area factor" (m^2/m^2) of the angle gauge. The estimation of $\rho_{\mathbf{B}}$ without tree circles would be very inefficient. A point plot would yield an estimate of either 1 or 0 m^2 of basal area per square metre of land area, depending on whether a tree stood over the point plot or not.

To estimate aggregate volume, we require the depth of wood over the point plot. We assume that a volume of wood equal to the volume of the ith tree, T_i, is distributed uniformly over the land area of the tree circle to a depth of $T_i/(\pi\,\alpha^2 r_i^2)$ or $T_i/(\alpha^2 B_i)$. Thus, the estimators of tree volume on the tract, **T**, and per unit area, $\rho_{\mathbf{T}}$, are

$$\dot{\mathbf{T}} = \frac{\mathbf{A}}{\alpha^2}\sum_i^m \frac{T_i}{B_i}; \qquad \dot{\rho}_{\mathbf{T}} = \frac{1}{\alpha^2}\sum_i^m \frac{T_i}{B_i}$$

In critical height sampling, a variant of variable-radius plot sampling (see, e.g., Kitamura 1962; Iles 1990; Schreuder et al. 1993), **T** and $\rho_{\mathbf{T}}$ are estimated by substituting a measurement of a critical height for T_i/B_i in the above estimators. Tree count, **P**, and tree density, $\rho_{\mathbf{P}}$, are estimated by substituting 1 for T_i.

Point relascope sampling

Like transect relascope sampling, point relascope sampling was developed to estimate quantities of coarse woody debris on a tract or per unit area (Gove et al. 1999). Whereas transect relascope sampling is quite similar, in theory and operation, to horizontal line sampling, point relascope sampling is similar, in theory and operation, to horizontal point sampling. In our Monte Carlo approach to horizontal point sampling (or variable-radius plot sampling as it is better known), we assumed that each tree is endowed with a tree circle; in our approach to point relascope sampling we shall assume that each piece of coarse woody debris is endowed with a dual circle.

The coverage area of the dual circle is determined by the true length-square of the piece of debris and the angle ν (°) of an angle gauge. The angle gauge indicates whether a dual circle overlaps a point plot. The coverage area of a dual circle is φL_i^2 (Gove et al. 1999), where

$$\varphi = \frac{\pi - \tilde{\nu} + \sin\nu\cos\nu}{2\sin^2\nu}$$

and $\tilde{\nu}$ is the angle of the angle gauge in radians.

The count, m, of dual circles that overlap the point plot is an estimator of the coverage area of dual circles per unit land area. Because φ is a constant of proportionality between the length-square of a piece of debris and the coverage area of its dual circle, m/φ is an estimator of aggregate length-square of coarse woody debris per unit land area. The total true length of coarse woody debris on the tract and per unit area are estimated by

$$\dot{\mathbf{L}} = \frac{\mathbf{A}}{\varphi}\sum_i^m \frac{L_i}{L_i^2} = \frac{\mathbf{A}}{\varphi}\sum_i^m \frac{1}{L_i}$$

and

$$\dot{\rho}_{\mathbf{L}} = \frac{1}{\varphi}\sum_i^m \frac{1}{L_i}$$

The ratio of the true coverage area of the ith piece of debris to the coverage area of its dual circle is $C_i/(\varphi L_i^2)$. Thus, **C** and $\rho_{\mathbf{C}}$ are estimated by

$$\dot{\mathbf{C}} = \frac{\mathbf{A}}{\varphi}\sum_i^m \frac{C_i}{L_i^2}; \qquad \dot{\rho}_{\mathbf{C}} = \frac{1}{\varphi}\sum_i^m \frac{C_i}{L_i^2}$$

Coverage area of the ith piece of debris can be unbiasedly estimated by $d_i \times L_i$, where d_i is the calipered diameter at a random point, uL_i, along the axis of true length. Substituting into the above estimators,

$$\dot{\mathbf{C}} = \frac{\mathbf{A}}{\varphi}\sum_i^m \frac{d_i}{L_i}; \qquad \dot{\rho}_{\mathbf{C}} = \frac{1}{\varphi}\sum_i^m \frac{d_i}{L_i}$$

To estimate aggregate volume, let us assume that the volume (V_i) of the ith piece of debris is distributed over the coverage area of the dual circle to a uniform depth of $V_i/(\varphi L_i^2)$. Thus, $\mathbf{V}$ and ρ_V are estimated by

$$\dot{\mathbf{V}} = \frac{\mathbf{A}}{\varphi}\sum_i^m \frac{V_i}{L_i^2}; \qquad \dot{\rho}_V = \frac{1}{\varphi}\sum_i^m \frac{V_i}{L_i^2}$$

Volume of the ith piece of debris is unbiasedly estimated by $A_i \times L_i$, where A_i is measured at uL_i. Therefore,

$$\dot{\mathbf{V}} = \frac{\mathbf{A}}{\varphi}\sum_i^m \frac{A_i}{L_i}; \qquad \dot{\rho}_V = \frac{1}{\varphi}\sum_i^m \frac{A_i}{L_i}$$

Fixed-radius plot sampling

Fixed-radius plot sampling for the purpose of estimating the attributes of trees per unit land area can be viewed as a special case of variable-radius plot sampling. As was noted, in a Monte Carlo approach to variable-radius plot sampling, we assume that each tree is endowed with a proportional tree circle. In a Monte Carlo approach to fixed-radius plot sampling, we also assume that each tree is endowed with a tree circle, but all tree circles are the same size, i.e., all tree circles have the same fixed radius. Obviously, those trees whose tree circles encircle the point plot are the trees that would occupy a circular plot, with the same fixed radius, centered at the point plot. The use of fixed-radius tree circles may have originated with Grosenbaugh (1958).

The count (m) of tree circles that encircle the point plot is an estimator of the coverage area of tree circles per unit land area. Division of the count by the area of a tree circle (f) provides m/f, the estimator of the number of trees per unit land area. Aggregate tree volume on the tract and per unit land area are estimated by

$$\dot{\mathbf{T}} = \mathbf{A}\sum_i^m \frac{T_i}{f}; \qquad \dot{\rho}_T = \sum_i^m \frac{T_i}{f}$$

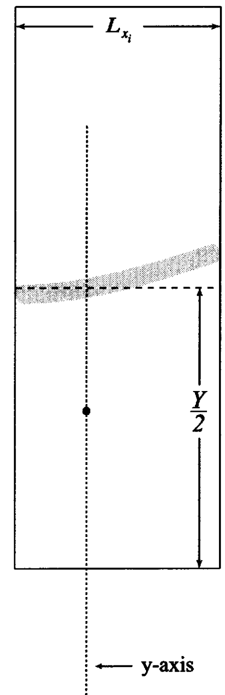
where T_i/f is the assumed uniform depth of wood over the coverage area of the ith tree circle.

Point-based approaches to line methods

As strange as it may seem, the most common implementation of line intersect sampling can be classified as a point method. A point within the tract is selected at random to serve as the midpoint of a transect line. The azimuth of the direction or orientation of the transect line may be selected at random (i.e., $u \times \pi$ (rad) or $u \times 180$ (°) or fixed in advance of the sampling. The length of the transect line, Y, ordinarily is fixed in advance of the sampling (e.g., Gregoire and Monkevich 1994). Line intersect sampling, conducted in this way, is a point method because only the midpoint and, possibly, the orientation of the transect line are selected at random; the intersect line, as a whole, is not selected at random from within the tract by importance sampling. A boundary problem, which is discussed below, may arise when the point method is used, but this problem is easily solved.

In the Monte Carlo approach to this method, the coordinates of a corner of a point plot are selected at random. As usual, the rectangular point plot has length Δy and width Δx, where both Δy and Δx approach 0. The azimuth of the y axis may be fixed or selected at random, in keeping with the established procedures of the point-based approach. Let us assume that one side of the point plot (of length Δy) is coincident with the y axis. The y axis extends across the tract, intersecting pieces of coarse woody debris. Our interest is limited to those intersected pieces of debris that are both on the tract and within a distance of $Y/2$ of the point plot in either direction. Such pieces are intersected by a line of length Y, centered at the point plot.

Fig. 6. Line intersect as a point method. Each piece of debris is assumed to occupy a $L_{x_i}Y$ quadrat. If the quadrat overlaps the point plot, the piece is measured.



Let us assume that the ith piece of debris is endowed with an enlarged coverage area equal to the area of a rectangular quadrat with dimensions $Y \times L_{x_i}$, where L_{x_i} is the projected length of the piece of debris on the x axis. The sides of the quadrat of length Y are parallel to the y axis and centered with respect to the intersection of the piece of debris with the y axis or intersect line (Fig. 6). Thus, if the piece of debris is within a distance of $Y/2$ of the point plot, then the quadrat overlaps the point plot.

The volume of coarse woody debris on the tract, $\mathbf{V}$, is estimated by the depth of wood over the point plot. Following our standard practice, we assume that a volume of wood equal to the volume of the ith piece of debris is distributed uniformly over the ith quadrat to a depth of $V_i/(YL_{x_i})$. Thus, $\mathbf{V}$ is estimated by

$$\dot{\mathbf{V}} = \frac{\mathbf{A}}{Y}\sum_i^m \frac{V_i}{L_{x_i}}$$

However, if we let the y axis define a line plot within the quadrat, then A_i, the cross-sectional area intersected by the y axis, is a crude Monte Carlo estimator of V_i/L_{x_i}. Substituting into the previous equation,

$$\dot{\mathbf{V}} = \frac{\mathbf{A}}{Y}\sum_{i}^{m} A_i$$

and, therefore,

$$\dot{\rho}_{\mathrm{V}} = \frac{1}{Y}\sum_{i}^{m} A_i$$

The area of the portion of a quadrat that covers the point plot is $\Delta x\ \Delta y$. The coverage area of the ith piece of debris per unit coverage area of quadrat is $C_i/(YL_{x_i})$. Thus, the estimator of $\mathbf{C}$, the coverage area of coarse woody debris on the tract, is

$$\dot{\mathbf{C}} = \mathbf{A}\sum_{i}^{m} \frac{[C_i/(YL_{x_i})]\Delta x\ \Delta y}{\Delta x\ \Delta y}$$

$$= \frac{\mathbf{A}}{Y}\sum_{i}^{m} \frac{C_i}{L_{x_i}}$$

However, C_i is unbiasedly estimated by $\Omega_i L_{x_i}$, where Ω_i is the calipered width of the piece of debris where it intersects the y axis. Substituting into the previous equation,

$$\dot{\mathbf{C}} = \frac{\mathbf{A}}{Y}\sum_{i}^{m} \Omega_i$$

and, therefore

$$\dot{\rho}_{\mathrm{C}} = \frac{1}{Y}\sum_{i}^{m} \Omega_i$$

It is evident that the estimators that derive from a point-based approach to line intersect sampling are the same as those that derive from a line-based approach. True length, mass, and the number of pieces of debris also can be estimated.

Horizontal line sampling and transect relascope sampling also can be implemented as point methods in a fashion analogous to line intersect sampling. In the former, the attributes of the ith tree are assumed to be distributed uniformly over the coverage area of a quadrat of dimensions $\alpha D_i \times Y$ and, in the latter, the attributes of the ith piece of debris are assumed to be distributed over a quadrat with dimensions $h_i \times Y$.

Sampling error

Estimates from two or more plots are needed to estimate sampling error. For example, let $\dot{\mathbf{T}}_j$ denote an estimate of $\mathbf{T}$ from the jth of n point plots. A combined estimate of $\mathbf{T}$ is provided by

$$\widehat{\mathbf{T}} = \frac{\sum \dot{\mathbf{T}}_j}{n}$$

and a sample-based estimate of the variance of $\dot{\mathbf{T}}$ is provided by

$$\mathrm{var}(\widehat{\mathbf{T}}) = \frac{\sum(\dot{\mathbf{T}}_j - \widehat{\mathbf{T}})^2}{n(n-1)}$$

The same formulas hold for estimates from line plots and for estimates from a combination of point plots and line plots, provided the selection of each plot is independent from all others.

Boundary problems

Boundary problems arise in both conventional and Monte Carlo approaches to line and point methods. In Monte Carlo approaches, tract boundaries are problematic when we assume that attributes of a piece of debris are distributed uniformly over the coverage area of a dual circle or quadrat, or along a line; or when we assume that the attributes of a tree are distributed uniformly over the coverage area of a tree circle or quadrat, or along a line.

To develop an intuitive grasp of the problem, let us assume the following: (1) the target parameter is the aggregate tree volume per unit land area of the tract, (2) the method of sampling is the Monte Carlo approach to variable-radius plot sampling, and (3) the estimator of the target parameter is the depth of wood on the point plot (or the average of the depths of wood on several point plots).

In a Monte Carlo approach to variable-radius plot sampling, we assume that each tree on the tract is endowed with a proportional tree circle. If the distance from a tree to the boundary is less than the radius (αr) of the tree circle, then part of the tree circle will be off the tract. When volume is the target parameter, we assume that an amount of wood equal to the volume of the tree is distributed uniformly over the area of the tree circle to a depth of $T_i/(\pi\alpha^2 r_i{}^2)$. In effect, we implicitly assume that a portion of volume of a tree is off the tract if the boundary runs through the tree circle, and so for all such trees.

Thus, owing to the artifice of the tree circle, some of the volume is missed by the sampling because the locations of point plots are restricted to the confines of the tract. Consequently, the "actual volume" per unit land area is expected to be underestimated. To develop an intuitive grasp of the solution of this problem, let us imagine that the tree circle is a piece of circular cloth. The portion of cloth that occurs outside of the tract is folded over onto the tract. At a square corner, two folds are necessary to get all of the cloth on the tract and a point plot may be covered by one, two, or four layers of cloth. If m layers of cloth from particular tree circle cover a point plot, the depth of wood is counted m times, and so for all trees whose cloth circles overlap the point plot. Because all of the volume is folded back onto the tract, the expectation of underestimation is eliminated. Operationally, the folding is accomplished implicitly with the mirage method (Schmid-Haas 1969; Gregoire 1982). The mirage method is easy to implement in the field, though, ordinarily, its use is limited to straight boundaries with square corners. The method works whether the tree circles are fixed-radius or proportional, so the method solves the boundary problem for fixed-radius as well as variable-radius plot sampling.

Dual circles cause boundary problems analogous to those caused by tree circles, i.e., they may overlap the boundary, and there is the added problem of the piece of debris that lies across a boundary. This second problem is easily solved by cutting off the out-of-bounds portion, either literally with a saw, or figuratively by erecting a range pole at the boundary to indicate a "new end". The size of the dual circle is determined by the length of the segment that is on the tract. The solution to the first problem is somewhat complicated because the orientation of a dual circle relative to the boundary comes into play, rendering the mirage method ineffective. A solution is provided by the "boundary reflection method" (Gove et al. 1999).

A boundary problem also arises in the point-based approach to line intersect sampling. When this method is used to estimate coarse woody debris, we assume that the attributes of each piece of debris are spread uniformly over the coverage area of a quadrat. Pieces of debris close to the boundary may occupy quadrats that overlap the boundary. This problem is simplified by the fact that we, in effect, subsample the quadrat with a line plot. The solution to this problem is to fold the portion of the line plot that extends across the boundary back upon itself (Gregoire and Monkevich 1994). If the folded portion of the line plot reaches the point plot, the measurements of the piece of debris are used twice. However, if a line plot centered at a piece of debris folds back from the border as far as the point plot, then a transect line of the same length centered at the point plot will fold back from the border as far as the piece of debris. Thus, we need only fold back the transect line at the border (Gregoire and Monkevich 1994). Measurements of all pieces of debris on the folded portion of the transect are used twice. This method also works for point-based approaches to horizontal line sampling and transect relascope sampling.

Horizontal line sampling or transect relascope sampling may have additional boundary problems, regardless of whether they are implemented as line or point methods. If a tree is close to the boundary, its tree-circle diameter may extend across the boundary. An analogous problem arises in transect relascope sampling when the projection line of a piece of coarse woody debris extends across a boundary. One solution is to fold the diameter line or projection line back onto the tract and use the measurements twice if the folded portion of the diameter line or projection line crosses the transect line. The measurements are used four times if a folded diameter line or projection line crosses a folded transect line. This solution would require a lot of extra work if the transect line runs closely beside a boundary.

Iles (2000, 2001) recently posted a "toss-back method" on the World Wide Web. The toss-back method solves boundary problems for all point methods, and it works for both curved and straight boundaries.

Discussion

We have shown that three line methods, i.e., line intersect sampling, horizontal line sampling, and transect relascope sampling, can be considered applications of importance sampling, where a line is selected from an area with probability (density) proportional to length. We also have shown that three point methods, variable-radius plot sampling, point relascope sampling, and fixed-radius plot sampling, can be considered as two-stage applications of importance sampling and crude Monte Carlo, where the second-stage application of crude Monte Carlo subsamples the line with a point. An essentially similar point method can substitute for each of the three line methods, which is fortunate because point methods reduce to a very simple recipe: (1) select the coordinates of a point plot at random, and (2) measure the dimensions, real or contrived, of the objects of interest that cover the point plot.

It is the contrived dimensions that lend efficiency to the sampling. Sampling by any of the six methods is most efficient when the attributes of interest are spread uniformly over the forest floor. Trees or pieces of coarse woody debris generally are spotty, covering a very small fraction of the area of forest floor. The depth of wood, to use an example, is either great or zero, depending upon whether a tree stands over a given point. Uniformity is increased by distributing the attributes of a tree uniformly over the coverage area of its tree circle or by distributing the attributes of a piece of debris uniformly over its dual circle or quadrat.

The coverage areas of the tree circles, dual circles, or quadrats are the so-called "inclusion areas" of the conventional approaches to line point methods that treat the trees or pieces of debris as populations of discrete elements (e.g., Kaiser 1983; Eriksson 1995). The inclusion probability, i.e., the probability that a tree or piece of debris will be sampled by a point or a line of a given length thrown down at random, is the inclusion area divided by the land area of the tract. In the Monte Carlo approach, we have no need of inclusion probabilities since only the location of a point or line within a tract is selected at random.

Variable-radius plot sampling, perhaps the most widely used sampling method in forestry, can be approached from at least four different theories (Eriksson 1995). Our Monte Carlo approach to this method is basically the same as the "infinite population approach" of Eriksson. Apart from labeling, the only difference between the two approaches seems to be the starting point. Whereas we derived the estimators directly from simple random sampling with replacement, Eriksson derived the estimators from a more general statistical theory. However, Monte Carlo falls under this same general statistical theory, so there is no real difference.

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Appendix A: Symbol table

$A(l)$	Cross-sectional area of a piece of debris at $0 \leq l \leq L$ (m^2)
A_i	Cross-sectional area of a piece of debris over a line plot (m^2)
$\mathcal{A}$	Aggregate cross-sectional area of pieces of debris over a line plot (m^2)
$\mathcal{A}(\phi)$	Aggregate cross-sectional area of pieces of debris at $0 \leq \phi \leq X$ (m^2)
$\mathbf{A}$	Area of the forested tract (m^2)
B_i	Basal area (cross-sectional area at 1.37 m) of a tree (m^2)
$\mathbf{B}$	Aggregate basal area of the trees on the forested tract (m^2)
$\dot{\mathbf{B}}$	Estimate of the aggregate basal area of the trees on the forested tract (m^2)
$\mathbf{C}$	Coverage area of coarse woody debris on the forested tract (m^2)
$\widetilde{\mathbf{C}}, \dot{\mathbf{C}}$	Estimate of coverage area of coarse woody debris on the forested tract (m^2)
d_i	Diameter, of a piece of debris, measured horizontally and perpendicular to the axis of true length (m)
D_i	Diameter of a tree at 1.37 m (m)
$\mathcal{D}$	Depth of wood over a point plot (m)
$\mathcal{D}(x, y)$	Depth of wood at location (x, y) (m)
$\mathbf{D}$	Sum of diameters of trees on the forested tract (m)
$\widetilde{\mathbf{D}}$	Estimate of sum of diameters of trees on the forested tract (m)
f	Land area of a fixed-radius plot or a fixed-radius tree circle (m^2)
h_i	Projected width (perpendicular to a line plot) of a dual circle (m)
l, l_k	Location ordinate on the axis of true length of a piece of debris (m)
Δl	Length of a segment of a piece of debris (m)
L, L_i	True length of a log or piece of debris (m)

$\mathbf{L}$	Sum of the true lengths of the pieces of debris on the forested tract (m)
$\widetilde{\mathbf{L}}$	Estimate of the sum of the true lengths of the pieces of debris on the forested tract (m)
L_{x_i}	Projected length of a piece of debris (m)
$\mathbf{L}_x$	Sum of projected lengths of the pieces of debris on the forested tract (m)
$\widetilde{\mathbf{L}}_x$	Estimate of the sum of projected lengths of the pieces of debris on the forested tract (m)
m	Count of objects crossing a line plot or covering a point plot
n	Count of line plots or point plots
N	Count of subplots or segments
$\mathbf{P}$	Count of pieces of debris on the forested tract
r_i	Radius of the bole of a tree at 1.37 m (m)
T_i	Volume of a tree (m^3)
$\mathbf{T}$	Aggregate volume of the trees on the forested tract (m^3)
$\widetilde{\mathbf{T}}, \dot{\mathbf{T}}$	Estimate of the aggregate volume of the trees on the forested tract (m^3)
$\widehat{\mathbf{T}}$	Combined estimate of the aggregate volume of the trees on the forested tract (m^3)
u, u_i	A random number from a uniform [0,1] distribution.
$V(l_i)$	Volume, of a piece of debris, from $l = 0$ to $l = l_i$ (m^3)
V_i	Total volume of a piece of debris (m^3)
ΔV_k	Volume of a segment of a piece of debris (m^3)
$\mathcal{V}(x)$	Aggregate volume of coarse woody debris on a quadrat with opposite corners at $(0, 0)$ and (x, Y) (m^3)
$\mathcal{V}_k$	Aggregate volume of coarse woody debris on a subplot (m^3)
$\mathbf{V}$	Aggregate volume of coarse woody debris on the forested tract (m^3)
$\widetilde{\mathbf{V}}, \dot{\mathbf{V}}$	Estimate of aggregate volume of coarse woody debris on the forested tract (m^3)
x_i	Location ordinate (m)
Δx	Width of a rectangular subplot (m)
X	Width of a rectangular plot (m)
$\mathbf{X}$	Projected width of the forested tract (m)
Δy	Length of a rectangular subplot (m)
Y	Length of a rectangular plot or line plot (m)
$\mathcal{Y}(x)$	Distance from the southern to the northern border of the forested tract, given the ordinate x (m)
$\mathbf{Y}$	Projected length of the forested tract (m)
α	Ratio of the diameter of a proportional tree cricle to the diameter of the tree
θ, θ_i	Location ordinate (m)
ν	Angle of an angle gauge (°)
$\widetilde{\nu}$	Angle of an angle gauge (rad)
ϕ, ϕ_i	Location ordinate (m)
$\rho_{\mathbf{Z}}$	Per unit land area density of $\mathbf{Z} = \mathbf{B}, \mathbf{C}, \mathbf{D}, \mathbf{L}, \mathbf{L}_x, \mathbf{P}, \mathbf{T}$, or $\mathbf{V}$

$\widetilde{\rho}_{\mathbf{Z}}, \dot{\rho}_{\mathbf{Z}}$ Estimate of $\rho_{\mathbf{Z}}$

φ Ratio of the area of a dual circle to the length-square of a piece of debris

ψ_i Acute angle formed by a line plot and the perpendicular to the axis of true length of a piece of debris (°)

ψ_i' Acute angle formed by a line plot and the axis of true length of a piece of debris (°)

$\omega(x)$ Width of a piece of debris calipered horizontally at x (m)

Ω_i Width of a piece of debris calipered horizontally over a line plot (m)

Ω_i' Thickness of a piece of debris calipered vertically over a line plot (m)

Appendix B: Methods for estimating cross-sectional area

Here we suggest five different ways to estimate A_i, the cross-sectional area of a piece of coarse woody debris, where it spans a line plot (intersect line):

(1) Under the assumption that the cross section of a piece of debris is round when viewed perpendicular to the axis of true length, the cross section over the line plot will be elliptical. As noted, Ω_i is the width of the ith piece of wood along the line plot. Let Ω_i' denote depth of wood calipered vertically over the line plot. The cross sectional area over the line plot is estimated by

$$A_i \approx \frac{\pi}{4} \times \Omega_i \Omega_i'$$

(2) If Ω_i' can not be measured, substitute d_i, which is the diameter calipered horizontally and perpendicular to the axis of true length. If the piece of debris is round and not tilted, then d_i should equal Ω_i'.

(3) The ratio of the projected length to true length of a piece of debris (L_{x_i}/L_i) is equivalent to the cosine of the acute angle (ψ_i) between the line plot and a line perpendicular to the axis of true length. Therefore, $\Omega_i = d_i/(L_{x_i}/L_i)$ and, since $\Omega_i' = d_i$,

$$A_i \approx \pi \left(\frac{d_i}{2}\right)^2 \frac{L_i}{L_{x_i}}$$

(4) Alternatively, and most likely less accurately, measure the angle ψ_i directly, hence,

$$A_i \approx \frac{\pi}{\cos \psi_i} \left(\frac{d_i}{2}\right)^2$$

or, measure the acute angle (ψ_i') between the axis of true length and the line plot, hence,

$$A_i \approx \frac{\pi}{\sin \psi_i'} \left(\frac{d_i}{2}\right)^2$$

(5) Substitute total volume per unit projected length for cross-sectional area, where the total volume (V_i) of a piece of debris is estimated by crude Monte Carlo (note that $(\pi/4)\, d_i{}^2\, L_i$ from (3) is a crude Monte Carlo estimate of V_i, but additional estimates can be averaged). Alternatively, instead of random diameters, use the diameters at the two ends (d_1 and d_2) to calculate volume with Smalian's formula, i.e., $\widetilde{V}_i = [(\pi L)/8](d_1{}^2 + d_2{}^2)$. Hence,

$$A_i \approx \frac{\widetilde{V}_i}{L_{x_i}}$$

Method (1) is a conventional method found in forest mensuration texts for estimating elliptical cross-sectional areas. Methods (2) and (3) may be original, though they are rather obvious, and method (5) may have originated with Warren and Olsen (1964).

Appendix C: Proofs and variances

All of the estimators for the line methods are unbiased. Let us consider a tract with area $\mathbf{A}$ whose most western point is denoted by $x = 0$ and whose most eastern point is denoted by $x = \mathbf{X}$. The distance from the southern to northern border at $x = \phi$ is $\mathcal{Y}(\phi)$ for $0 \le \phi \le \mathbf{X}$. An ordinate, $x = \phi$, is selected at random with probability density $\mathcal{Y}(\phi)/\mathbf{A}$. The line plot at $x = \phi$, with length $\mathcal{Y}(\phi)$, runs perpendicular to the x axis from the southern to the northern border.

Let $\mathbf{Z}$ denote an attribute of interest concerning the coarse woody debris or standing trees on the tract, i.e., $\mathbf{Z} = \mathbf{B}$, $\mathbf{C}$, $\mathbf{D}$, $\mathbf{L}$, $\mathbf{L}_x$, $\mathbf{P}$, $\mathbf{T}$, or $\mathbf{V}$ (see Appendix A). To facilitate the following proof, let us refine our definition of $\mathbf{Z}$ somewhat. Let $\mathbf{Z}(\phi)$ denote the aggregate quantity of the attribute between $x = 0$ and $x = \phi$. Thus, the aggregate quantity of the attribute on the entire tract is $\mathbf{Z} \equiv \mathbf{Z}(\mathbf{X})$. The aggregate quantity of attribute covering a line plot at $x = \phi$ is

$$\frac{\mathrm{d}\,\mathbf{Z}(\phi)}{\mathrm{d}x} = \lim_{\Delta x \to 0} \frac{\mathbf{Z}(\phi + \Delta x) - \mathbf{Z}(\phi)}{\Delta x}$$

The generic estimator of the aggregate quantity of an attribute on the tract is

$$\widetilde{\mathbf{Z}} = \frac{\mathbf{A}}{\mathcal{Y}(\phi)} \frac{\mathrm{d}\,\mathbf{Z}(\phi)}{\mathrm{d}x}$$

For example, the aggregate true length of coarse woody debris on the tract is denoted by $\mathbf{L}$. The estimator of $\mathbf{L}$ is

$$\widetilde{\mathbf{L}} = \frac{\mathbf{A}}{\mathcal{Y}(\phi)} \frac{\mathrm{d}\,\mathbf{L}(\phi)}{\mathrm{d}x}$$

where

$$\frac{\mathrm{d}\,\mathbf{L}(\phi)}{\mathrm{d}\,x} = \begin{cases} \sum_i^{m(\phi)} \frac{L_i}{L_{x_i}}, & \text{line intersect sampling} \\ \sum_i^{m(\phi)} \frac{L_i}{h_i}, & \text{transect relascope sampling} \end{cases}$$

and $m(\phi)$ is the number of pieces of debris or the number of projection lines of dual circles that cross the line plot at $x = \phi$. The expectation of $\widetilde{\mathbf{L}}$ is

$$\mathrm{E}(\widetilde{\mathbf{L}}) = \int_0^{\mathbf{X}} \frac{\mathcal{Y}(x)}{\mathbf{A}} \left(\frac{\mathbf{A}}{\mathcal{Y}(x)} \frac{\mathrm{d}\,\mathbf{L}(x)}{\mathrm{d}\,x} \right) \mathrm{d}\,x$$

$$= \int_0^{\mathbf{X}} \frac{\mathrm{d}\,\mathbf{L}(x)}{\mathrm{d}\,x} \mathrm{d}\,x$$

$$= \mathbf{L}$$

The variance of $\widetilde{\mathbf{L}}$ is

$$\mathrm{Var}(\widetilde{\mathbf{L}}) = \frac{1}{n} \int_0^{\mathbf{X}} \frac{\mathcal{Y}(x)}{\mathbf{A}} \left(\widetilde{\mathbf{L}} - \mathbf{L}\right)^2 \mathrm{d}\,x$$

Proofs of unbiasedness and variances for the other estimators are analogous. All of the estimators for the point methods also are unbiased. Let us assume standard procedures, where the x ordinate of a point plot, ϕ, is selected with probability density $\mathcal{Y}(\phi)/\mathbf{A}$ and the y ordinate, θ, is selected with probability density $1/\mathcal{Y}(\phi)$. Thus, the coordinates (ϕ, θ) are selected with probability density $1/\mathbf{A}$. The generic estimator of $\mathbf{Z}$ is

$$\dot{\mathbf{Z}} = \mathbf{A} \times \frac{\mathrm{d}}{\mathrm{d}y} \left(\frac{\mathrm{d}\,\mathbf{Z}(\phi)}{\mathrm{d}\,x} \right)$$

For sake of example, we prove that the estimator of aggregate volume of coarse woody debris on the tract is unbiased. In point relascope sampling, the estimator of $\mathbf{V}$ is

$$\dot{\mathbf{V}} = \mathbf{A} \times \frac{\mathrm{d}}{\mathrm{d}y} \left(\frac{\mathrm{d}\,\mathbf{V}(\phi)}{\mathrm{d}\,x} \right)$$

$$= \mathbf{A} \times \frac{\mathrm{d}\,\mathcal{A}(\phi)}{\mathrm{d}\,y}$$

$$= \mathbf{A} \times \mathcal{D}(\phi, \theta)$$

where

$$\mathcal{D}(\phi, \theta) = \frac{1}{\varphi} \sum_i^{m(\phi,\theta)} \frac{V_i}{{L_i}^2}$$

is the depth of wood at (ϕ, θ), $m(\phi, \theta)$ is the number of dual circles that overlap the point plot, and $\mathcal{A}(\phi)$ is the cross-sectional area of wood on a line plot at $x = \phi$. The expectation of $\dot{\mathbf{V}}$ is

$$\mathrm{E}(\dot{\mathbf{V}}) = \mathrm{E}\left[\mathbf{A} \times \mathcal{D}(x, y)\right]$$

$$= \mathrm{E}\left(\frac{\mathbf{A}}{\mathcal{Y}(x)} \mathcal{Y}(x) \times \mathcal{D}(x, y) \right)$$

$$= \int_0^{\mathbf{X}} \frac{\mathcal{Y}(x)}{\mathbf{A}} \left\{ \frac{\mathbf{A}}{\mathcal{Y}(x)} \times \int_0^{\mathcal{Y}(x)} \frac{1}{\mathcal{Y}(x)} \left[\mathcal{Y}(x) \times \mathcal{D}(x, y)\right] \mathrm{d}y \right\} \mathrm{d}\,x$$

$$= \int_0^{\mathbf{X}} \frac{\mathcal{Y}(x)}{\mathbf{A}} \left\{ \frac{\mathbf{A}}{\mathcal{Y}(x)} \int_0^{\mathcal{Y}(x)} \mathcal{D}(x, y)\, \mathrm{d}y \right\} \mathrm{d}\,x$$

$$= \int_0^{\mathbf{X}} \frac{\mathcal{Y}(x)}{\mathbf{A}} \left\{ \frac{\mathbf{A}}{\mathcal{Y}(x)} \mathcal{A}(x) \right\} \mathrm{d}\,x$$

$$= \int_0^{\mathbf{X}} \mathcal{A}(x)\, \mathrm{d}\,x$$

$$= \mathbf{V}$$

The variance of $\dot{\mathbf{V}}$ is

$$\mathrm{Var}\left(\dot{\mathbf{V}}\right) = \frac{1}{n\mathbf{A}} \int_0^{\mathbf{X}} \int_0^{\mathcal{Y}(x)} \left(\dot{\mathbf{V}} - \mathbf{V}\right)^2 \mathrm{d}\,y\, \mathrm{d}\,x$$

Proofs of unbiasedness and variances for the other estimators of the point methods are analogous.