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A Comparison of Dynamic and Static Economic Models of Uneven-Aged Stand Management

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ABSTRACT. Numerical techniques have been used to compute the discrete-time sequence of residual diameter distributions that maximize the present net worth (PNW) of harvestable volume from an uneven-aged stand. Results contradicted optimal steady-state diameter distributions determined with static analysis. In this paper, optimality conditions for solutions to dynamic and static harvesting problems are established. Comparison of these conditions shows that for a stand with any diameter distribution: (1) the optimal transition regime does not converge to the steady state that maximizes land expectation value (LEV) using the Faustmann equation; (2) the PNW of the optimal transition and steady-state regime is greater than the PNW of the statically determined steady-state regime; and (3) the optimal steady-state regime is invariant. A refined version of a recently published dynamic optimization algorithm is provided and demonstrated with a whole-stand/diameter-class simulator for hardwood stands in Wisconsin. Optimal management regimes are computed for comparison with a static equilibrium management regime and for analysis of the effect of cutting-cycle length on harvest pattern and PNW. FOREST SCI. 31:957–974.

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HARVESTING DECISIONS for an uneven-aged stand are dynamic because stand structure changes over time as a result of tree removals. Dynamic optimization has not been used to determine harvest schedules because of the large number of decision variables involved. Instead, studies in uneven-aged management have concentrated on the smaller problem of determining optimal equilibrium management regimes.

Equilibrium management regimes have been developed with static optimization, which maximizes the present value of equilibrium harvests minus the value of residual growing stock. Adams (1976) developed investment-efficient diameter distributions for a fixed cutting cycle. Buongiorno and Michie (1980) and Martin (1982) extended the analysis to consider alternative cutting cycles. Chang (1981) and Hall (1983) derived marginal production rules for optimal equilibrium growing-stock levels and cutting-cycle lengths. Chang (1981) demonstrated a method for solving these equations.

Solutions obtained with static analysis have been used to develop transition strategies and to evaluate alternative management systems. Adams and Ek (1974) demonstrated a method for determining optimal transition regimes that converge

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to steady-state diameter distributions in three periods or less. Chang (1981) and Hall (1983) showed that the objective function for determining the optimal equilibrium growing-stock level and cutting cycle is equivalent to maximizing land expectation value (LEV) with the Faustmann (1849) equation. Chang (1981) then used LEV as a measure for comparing the profitability of uneven-aged and even-aged management. Martin (1982) measured the profitability of uneven-aged management on different sites with LEV.

In a departure from static analysis, Haight and others (1985) used dynamic optimization to determine the optimal sequence of diameter distributions and selection harvests for an existing stand. They reported a 150-year management regime for a stand with a pre-harvest investment-efficient diameter distribution. This regime exhibited a pulselike transition away from the steady-state diameter distribution and had a higher present net worth (PNW) than the equilibrium management regime. These results called into question the use of static analysis for developing optimal steady-state stand structures and the use of LEV for evaluating the profitability of uneven-aged management.

In the first three parts of this paper, I resolve the dynamic and static optimization problems for the diameter distribution model of uneven-aged management. In parts 1 and 2, I formulate the problems and derive marginal production rules that are satisfied by optimal solutions. In part 3, I compare the marginal production rules and steady-state solutions to each problem and discuss how solutions to each problem should be used.

In part 4, I use the solution procedure given by Haight and others (1985) to develop an algorithm that provides stable solutions for transition and equilibrium management regimes. In the next two parts, I demonstrate the algorithm by computing optimal management regimes for comparison with a statically determined equilibrium regime and for analysis of the effect of increasing the cutting-cycle length. Part 7 provides a summary and discussion.

Problem 1: Dynamic Optimization

Consider the problem of determining the optimal sequence of selection harvests for an uneven-aged stand:

$$\underset{\{S_{ij}\}}{\text{Max}}\ \text{PNW} = \sum_{i=0}^{\infty} \sum_{j=1}^{M} \frac{(X_{ij} - S_{ij})P_j}{(1 + r)^{it}} \tag{1.1}$$

subject to, for all $i = 0, \ldots, \infty$ and for $j = 1, \ldots, M$,

$$X_{i+1j} = S_{ij} + f_{ij}(S_{i1}, \ldots, S_{iM}) \tag{1.2}$$

$$S_{ij} \geq 0 \tag{1.3}$$

$$X_{ij} - S_{ij} \geq 0 \tag{1.4}$$

$$X_{0j} \tag{1.5}$$

where

- X_{ij} = a state variable representing the number of trees per acre in diameter class j at the beginning of period i before cutting
- S_{ij} = a control variable representing the number of trees per acre in diameter class j at the beginning of period i after cutting
- P_j = the net price per tree in diameter class j
- f_{ij} = a continuous nonlinear function with continuous partial derivatives representing the change in number of trees per acre in diameter class j during period i
- M = the number of diameter classes

r = the discount rate
t = the number of years in a time period.

The objective function, equation (1.1), is formulated to seek the set $\{S_{ij}\}$ representing the residual number of trees in each diameter class and time period that maximizes the PNW of harvests over an infinite planning horizon. This formulation is equivalent to the PNW maximization problem presented and numerically solved by Haight and others (1985), in which the control variable set represented the percentage of the number of trees in each diameter class that were cut at the beginning of each time period. I use residual numbers of trees as control variables in the formulation for the dynamic problem presented here to facilitate comparison with the static optimization problem.

The motion equation set (1.2) represents the diameter-class growth dynamics. The change in number of trees per acre f_{ij} is the sum of functions for ingrowth from smaller diameter classes, upgrowth into larger diameter classes, and mortality. Each of these components is a continuous and differentiable nonlinear function of the residual diameter distribution $S_{i1}, \ldots, S_{iM}$. Uneven-aged stand simulators with this structure have been developed for Wisconsin hardwood stands (Adams and Ek 1974) and for Arizona ponderosa pine stands (Hann 1980).

Equation sets (1.3) and (1.4) require that the residual numbers of trees and harvest levels be non-negative. The initial stand structure is given by equation set (1.5).

Knapp (1983) derived Kuhn-Tucker conditions that are satisfied by optimal transition and steady-state solutions to a general dynamic optimization problem. I apply his analysis to the dynamic uneven-aged management problem given by equation sets (1.1) to (1.5) to obtain marginal production rules that are satisfied by optimal transition and steady-state management regimes.

Kuhn-Tucker Conditions.—I define an optimal value function $T_i^*(X_{i1}, \ldots, X_{iM})$ as the discounted value of the optimal sequence of selection harvests in periods i to ∞ when starting period i in state $X_{i1}, \ldots, X_{iM}$. Using the principle of optimality (Dreyfus and Law 1977, p. 101), T_i^* satisfies the functional equation:

$$T_i^*(X_{i1}, \ldots, X_{iM}) = \underset{\{S_{i1}, \ldots, S_{iM}\}}{\mathrm{Max}} \left[\sum_{j=1}^{M} \frac{(X_{ij} - S_{ij})P_j}{(1 + r)^{it}} + T_{i+1}^*(S_{i1} + f_{i1}, \ldots, S_{iM} + f_{iM}) \right] \quad (1.6)$$

subject to

$$S_{ij} \geq 0 \quad \text{and} \quad X_{ij} - S_{ij} \geq 0 \qquad j = 1, \ldots, M.$$

For $S_{i1}^*, \ldots, S_{iM}^*$ to solve the constrained maximization problem (1.6), I can define Lagrange multipliers $u_{i1}^*, \ldots, u_{iM}^*$ and $v_{i1}^*, \ldots, v_{iM}^*$ that satisfy the following Kuhn-Tucker conditions:

$$\frac{-P_j}{(1 + r)^{it}} + \frac{\partial T_{i+1}^*}{\partial X_{i+1j}} + \sum_{k=1}^{M} \frac{\partial T_{i+1}^*}{\partial X_{i+1k}} \frac{\partial f_{ik}}{\partial S_{ij}} + u_{ij}^* - v_{ij}^* = 0 \qquad j = 1, \ldots, M \quad (1.7a)$$

$$u_{ij}^* S_{ij}^* = 0 \quad \text{and} \quad v_{ij}^*(X_{ij} - S_{ij}^*) = 0 \qquad j = 1, \ldots, M \quad (1.7b)$$

$$S_{ij}^* \geq 0 \quad \text{and} \quad X_{ij} - S_{ij}^* \geq 0 \qquad j = 1, \ldots, M \quad (1.7c)$$

$$u_{ij}^* \geq 0 \quad \text{and} \quad v_{ij}^* \geq 0 \qquad j = 1, \ldots, M. \quad (1.7d)$$

The dependence of the optimal control variable values $S_{i1}^*, \ldots, S_{iM}^*$ on the values of the state variables $X_{i1}, \ldots, X_{iM}$ is recognized with the following equation set:

$$\frac{\partial T_i^*}{\partial X_{ij}} = \frac{P_j}{(1 + r)^{it}} + v_{ij}^* \qquad j = 1, \ldots, M. \tag{1.7e}$$

The optimal value function T_i^* and conditions (1.7a)–(1.7e) are defined for each time period $i = 0, \ldots, \infty$. An optimal management regime must satisfy this complete set of interperiodic conditions.

Marginal Production Rules.—To more easily derive and interpret the marginal production rules, I first define functional relationships for the Lagrange multipliers u_{ij}^* and v_{ij}^*. Examination of equation sets (1.7a), (1.7b), and (1.7c) yields the following conditions for the Lagrange multipliers:

$$u_{ij}^* = \begin{cases} 0 & \text{for} \quad S_{ij}^* > 0 \\ -\partial T_i^*/\partial S_{ij} & \text{for} \quad S_{ij}^* = 0 \end{cases} \qquad j = 1, \ldots, M \tag{1.8a}$$

$$v_{ij}^* = \begin{cases} 0 & \text{for} \quad X_{ij} - S_{ij}^* > 0 \\ \partial T_i^*/\partial S_{ij} & \text{for} \quad X_{ij} - S_{ij}^* = 0 \end{cases} \qquad j = 1, \ldots, M. \tag{1.8b}$$

Each Lagrange multiplier is a shadow price, which measures the change in the value of the objective function corresponding to a small change in the constraint. When $S_{ij}^* = 0$, the multiplier u_{ij}^* is the increase in benefits that would result from having one less unit of growing stock in diameter class j in period i. Similarly, when $X_{ij} - S_{ij}^* = 0$, the multiplier v_{ij}^* is the increase in benefits due to an increase in the residual number of trees in diameter class j period i. When a constraint is nonbinding in the optimal solution, the corresponding shadow price is zero.

Equation (1.7e) gives the value of an additional tree in diameter class j at the beginning of period i, $\partial T_i^*/\partial X_{ij}$. When $X_{ij} - S_{ij}^* > 0$, the marginal tree is cut with value $P_j/(1 + r)^{it}$. When $X_{ij} - S_{ij}^* = 0$, the marginal value includes the shadow price for an additional unit of growing stock in diameter class j at the beginning of period i.

With these definitions, condition (1.7a) can be rearranged to obtain marginal production rules that are defined over the set of diameter classes in period i:

$$\frac{P_j}{(1 + r)^{it}} - u_{ij}^* + v_{ij}^* = \sum_{k=1}^{M} \left[\frac{P_k}{(1 + r)^{(i+1)t}} + v_{i+1k}^* \right] \frac{\partial f_{ik}}{\partial S_{ij}} + \frac{P_j}{(1 + r)^{(i+1)t}} + v_{i+1j}^* \qquad j = 1, \ldots, M. \tag{1.9}$$

The left-hand side of equation (1.9) represents the opportunity cost of leaving an additional tree in diameter class j. When $0 < S_{ij}^* < X_{ij}$, the marginal input cost equals the discounted tree price. When $S_{ij}^* - X_{ij} = 0$, the marginal cost equals the sum of the discounted tree price and the shadow price (discounted) of an additional tree. When $S_{ij}^* = 0$, the marginal cost equals the discounted sum of the tree price and a price reduction due to an additional tree. The price reduction means that we would prefer to harvest more trees to improve the objective function value.

The right-hand side of equation (1.9) gives the marginal revenue product of leaving an additional tree in diameter class j. This equals the discounted value growth resulting from the additional tree summed over all diameter classes plus the discounted value of the tree. The value of the growth in period $i + 1$ is the tree price if $X_{i+1k} - S_{i+1k}^* > 0$. If $X_{i+1k} - S_{i+1k}^* = 0$, the value of the marginal growth includes v_{i+1k}^*, the shadow price for an additional tree in diameter class k.

The rule for resource production states that the residual number of trees in diameter class j period i is increased until the marginal input cost of one more tree equals the marginal revenue product associated with leaving the tree. An optimal management regime for an existing stand has satisfied this rule simultaneously over all diameter classes and periods in the planning horizon. The marginal production rules and definitions of the shadow prices are summarized in the Appendix.

Problem 2: Static Optimization

To derive the static optimization problem, I modify the original dynamic formulation given by equations (1.1) to (1.5) by including an equilibrium constraint:

$$\operatorname*{Max}_{\{S_{ij}\}} \text{PNW} = \sum_{i=0}^{\infty} \sum_{j=1}^{M} \frac{(X_{ij} - S_{ij})P_j}{(1 + r)^{it}} \tag{2.1}$$

subject to, for all $i = 0, \ldots, \infty$ and for $j = 1, \ldots, M$,

$$X_{i+1j} = S_{ij} + f_{ij}(S_{i1}, \ldots, S_{iM}) \tag{2.2}$$

$$S_{ij} \geq 0 \tag{2.3}$$

$$X_{ij} - S_{ij} \geq 0 \tag{2.4}$$

$$X_{0j} \tag{2.5}$$

and

$$X_{i+1j} = X_{ij} \text{ for all } i = 1, \ldots, \infty \quad \text{and} \quad j = 1, \ldots, M. \tag{2.6}$$

Constraint (2.6) requires that an equilibrium harvest regime be achieved after one harvest from an existing stand. Using this constraint, I can rearrange the objective function (2.1) to obtain

$$\operatorname*{Max}_{\{S_{0j}\}} \text{PNW} = \sum_{j=1}^{M} (X_{0j} - S_{0j})P_j + \sum_{j=1}^{M} \frac{(X_{1j} - S_{0j})P_j}{(1 + r)^t - 1}. \tag{2.7}$$

Substituting the growth dynamics (2.2) into (2.7) results in the static optimization problem:

$$\operatorname*{Max}_{\{S_{0j}\}} \text{PNW} = \sum_{j=1}^{M} (X_{0j} - S_{0j})P_j + \sum_{j=1}^{M} \frac{f_{0j}(S_{01}, \ldots, S_{0M})P_j}{(1 + r)^t - 1} \tag{2.8}$$

subject to, for $j = 1, \ldots, M$,

$$S_{0j}(1 + r)^t/[(1 + r)^t - 1] \geq 0 \tag{2.9}$$

$$f_{0j}(S_{01}, \ldots, S_{0M})/[(1 + r)^t - 1] \geq 0 \tag{2.10}$$

$$X_{0j} - S_{0j} \geq 0 \tag{2.11}$$

$$X_{0j}. \tag{2.12}$$

The objective function (2.8) seeks a steady-state diameter distribution $S_{01}, \ldots, S_{0M}$ that maximizes the value of the initial harvest plus the present value of equilibrium harvests in perpetuity. This is equivalent to the objective function for forest value given by Chang (1981), except that we consider diameter-class growth dynamics.

In contrast to Chang's (1981) formulation, three sets of constraints are needed to determine the optimal steady-state management regime. Constraint set (2.9) requires a non-negative residual stocking level and constraint sets (2.10) and (2.11) require non-negative harvest levels. Constraint sets (2.9) and (2.10) are multiplied by the constants $(1 + r)^t/[(1 + r)^t - 1]$ and $1/[(1 + r)^t - 1]$ to simplify the derivation and interpretation of marginal production rules. The initial diameter distribution is given in equation set (2.12).

When the initial harvest constraints (equation set (2.11)) are not binding and the value of the initial stand structure is ignored, the problem is equivalent to maximizing LEV. Solutions are investment-efficient diameter distributions that have been developed by Adams (1976), Buongiorno and Michie (1980), and Martin (1982).

Kuhn-Tucker Conditions.—For $S_{01}^*, \ldots, S_{0M}^*$ to solve the constrained maximization problem (2.8) to (2.12), I can define Lagrange multipliers $a_1^*, \ldots, a_M^*$, $b_1^*, \ldots, b_M^*$, and $c_1^*, \ldots, c_M^*$ that satisfy the following Kuhn-Tucker conditions:

$$-P_j + \sum_{k=1}^{M} \frac{(P_k + b_k^*)}{(1 + r)^t - 1} \frac{\partial f_{0k}}{\partial S_{0j}} + \frac{a_j^*(1 + r)^t}{(1 + r)^t - 1} - c_j^* = 0 \qquad j = 1, \ldots, M \quad (2.13a)$$

$$a_j^* S_{0j}^* = 0, \quad b_j^* f_{0j} = 0, \quad \text{and} \quad c_j^*(X_{0j} - S_{0j}^*) = 0 \qquad j = 1, \ldots, M \quad (2.13b)$$

$$S_{0j}^* \geq 0, \quad f_{0j} \geq 0, \quad \text{and} \quad X_{0j} - S_{0j}^* \geq 0 \qquad j = 1, \ldots, M \quad (2.13c)$$

$$a_j^* \geq 0, \quad b_j^* \geq 0, \quad \text{and} \quad c_j^* \geq 0 \qquad j = 1, \ldots, M. \quad (2.13d)$$

Marginal Production Rules.—The Lagrange multipliers a_j^*, b_j^*, and c_j^* are shadow prices, which measure the change in value of the objective function corresponding to a change in the associated constraint. When $S_{0j}^* = 0$, $a_j^*(1 + r)^t/[(1 + r)^t - 1]$ is the increase in PNW resulting from a decrease in residual stocking in diameter class j. When $f_{0j} = 0$, $b_j^*/[(1 + r)^t - 1]$ is the increase in PNW due to an increase in the number of trees in diameter class j in the equilibrium stand structure. Finally, when $X_{0j} - S_{0j} = 0$, c_j^* is the increase in PNW due to an increase in residual stocking in diameter class j. The shadow prices a_j^* and b_j^* must be multiplied by the scaling factors used in constraints (2.9) and (2.10) to obtain these definitions. Whenever a constraint is nonbinding in the optimal solution, the corresponding shadow price is zero.

Rearranging (2.13a) gives the marginal production rules for an optimal equilibrium diameter distribution:

$$P_j - a_j^* + c_j^* = \sum_{k=1}^{M} \frac{(P_k + b_k^*)}{(1 + r)^t} \frac{\partial f_{0k}}{\partial S_{0j}} + \frac{P_j}{(1 + r)^t} + \frac{c_j^*}{(1 + r)^t} \qquad j = 1, \ldots, M. \quad (2.14)$$

The marginal input cost of leaving an additional tree in diameter class j is the tree price. Marginal input cost is reduced by a_j^* if $S_{0j}^* = 0$ and is increased by c_j^* if $X_{0j} - S_{0j}^* = 0$. Marginal revenue product equals the discounted value growth that results from the additional tree in diameter class j plus the discounted value of the tree. Prices of the growing stock are increased or reduced according to whether the growing stock and harvest constraints are binding.

The marginal production rule stated by equation set (2.14) is to increase the number of trees in diameter class j until the marginal input cost of one more tree equals the marginal revenue product associated with leaving the tree. An optimal steady-state diameter distribution satisfies this rule simultaneously over all diameter classes. Production rules and shadow price definitions are summarized in the Appendix.

PROBLEM COMPARISON

Both the dynamic and static optimization problems seek to maximize PNW, but the static problem includes an equilibrium harvest constraint. For any initial stand, problem 1 seeks the optimal sequence of diameter distributions and selection harvests over an extended planning horizon, while problem 2 seeks the optimal equilibrium diameter distribution that can be established with one harvest from the stand. Although problems 1 and 2 have different constraint sets, the set of feasible solutions to problem 2 is a subset of the feasible solutions to problem 1.

Because the problems have different constraint sets, they have differences in marginal production rules. Problem 1 defines interperiodic production rules for both transition and equilibrium management regimes, while problem 2 defines production rules for equilibrium management regimes only. Thus, we can only compare production rules for steady-state stand structures. The conditions for the equilibrium stand structures are given in the Appendix. I make two comparisons. The first comparison is made with the assumption that the initial diameter distribution is arbitrarily large. In the second comparison this assumption is dropped.

Comparison 1.—Will an investment-efficient diameter distribution satisfy the marginal production rules for a dynamic equilibrium stand structure? An investment-efficient diameter distribution is found by solving problem 2 with an arbitrarily large initial stand structure. Thus, $c_j^* = 0$ for $j = 1, \ldots, M$. In this case, the marginal input cost and marginal revenue product of the conditions for the static equilibrium do not contain Lagrange multipliers for binding non-negative harvest constraints. As a result, whenever an equilibrium harvest regime found by solving the static optimization problem contains no harvest in diameter class k, so that $b_k^* = 0$, the management regime will not satisfy the conditions for a dynamic equilibrium. The converse of this is also true.

Suppose an investment-efficient diameter distribution contains positive harvest levels in all diameter classes so that $b_k^* = 0$ for $k = 1, \ldots, M$, and all trees growing into the largest diameter classes are cut so that $a_M^* > 0$. In this case, the rules for static and dynamic equilibrium production are equivalent. The converse of this is also true.

How often will investment-efficient diameter distributions contain positive harvests in all diameter classes? Getz (1980) investigated the structure of maximum sustained-yield harvest policies for age-structured populations with linear growth dynamics and arbitrary stock-recruitment functions. He showed that, at most, two age classes are harvested, the younger partially and the older completely. The remaining age classes that contain individuals are not harvested. The structure of the Adams and Ek (1974) simulator is similar to the age-class population model analyzed by Getz (1980), except that movement of trees to the next larger diameter class is predicted with a nonlinear function. Nevertheless, the dynamic equilibria obtained by numerically solving problem 1 with the Adams and Ek (1974) simulator exhibit bimodal harvest policies. If optimal equilibrium harvest policies for diameter-class simulators with nonlinear growth dynamics are always bimodal, dynamic and static equilibrium regimes will be equivalent only when the residual diameter distributions contain trees in the smallest diameter class.

These results restrict the use of statically determined steady-state diameter distributions. Adams (1976), Buongiorno and Michie (1980), and Martin (1982) solve problem 2 with an assumption that the first-period harvest constraints are not binding. They call their solutions investment-efficient diameter distributions; these are used as goals in transition management. Because investment-efficient diameter distributions do not satisfy marginal production rules for dynamic equilibrium stand structures, they should not be used as goals for long-term management. They should be used only when the objective is to convert the stand to an equilibrium structure in one harvest.

Chang (1981) used static analysis to determine the optimal steady-state growing stock and cutting cycle without considering diameter-class growth dynamics. The management objective was to maximize forest value, which he defined as the PNW of harvests when converting an existing stand to an equilibrium growing-stock level in one cut. This objective is equivalent to maximizing LEV with the Faustmann equation where the opportunity cost on the residual growing stock represents the regeneration costs of the stand. As a result, Chang (1981) concluded that the forest value model is the only correct method of determining equilibrium growing stock and cutting cycle.

The static optimization problem that I present is equivalent to Chang's (1981) forest value model, except that it seeks the optimal equilibrium stand structure and includes constraints on the diameter-class growth dynamics and bounds on the residual diameter distribution. Investment-efficient diameter distributions that solve problem 2 can also be viewed as maximizing LEV with the Faustmann equation. However, investment-efficient management regimes do not maximize the present value of returns to the initial growing stock. Thus, maximizing LEV with the Faustmann equation is not the correct criterion for determining optimal steady-state stand structures.

Why does the dynamically determined harvest regime converge to a steady state that does not maximize LEV? An intuitive explanation can be made by viewing an optimal management regime as having transition and equilibrium phases. Since the equilibrium phase takes place later in the planning horizon, its value is discounted more than the value of the transition phase. As a result, it is possible to convert a stand to an equilibrium phase that has less than maximum LEV with a transition regime. The value of the transition regime would more than offset the discounted losses during the equilibrium phase.

Comparison 2.—Here I make no assumptions about the form of the initial stand structure. Does the initial stand structure affect the form of the equilibrium found by solving problems 1 or 2? The rules for dynamic equilibrium production do not contain any conditions regarding the initial stand structure. Thus, under the assumption that the stumpage price function and interest rate are constant over time, optimal transition strategies for different stands will converge to the same equilibrium.

In contrast, the equilibrium found by solving the static optimization problem does depend on the initial stand structure, since the objective is to maximize PNW of harvests when converting to an equilibrium structure in one cut.

For a given stumpage value function and interest rate, is there an initial stand structure that results in the same equilibrium solution to both problems 1 and 2? Suppose the initial stand has the pre-harvest dynamic equilibrium structure. Equating the Lagrange multipliers for the rules of static and dynamic production shows that the optimal solution to problem 2 is the dynamic equilibrium harvest regime. For any other initial stand structure, the management regime found by solving problem 2 will not be the same as the optimal transition and equilibrium regime found by solving problem 1.

This result changes the method for measuring the profitability of uneven-aged management. Both dynamic and static optimization problems can be used to develop management regimes for an existing stand. The steady-state regime found by solving problem 2 will not, except in special cases, be the optimal solution to problem 1. Therefore, the optimal transition and steady-state management regime will have a PNW greater than the PNW of the optimal steady-state management

TABLE 1. Initial diameter distributions and stumpage values for a northern hardwood stand (site index 60).

Diameter class midpoint (inches)	Initial diameter distribution		Stumpage value	
	Regime 1	Regime 2		
	Trees/acre		*$/ft³*	*$/tree*
6	69.9	15.0	0.01	0.04
8	49.3	11.5	.01	.10
10	37.5	8.5	.01	.18
12	29.7	6.5	.01	.26
14	24.1	5.5	.025	.97
16	10.7	4.5	.05	2.80
18	2.1	1.5	.08	5.04
20	.3	.0	.10	7.90

regime that solves problem 2. This makes economic sense, since the dynamic formulation is not constrained to achieve a steady state in any future time period. Thus, the profitability of uneven-aged management should be measured by the PNW of the optimal transition and equilibrium management regime found by solving problem 1.

Solution Procedure for Dynamic Optimization

In the analysis presented above, the dynamic optimization formulation and the marginal production rules for optimal transition and equilibrium harvesting were used to point out the limitations of investment-efficient stand structures that solve the traditional static optimization problem. In this section I develop an algorithm for solving the dynamic harvesting problem that improves the unconstrained nonlinear programming approach used by Haight and others (1985).

Haight and others (1985) used a gradient-based procedure called *the method of steepest descent.* The algorithm starts with an initial guess of the control variable values and seeks to improve the objective function value by successive approximations of the control variables. Each approximation is found by moving in the direction of the gradient, and the process terminates when improvements are less than a set tolerance. The solution obtained at termination is an approximation to a stationary point that satisfies Kuhn-Tucker conditions for optimality.

Haight and others (1985) developed optimal management regimes with 5-year cutting cycles over a 150-year time horizon, with the assumption that all remaining trees were harvested at the end of the time horizon. For time horizons 150 years or longer and a 4 percent discount rate, the discounted value of the terminal diameter distribution was so small that it had no effect on the determination of the control variable values in earlier periods.

The discount factors for distant-period harvests did affect the ability of the gradient method to reach stationary solutions. Because the marginal values of distant-period control variables were much smaller than the termination criterion, the algorithm terminated before reaching a stationary point. As a result, final solutions for distant-period control variables were sensitive to values initially given to the control variables. Nevertheless, the gradient method produced stable solutions for the first several periods.

In order to obtain stationary solutions for transition and equilibrium management regimes, I have developed an algorithm that takes advantage of the stability of the first-period solution and the sequential nature of the problem. The method

TABLE 2. Management regime 1 (PNW = $171.29/acre).

Year	Diameter class midpoint (inches) 6	8	10	12	14	16	18	20	Totals Trees	Ft²	$
	Trees per acre cut										
0	69.8	37.6	0.0	0.0	0.0	0.5	0.6	0.3	108.9	29.3	13.7
5	4.1	.0	.0	.0	.0	2.8	3.5	.9	11.4	13.0	33.2
10	3.4	.0	.0	.0	.0	2.9	3.9	.9	11.2	13.7	35.5
15	1.2	.0	.0	.0	.0	1.7	4.1	.9	8.0	12.0	33.4
20	.0	.0	.0	.0	.0	.6	4.8	1.1	6.6	11.9	35.1
25	.0	.0	.0	.0	.0	.0	5.3	1.1	6.4	11.9	35.9
30	.0	.0	.0	.0	.0	.0	5.9	1.3	7.2	13.4	40.5
35	.0	.0	.0	.0	.0	.0	5.5	1.3	6.8	12.7	38.5
40	.0	.0	.0	.0	.0	.0	5.1	1.5	6.7	12.5	38.3
45	.0	.0	.0	.0	.0	.0	4.0	1.7	5.7	10.9	33.9
50	.0	.0	.0	.0	.0	.0	3.5	2.0	5.6	10.7	33.9
55	.0	.0	.0	.0	.0	.0	2.8	2.0	4.8	9.4	30.1
60	8.7	.0	.0	.0	.0	.0	2.2	2.0	13.0	10.1	27.7
65	10.2	.0	.0	.0	.0	.0	1.8	2.0	14.1	9.7	25.9
70	10.0	.0	.0	.0	.0	.0	1.7	2.0	13.8	9.5	25.3
75	8.7	.0	.0	.0	.0	.0	1.7	1.9	12.4	9.0	24.6
80	8.6	.0	.0	.0	.0	.0	1.8	1.9	12.3	9.1	24.7
85	7.9	.0	.0	.0	.0	.0	2.0	1.8	11.8	9.1	25.1
90	6.5	.0	.0	.0	.0	.0	2.2	1.8	10.6	9.2	25.9
95	5.2	.0	.0	.0	.0	.0	2.5	1.7	9.5	9.3	26.9
100	4.6	.0	.0	.0	.0	.0	2.7	1.7	9.0	9.5	27.7
105	4.1	.0	.0	.0	.0	.0	2.9	1.7	8.7	9.7	28.5
110	4.1	.0	.0	.0	.0	.0	3.0	1.7	8.9	9.9	29.0
115	4.3	.0	.0	.0	.0	.0	3.0	1.7	9.1	10.0	29.4
120	3.5	.0	.0	.0	.0	.0	3.0	1.7	8.3	9.9	29.4
125	3.6	.0	.0	.0	.0	.0	2.9	1.7	8.3	9.7	29.0
130	3.8	.0	.0	.0	.0	.0	3.0	1.8	8.7	10.0	29.7
135	5.4	.0	.0	.0	.0	.0	2.8	1.7	10.0	10.0	28.6
140	5.5	.0	.0	.0	.0	.0	2.7	1.8	10.0	9.9	28.4
145	6.7	.0	.0	.0	.0	.0	2.6	1.8	11.2	10.0	28.1
150	4.8	.0	.0	.0	.0	.0	2.6	1.8	9.3	9.6	27.9
	Residual trees per acre										
0	0.0	11.6	37.5	29.7	24.1	10.1	1.5	.0	114.7	90.5	75.1
5	5.7	7.5	28.2	29.7	24.1	11.4	1.5	.0	108.4	87.0	76.9
10	10.0	6.4	20.6	27.2	24.2	12.1	1.6	.0	102.4	82.6	77.6
15	15.1	6.8	15.1	23.3	23.5	13.8	1.8	.0	99.7	79.7	81.1
20	20.0	8.3	11.5	19.2	21.8	15.6	1.9	.0	98.6	76.9	83.5
25	23.9	10.4	9.7	15.4	19.4	16.7	2.2	.0	98.1	74.0	85.0
30	27.1	12.8	9.1	12.4	16.7	16.6	2.2	.0	97.3	69.7	81.4
35	30.2	15.2	9.5	10.4	14.0	15.5	2.6	.0	97.7	66.1	77.8
40	33.3	17.5	10.4	9.3	11.7	13.9	2.9	.0	99.2	62.8	72.4
45	36.7	19.8	11.6	9.0	10.0	12.1	3.4	.0	102.8	61.3	69.0
50	40.3	22.1	13.1	9.2	8.9	10.4	3.4	.0	107.6	60.2	63.8
55	44.2	24.5	14.6	9.8	8.3	9.0	3.5	.0	114.2	60.8	60.6
60	39.6	27.1	16.3	10.6	8.2	8.1	3.5	.0	113.6	61.0	58.7
65	34.4	27.8	18.0	11.7	8.5	7.6	3.5	.0	111.7	61.5	57.9
70	30.3	27.1	19.3	12.8	9.1	7.4	3.4	.0	109.6	62.2	57.6
75	27.9	25.6	19.9	13.9	9.8	7.5	3.3	.0	108.3	63.2	58.3
80	25.8	24.1	20.0	14.8	10.6	7.9	3.2	.0	106.5	64.1	59.5
85	24.3	22.5	19.6	15.3	11.3	8.4	3.1	.0	104.7	64.8	61.1
90	24.0	21.1	18.9	15.5	11.9	8.9	3.0	.0	103.6	65.3	62.6

TABLE 2. Continued.

Year	Diameter class midpoint (inches)								Totals		
	6	8	10	12	14	16	18	20	Trees	Ft²	$
95	24.9	20.1	18.0	15.4	12.3	9.5	2.9	.0	103.3	65.6	63.9
100	25.9	19.6	17.2	15.1	12.5	9.9	2.9	.0	103.4	65.7	64.9
105	27.2	19.5	16.5	14.6	12.5	10.2	2.9	.0	103.7	65.6	65.6
110	28.1	19.7	16.1	14.1	12.4	10.4	2.9	.0	103.9	65.4	65.8
115	28.7	20.1	15.8	13.7	12.1	10.4	3.0	.0	104.1	64.9	65.7
120	30.1	20.5	15.8	13.3	11.8	10.3	3.0	.0	105.1	64.7	65.4
125	31.3	21.1	15.9	13.1	11.5	10.2	3.1	.0	106.4	64.6	65.2
130	32.2	21.8	16.1	13.0	11.2	10.0	3.0	.0	107.5	64.4	64.1
135	31.5	22.4	16.4	13.0	11.1	9.7	3.1	.0	107.6	64.3	63.7
140	31.1	22.7	16.8	13.1	11.0	9.6	3.1	.0	107.6	64.3	63.3
145	29.5	22.8	17.2	13.3	10.9	9.4	3.1	.0	106.5	64.1	63.0
150	30.1	22.5	17.4	13.6	11.0	9.4	3.1	.0	107.3	64.4	62.8

uses repeated application of the gradient method described by Haight and others (1985).

In equation (1.6) I defined the optimal value function $T_i^*(X_{i1}, \ldots, X_{iM})$ as the discounted value of the optimal sequence of selection harvests in periods i to ∞ when starting period i in state $X_{i1}, \ldots, X_{iM}$. Using equation (1.6), the optimal value function for period 0 is as follows:

$$T_0^* = \underset{\{S_{01}, \ldots, S_{0M}\}}{\mathrm{Max}} \left[\sum_{j=1}^{M} (X_{0j} - S_{0j})P_j + T_1^*(S_{01} + f_{01}, \ldots, S_{0M} + f_{0M}) \right] \quad (4.1)$$

subject to, for $j = 1, \ldots, M,$

$$S_{0j} \geq 0$$
$$X_{0j} - S_{0j} \geq 0$$
$$X_{0j}.$$

The gradient method can be used to find a stationary solution $S_{01}^*, \ldots, S_{0M}^*$ to the constrained maximization problem defined by (4.1) for a finite-period time horizon (N) 150 years or longer. Using $S_{01}^*, \ldots, S_{0M}^*$, we can write the optimal value function for period 1:

$$T_1^* = \underset{\{S_{11}, \ldots, S_{1M}\}}{\mathrm{Max}} \left[\sum_{j=1}^{M} \frac{(X_{1j} - S_{1j})P_j}{(1 + r)^t} + T_2^*(S_{11} + f_{11}, \ldots, S_{1M} + f_{1M}) \right] \quad (4.2)$$

subject to, for $j = 1, \ldots, M,$

$$S_{1j} \geq 0$$
$$X_{1j} - S_{1j} \geq 0$$
$$X_{1j} = S_{0j}^* + f_{0j}(S_{01}^*, \ldots, S_{0M}^*).$$

A stationary solution $S_{11}^*, \ldots, S_{1M}^*$ to the constrained optimization problem defined by (4.2) can be obtained by multiplying each side of the objective function by the constant $(1 + r)^t$ and applying the gradient method for an N-period time horizon. Multiplication of the objective function by $(1 + r)^t$ shifts the discount factors back one period and allows the use of the same termination criterion. Repeating this process for a predetermined number of periods (L) gives a stationary solution for the L-period management regime.

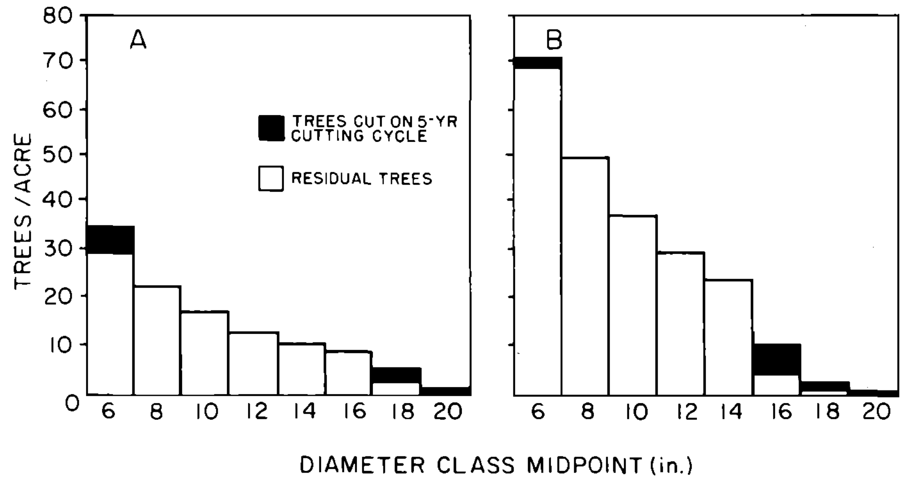

FIGURE 1. Residual diameter distributions for equilibrium harvest determined with (*a*) dynamic and (*b*) static optimization. Equilibrium structures were determined with the same initial stand structure, stumpage value, and interest rate.

The algorithm is summarized as follows:

1. Set the number of periods in the planning horizon equal to L and the current period (K) equal to 0. To operate the gradient method, specify the number of periods in the time horizon (N) and the termination criterion. To start the gradient method in period 0, specify the initial diameter distribution and the starting values for the control variables.
2. Solve the N-period control problem using the gradient method. Save the control variable values for period 0 and the vector of state variables for period 1 from this solution.
3. If $K < L$, replace K by $K + 1$ and set the initial diameter distribution equal to the vector of state variables that were saved in step 2. Reset the starting values for the control variables and go to step 2. If $K = L$, stop. The optimal control sequence for an L-period management regime has been determined.

I demonstrate this algorithm with a whole-stand/diameter-class simulator for mixed hardwood stands in Wisconsin. The simulator has been incorporated into static optimization algorithms (Adams and Ek 1974, Adams 1976, Martin 1982) and a dynamic optimization program (Haight and others 1985). The simulator includes nonlinear functions f_{ij} for the change in number of trees per acre in a 2-inch diameter class j during a 5-year growth period i. Growth function arguments are the residual stand structure at the start of the growth period. The smallest diameter class is 6 inches.

The algorithm applies the gradient method in each period of an L-period time horizon. This method requires (1) predetermined values for the number of periods (N) over which the gradient method will be applied and (2) the termination criterion, which is the minimum acceptable improvement in the objective function value used to obtain the next approximation of the control variable values.

After sensitivity analysis, I set the time horizon equal to 31 5-year periods and the termination criterion equal to \$0.01/acre PNW. With these parameters, the algorithm produced stable solutions for transition and equilibrium management regimes. Management regimes developed with random values given to control variables varied by less than 1 tree per acre per diameter class in residual diameter

TABLE 3. Dynamically and statically determined steady-state residual diameter distributions and harvests (5-year cutting cycle).

Diameter class midpoint (inches)	Dynamic equilibrium diameter distribution		Investment-efficient diameter distribution	
	Residual	Cut	Residual	Cut
	Trees/acre			
6	29.7	5.2	68.5	1.4
8	22.5	.0	49.3	.0
10	17.4	.0	37.5	.0
12	13.6	.0	29.7	.0
14	11.0	.0	24.1	.0
16	9.4	.0	4.2	6.5
18	3.1	2.6	.5	1.6
20	.0	1.8	.0	.3
Value of residual ($/acre)	62.8		60.0	
Value of harvest ($/acre)	27.9		29.1	
Land expectation value, LEV ($/acre)	66.0		74.3	

distributions over a 31-period time horizon. Increasing the number of periods (N) and reducing the termination criterion did not change the solutions but increased execution times. Execution times for the examples in the next sections varied between 1 and 3 hours on an IBM PC-XT microcomputer.

Comparison of Dynamic and Static Harvest Regimes

Adams (1976) used static optimization to develop investment-efficient diameter distributions for hardwood stands (site index 60) managed with 5-year cutting cycles. He used a stumpage value function that assigned a constant price per cubic foot for trees between 6 and 12 inches in diameter at breast height and a rapidly increasing price for larger trees. Each distribution was constrained to a predetermined value of residual growing stock. Because of limits on the data base from which the simulator was constructed, the maximum diameter of residual trees was 18 inches.

For comparison, I use the dynamic optimization algorithm presented here to develop two management regimes for a hardwood stand, site index 60. The regimes are developed to maximize the PNW of selection harvests taken on a 5-year cutting cycle over a 150-year time horizon. The value of the terminal diameter distribution is not included in the objective function. I use the same stumpage value function (Table 1) and a 4 percent real interest rate.

The initial diameter distribution for management regime 1 is the pre-harvest investment-efficient diameter distribution, which has a marginal value growth of 4 percent (Adams 1976). Management regime 2 is developed for an initial diameter distribution with fewer trees in each diameter class.

Management regime 1 exhibits a pulse-transition harvest pattern during the first 55 years (Table 2). All of the 6-inch and 76 percent of the 8-inch trees are cut at the start. In the remaining periods, trees are harvested primarily from the two largest and most valuable diameter classes. Except for the first period, harvest

TABLE 4. *Management regime 2 (PNW = $82.03/acre).*

Year	Diameter class midpoint (inches) 6	8	10	12	14	16	18	20	Totals Trees	Ft²	$
	Trees per acre cut										
0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
5	.0	.0	.0	.0	.0	.0	.0	1.0	1.0	2.2	8.2
10	.0	.0	.0	.0	.0	.0	.0	1.9	1.9	4.1	15.0
15	.0	.0	.0	.0	.0	.0	.0	2.2	2.2	4.8	17.5
20	.0	.0	.0	.0	.0	.0	.0	2.3	2.3	5.0	18.4
25	.0	.0	.0	.0	.0	.0	.0	2.3	2.3	5.1	18.7
30	2.1	.0	.0	.0	.0	.0	.1	2.4	4.6	5.8	19.7
35	16.5	.0	.0	.0	.0	.0	.4	2.3	19.2	9.0	21.2
40	16.8	.0	.0	.0	.0	.0	.6	2.2	19.6	9.1	21.1
45	15.7	.0	.0	.0	.0	.0	.8	2.1	18.6	9.2	21.5
50	9.0	.0	.0	.0	.0	.0	1.2	2.0	12.2	8.3	22.3
55	8.6	.0	.0	.0	.0	.0	1.6	1.8	12.1	8.6	23.4
60	6.6	.0	.0	.0	.0	.0	2.0	1.7	10.5	8.8	24.6
65	5.4	.0	.0	.0	.0	.0	2.4	1.7	9.6	9.1	26.1
70	3.6	.0	.0	.0	.0	.0	2.8	1.6	8.1	9.3	27.5
75	4.3	.0	.0	.0	.0	.0	3.0	1.6	9.0	9.8	28.6
80	2.7	.0	.0	.0	.0	.0	3.2	1.6	7.7	10.0	29.8
85	3.0	.0	.0	.0	.0	.0	3.3	1.6	8.1	10.1	30.2
90	3.8	.0	.0	.0	.0	.0	3.3	1.7	8.8	10.3	30.4
95	2.4	.0	.0	.0	.0	.0	3.2	1.7	7.4	10.0	30.2
100	4.2	.0	.0	.0	.0	.0	3.1	1.7	9.1	10.1	29.8
105	5.4	.0	.0	.0	.0	.0	2.9	1.8	10.2	10.2	29.2
110	5.2	.0	.0	.0	.0	.0	2.7	1.8	9.9	9.9	28.6
115	5.7	.0	.0	.0	.0	.0	2.6	1.8	10.2	9.8	28.2
120	5.5	.0	.0	.0	.0	.0	2.6	1.8	9.9	9.6	27.8
125	6.4	.0	.0	.0	.0	.0	2.5	1.8	10.7	9.7	27.5
130	6.8	.0	.0	.0	.0	.0	2.5	1.8	11.1	9.8	27.4
135	5.4	.0	.0	.0	.0	.0	2.5	1.8	9.8	9.5	27.4
140	5.8	.0	.0	.0	.0	.0	2.6	1.7	10.2	9.6	27.6
145	5.5	.0	.0	.0	.0	.0	2.6	1.7	10.0	9.7	27.7
150	5.4	.0	.0	.0	.0	.0	2.7	1.7	9.9	9.8	28.0
	Residual trees per acre										
0	15.0	11.5	8.5	6.5	5.5	4.5	1.5	.0	53.0	31.5	30.4
5	24.3	11.4	8.7	6.7	5.4	4.6	3.0	.0	64.4	36.4	38.8
10	31.9	13.8	8.9	6.9	5.5	4.6	3.6	.0	75.5	40.3	42.7
15	38.7	17.3	9.7	7.1	5.7	4.7	3.8	.0	87.3	44.1	45.2
20	45.1	21.1	11.3	7.5	5.8	4.8	4.0	.0	99.8	48.4	47.3
25	50.9	25.1	13.3	8.2	6.0	4.9	4.1	.0	112.8	53.2	49.5
30	54.0	29.0	15.7	9.3	6.4	5.0	4.0	.0	123.8	57.8	51.2
35	42.4	32.4	18.2	10.7	7.1	5.3	3.8	.0	120.0	59.7	51.9
40	31.8	32.0	20.7	12.3	7.9	5.6	3.6	.0	114.2	61.3	53.3
45	23.3	29.4	22.2	14.1	9.0	6.2	3.4	.0	107.8	62.4	55.0
50	22.2	25.6	22.4	15.6	10.2	6.9	3.2	.0	106.3	64.1	57.2
55	21.0	22.7	21.6	16.6	11.4	7.7	3.0	.0	104.2	65.3	59.6
60	21.3	20.4	20.2	16.9	12.4	8.6	2.9	.0	103.0	66.2	62.1
65	22.3	19.0	18.7	16.7	13.1	9.5	2.8	.0	102.3	66.7	64.3
70	24.6	18.2	17.3	16.1	13.4	10.2	2.8	.0	102.8	66.9	66.0
75	25.6	18.2	16.2	15.3	13.4	10.7	2.8	.0	102.5	66.7	67.2
80	27.9	18.5	15.5	14.4	13.1	11.0	2.8	.0	103.5	66.2	67.5
85	29.6	19.2	15.2	13.7	12.6	11.0	2.9	.0	104.4	65.7	67.3
90	30.3	20.1	15.1	13.1	12.1	10.8	2.9	.0	104.8	65.0	66.6

TABLE 4. Continued.

Year	Diameter class midpoint (inches) 6	8	10	12	14	16	18	20	Totals Trees	Ft²	$
95	32.5	20.8	15.3	12.8	11.6	10.6	3.0	.0	106.8	64.7	65.7
100	32.8	21.9	15.7	12.6	11.2	10.2	3.0	.0	107.6	64.3	64.7
105	31.9	22.6	16.2	12.7	10.9	9.9	3.1	.0	107.5	64.0	63.8
110	31.6	23.0	16.7	12.8	10.7	9.6	3.1	.0	107.8	64.0	63.1
115	31.0	23.1	17.2	13.1	10.7	9.3	3.1	.0	107.8	64.0	62.6
120	30.7	23.1	17.5	13.4	10.8	9.2	3.1	.0	108.1	64.3	62.4
125	29.6	23.0	17.7	13.7	11.0	9.2	3.1	.0	107.5	64.4	62.4
130	28.3	22.7	17.8	14.0	11.1	9.2	3.0	.0	106.4	64.5	62.6
135	28.4	22.2	17.8	14.1	11.3	9.3	3.0	.0	106.4	64.8	62.9
140	28.1	21.8	17.6	14.2	11.5	9.4	3.0	.0	106.0	65.0	63.3
145	28.0	21.5	17.4	14.2	11.6	9.6	3.0	.0	105.7	65.1	63.8
150	28.0	21.5	17.2	14.2	11.7	9.7	3.0	.0	105.3	65.0	64.0

values vary between $30.1 and $40.5 per acre. The pulse-transition is nearly the same as the harvest pattern presented by Haight and others (1985) for the same initial diameter distribution.

Downward-sloping diameter distributions are maintained between years 60 and 150. Harvests take place in the 6-, 18-, and 20-inch diameter classes. By year 135, harvests stabilize so that during the last four periods residual diameter distributions differ by fewer than 1.0 tree per acre per diameter class for all but the 6-inch diameter class.

In part 3, I found that the PNW of a management regime developed with dynamic optimization will be greater than the PNW of an equilibrium management regime developed with static analysis for the same initial stand structure. In this example, the PNW of management regime 1 ($171.29/acre) is 5.1 percent greater than the PNW of equilibrium harvests taken on a 5-year cycle for 150 years from the initial investment-efficient diameter distribution ($163.05/acre).

I also applied the dynamic optimization algorithm using investment-efficient diameter distributions given by Martin (1982) as initial stand conditions. The resulting management regimes, which were developed using the corresponding stumpage value functions, interest rates, and cutting-cycle lengths given by Martin (1982), improved PNW by 30 to 50 percent relative to the PNW of the corresponding static equilibrium management regimes.

I also determined in part 3 that dynamically determined management regimes do not converge to steady-state stand structures determined with static analysis. In management regime 1, the residual diameter distribution at year 150 is essentially a steady state. This distribution has a flatter shape (Fig. 1) and a 4.7 percent higher growing-stock value than the investment-efficient distribution (Table 3). The value of harvests taken from the dynamically determined steady state is 4.1 percent less than the value growth from the investment-efficient stand structure. As a result, the LEV of the dynamic equilibrium management regime is 11.1 percent less than the LEV of the statically determined regime.

Management regime 2, in which the initial stand structure includes fewer trees in each diameter class relative to the steady state found in regime 1, demonstrates that the dynamically determined steady state is independent of the initial stand structure (Table 4). During the first 25 years, harvests take trees from the 20-inch diameter class only, while the numbers of trees in the remaining classes increase. Between years 30 and 55, a weak pulselike harvest pattern takes place, with large

TABLE 5. *Management regime 3 (PNW = $169.41/acre).*

Year	Diameter class midpoint (inches) 6	8	10	12	14	16	18	20	Totals Trees	Ft²	$
	Trees per acre cut										
0	69.8	34.0	0.0	0.0	0.0	3.7	2.1	0.3	110.1	35.3	30.0
10	8.9	.0	.0	.0	.0	6.5	6.8	1.7	24.0	26.7	66.7
20	.3	.0	.0	.0	.0	5.3	7.9	2.2	15.8	26.4	72.7
30	.0	.0	.0	.0	.0	1.0	8.5	2.7	12.3	22.5	67.6
40	.0	.0	.0	.0	.0	.0	9.3	3.4	12.7	23.9	74.1
50	.0	.0	.0	.0	.0	.0	8.4	3.3	11.7	22.2	69.0
60	8.4	.0	.0	.0	.0	.0	6.7	2.8	18.1	19.9	57.1
70	21.7	.0	.0	.0	.0	.0	5.7	2.4	29.8	19.6	48.6
80	21.2	.0	.0	.0	.0	.0	5.4	2.2	28.9	18.7	46.0
90	16.0	.0	.0	.0	.0	.0	5.8	2.2	24.2	18.5	48.3
100	7.4	.0	.0	.0	.0	.0	6.5	2.4	16.4	18.4	52.9
110	6.7	.0	.0	.0	.0	.0	7.1	2.7	16.5	19.7	57.3
120	7.6	.0	.0	.0	.0	.0	7.3	2.8	17.8	20.6	59.6
130	8.4	.0	.0	.0	.0	.0	7.2	2.8	18.5	20.6	59.2
140	10.2	.0	.0	.0	.0	.0	6.9	2.7	19.8	20.2	57.0
150	11.9	.0	.0	.0	.0	.0	6.6	2.6	21.2	19.8	54.8
	Residual trees per acre										
0	.0	15.2	37.5	29.7	24.1	6.9	.0	.0	113.5	84.5	58.8
10	8.2	9.3	22.0	27.5	24.2	9.1	.0	.0	100.5	77.2	61.6
20	20.0	9.3	13.1	20.1	22.0	11.0	.0	.0	95.9	69.2	61.7
30	27.9	13.5	10.2	13.5	17.3	14.3	.0	.0	97.0	65.0	64.8
40	34.0	18.2	11.1	10.2	12.5	13.5	.0	.0	99.9	59.6	58.2
50	40.7	22.9	13.6	9.8	9.6	10.8	.0	.0	107.7	56.6	48.7
60	40.4	27.8	16.8	11.2	8.8	8.6	.0	.0	113.8	57.2	43.2
70	28.8	30.3	19.8	13.3	9.5	7.9	.0	.0	109.9	58.8	42.7
80	20.6	27.4	21.3	15.4	10.9	8.3	.0	.0	104.2	60.7	45.4
90	18.3	22.9	20.3	16.5	12.4	9.3	.0	.0	100.0	62.1	49.2
100	23.8	19.8	18.1	16.2	13.2	10.4	.0	.0	101.6	62.9	52.4
110	27.4	20.1	16.4	14.9	13.1	11.0	.0	.0	103.1	62.6	53.6
120	29.0	21.5	16.0	13.8	12.4	10.9	.0	.0	103.9	61.5	52.6
130	29.9	22.8	16.5	13.4	11.6	10.5	.0	.0	104.9	60.5	50.7
140	29.4	23.9	17.2	13.5	11.3	9.9	.0	.0	105.5	60.2	49.1
150	27.9	24.2	17.9	13.9	11.3	9.6	.0	.0	105.1	60.3	48.5

numbers of trees cut from the 6-inch diameter class and small but increasing numbers of trees harvested from the 18-inch class. Downward-sloping diameter distributions are maintained for the remainder of the time horizon. The residual stand structure at year 150 is nearly stable and differs by fewer than 1 tree per acre per diameter class in all but the 6-inch diameter class from the steady-state structure in regime 1.

Cutting-Cycle Analysis

The dynamic optimization algorithm is easily expanded to analyze cutting cycles that are multiples of the 5-year projection interval. This is accomplished by constraining $X_{ij} - S_{ij} = 0$ for $j = 1, \ldots, M$ in periods when no harvest is desired.

Management regime 3 (Table 5) is developed for the same initial stand structure, stumpage value function, and interest rate as management regime 1; however, the cutting cycle is 10 years. Harvesting takes place in two pulse cycles during the first 90 years, with downward-sloping diameter distributions maintained thereafter. In contrast to management regime 1, the maximum diameter of residual trees is 16 inches and harvest values vary between $46.0 and $75.0 per acre.

Buongiorno and Michie (1980) demonstrated the importance of fixed entry costs in determining the economic cutting cycle. A break-even analysis with fixed entry costs can be used to compare the PNW of regimes with different cutting cycles. Similar to the results for thinning in even-aged stands (Brodie and others 1978), a fixed entry cost for selection harvests on a given cycle lowers the PNW, leaving the management regime unchanged. With no fixed entry cost, the PNW of management regime 1 exceeds the PNW of regime 3 by 1.1 percent. An entry cost of $0.75/acre would equate the values of the two management regimes. A similar comparison showed that an entry cost of $2.37/acre would be necessary for a 15-year cutting cycle to be optimal. In general, higher entry costs favor extended cutting cycles.

Summary and Conclusions

I have presented the marginal production rules and a computational solution method for the optimal transition and equilibrium management regimes for an uneven-aged stand. Comparing the production rules for a dynamic equilibrium stand structure with marginal production rules for the steady state determined with static optimization shows that solutions to the two problems will differ. Also, the PNW of the optimal transition and steady-state management regime will be greater than the PNW of the static equilibrium regime. Thus, for the diameter distribution model of uneven-aged management, maximization of LEV with the Faustmann equation is not the correct criterion for determining the optimal steady-state stand structure.

The solution method is the first to allow the simultaneous determination of optimal stand-specific transition and equilibrium management regimes. Numerical results show that while the steady-state solution is independent of the initial stand structure, the length and harvest pattern of the transition regime is not. Thus, generalizations about the relative profitability of uneven-aged and even-aged management cannot be made on the basis of steady-state management regimes alone, but must include the value of the stand-specific transition regime.

It would be very interesting to apply the solution method presented here to a simulator that projects growth and yield for both even-aged and uneven-aged stands. Then we could observe the effects of the initial stand structure and the timing and cost of regeneration on the relative profitability of the two fundamental management systems.

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APPENDIX

The marginal production rules and shadow price definitions for equilibrium harvest regimes in dynamic and static optimization problems are shown below.

Problem 1: Dynamic Optimization

$$P_j - u_{ij}^* + v_{ij}^* = \sum_{k=1}^{M} \frac{P_k + v_{ik}^*}{(1 + r)^t} \frac{\partial f_{ik}}{\partial S_{ij}} + \frac{P_j}{(1 + r)^t} + \frac{v_{ij}^*}{(1 + r)^t}$$

where

u_{ij}^* = the increase in PNW that would result from having one less unit of growing stock in diameter class j in the residual stand

v_{ij}^* = the increase in PNW that would result from having one more unit of growing stock in diameter class j in the residual stand.

Problem 2: Static Optimization

$$P_j - a_j^* + c_j^* = \sum_{k=1}^{M} \frac{P_k + b_k^*}{(1 + r)^t} \frac{\partial f_{0k}}{\partial S_{0j}} + \frac{P_j}{(1 + r)^t} + \frac{c_j^*}{(1 + r)^t}$$

where

$\dfrac{a_j^*(1 + r)^t}{[(1 + r)^t - 1]}$ = the increase in PNW that would result from having one less unit of growing stock in diameter class j in the residual stand

$\dfrac{b_j^*}{[(1 + r)^t - 1]}$ = the increase in PNW that would result from having one more unit of growing stock in diameter class j in the residual stand

c_j^* = the increase in PNW that would result from having one more unit of growing stock in diameter class j in the residual stand.