

United States
Department of
Agriculture

Forest Service

**Forest
Products
Laboratory**

Research
Paper
FPL 375

June 1980

Laminated Beams of Isotropic or Orthotropic Materials Subjected to Temperature Change

ABSTRACT

This paper considers laminated beams with layers of different isotropic or orthotropic materials fastened together by thin adhesives. The stresses that result from subjecting each component layer of the beam to different temperature or moisture stimuli which may also vary along the length of the beam, are calculated. Two-dimensional elasticity theory is used so that a wide range of problems, such as that of beams composed of layers of orthotropic materials like wood, can be studied, and accurate distributions of normal and shear stresses obtained. The stress intensity along the bearing surfaces of the layers of the beam is of particular importance because it is responsible for delamination failures of laminated structural elements. The distributions of inter-laminar normal and shear stresses measured along the longitudinal axis of the beam indicate that high stress intensity occurs in the end zones of the beam. Thus, delamination failure, when it occurs, will start at the ends of the beam.

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LAMINATED BEAMS OF ISOTROPIC OR ORTHOTROPIC MATERIALS
SUBJECTED TO TEMPERATURE CHANGE

By

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INTRODUCTION

Beams are among the most widely used structural elements. The application of laminated beams is expanding as they are studied and developed by the forest products industry as well as other industries [1,2,3]. However, the delamination failure or deformation (bow, twist, warp, etc.) induced in laminated beams by thermal or moisture stimuli has always been of major concern.

The use of elementary beam theory in solution of this problem does not allow evaluation of the shearing and normal stresses along the bearing surface. Thus, these stresses cannot be determined from Timoshenko's pioneering analysis [4] of a bimetal strip submitted to uniform heating along its length. Notable contributions on this subject due to Boley and others may be found in [5,6]. However, Grimado's analysis [7] is a further consideration, extension, and significant improvement of the same problem treated by Timoshenko. When Grimado [7] uses elementary beam theory the effect of the bonding material between the two layers of the strip is taken into account by treating the bonding material as a third layer. Grimado deduces a sixth-order governing differential equation [7] compared to the fourth-order biharmonic equation of plane stress problems. It is shown in [7] that the sixth-order equation reduces to a characteristic cubic equation which, unlike the biharmonic

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equation, may yield complex roots when solved. Thus, the analysis presented in [7] is not necessarily as simple and direct as that based on the two-dimensional elasticity theory used here.

In the current paper a beam of uniform cross-section made of layers of different isotropic or orthotropic materials (such as wood, fastened together by thin adhesives) is considered in accordance with two-dimensional elasticity theory. Instead of being subjected to a uniform heating, treated previously [4,7], each layer may have different temperature distributions along its length. The use of two-dimensional elasticity theory should accurately yield the distribution of shear and normal stresses in the beam. The interlaminar stresses between layers are known to be mainly responsible for delamination failures of laminated beams. If the stress distribution between layers can be determined, such failures may be minimized or eliminated by an appropriate choice of materials and beam-section properties. The use of two-dimensional elasticity theory can be further justified in the analysis of beams of orthotropic materials since in such a case elementary beam theory cannot take into account the effects of material properties on stresses and deformation.

It is difficult to accurately estimate the effect of the bonding material on the stress distribution and deformation of the laminated structural element without also treating the thin layers of the bonding material in the same way that the component layers of primary concern are treated. When the bonding material between two component layers is very thin, the effect of the bonding material on the stress distribution and deformation of laminated beams can be negligible. Thus, this paper considers two cases: (1) beams having two component layers rigidly bonded together and (2) beams having three layers where the bonding material is treated as a third layer. These are analyzed so that the effect of the bonding material can be revealed. Special attention is given to the stress intensity along the bearing surfaces of the layers of the laminated beams since it is responsible for delamination failures. Only laminated beams subjected to thermal stimulus are treated in detail. However, beams subjected to moisture stimulus can be analyzed in the same way simply by replacing the linear coefficient of thermal expansion multiplied by the change of temperature with a linear coefficient of moisture shrinkage multiplied by a relative moisture content [8, p. 3-11].

TWO-DIMENSIONAL THERMOELASTIC FORMULATION

Consider a beam of unit width which is made of two layers of different materials fastened together by a thin adhesive, where each layer is subjected to different arbitrary temperature distribution $T_i(x)$,

(i = 1,2) along its length. The laminated structure is initially free from stress. Let the x-axis be the longitudinal axis which lies along the bonding line of the two layers of the beam of length, l , total thickness, h , and place the y-axis at the left end of the beam. Let h_1 be the thickness of the top layer, h_2 the thickness of the bottom layer, and $d_1 = h_1$, $d_2 = -h_2$. Let E be the Young's modulus, G the shear modulus, and ν (Greek letter nu) the Poisson's ratio. Let σ_x and σ_y be the components of stress and strain, u and w the displacements along the x and y directions, and α_x and α_y the coefficients of thermal expansion in directions x and y , respectively. The next section, concerning laminated beams of isotropic materials, is limited to the consideration of a specific case where the beam is uniformly heated by raising its temperature t degrees. It is a fundamental case that can reveal the essential features of thermal-stress problems of laminated beams; the solution can be used to make comparisons with known results in the literature. A subsequent section, concerning laminated beams of orthotropic materials, treats the general case of temperature distribution.

LAMINATED BEAMS OF ISOTROPIC MATERIALS

In this section we consider a laminated beam consisting of two different isotropic materials. For plane stress distribution the stress-strain and strain-displacement relations, the governing differential equation, and the expressions for stresses in terms of a stress function are [9,10].

$$\epsilon_x = \frac{\partial u}{\partial x} = \frac{1}{E} (\sigma_x - \nu \sigma_y) + \alpha T, \quad \epsilon_y = \frac{\partial w}{\partial y} = \frac{1}{E} (\sigma_y - \nu \sigma_x) + \alpha T, \quad (1)$$

$$\gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial w}{\partial x} = \frac{1}{G} \tau_{xy}$$

$$\nabla^4 \phi = -E\alpha \nabla^2 T \quad (2)$$

$$\sigma_x = \frac{\partial^2 \phi}{\partial y^2}, \quad \sigma_y = \frac{\partial^2 \phi}{\partial x^2}, \quad \tau_{xy} = -\frac{\partial^2 \phi}{\partial x \partial y} \quad (3)$$

When the laminated beam is uniformly heated, changing the temperature by t degrees, T_i may be expressed as

$$T_i = t = \frac{4t}{\pi} \sum_{n=1}^{\infty} \frac{1}{n} \sin \frac{n\pi x}{\ell} \quad (n = 1, 3, 5, \dots) \quad (4)$$

Accordingly, the biharmonic function ϕ_i for the i th layer which satisfies equation (2) may be taken as

$$\begin{aligned} \phi_i = \sum_{n=1}^{\infty} (A_{ni} \cosh \gamma y + B_{ni} \gamma y \sinh \gamma y + C_{ni} \sinh \gamma y \\ + D_{ni} \gamma y \cosh \gamma y) \sin \gamma x \quad (n = 1, 3, 5, \dots) \end{aligned} \quad (5)$$

$$\text{where } \gamma = \frac{n\pi}{\ell}. \quad (6)$$

The boundary conditions on the top and bottom surfaces of the beam are

$$\sigma_y = 0, \quad \tau_{xy} = 0, \quad \text{on } y = d_i \quad (d_1 = h_1, \quad d_2 = -h_2) \quad (7)$$

In addition to the preceding boundary conditions, the conditions of continuity along the line of division of the two layers must be satisfied. These are the continuity of stress

$$\sigma_{y1} = \sigma_{y2}, \quad \tau_{xy1} = \tau_{xy2}, \quad \text{on } y = 0 \quad (8)$$

where σ_{y1} and τ_{xy1} represent stresses acting on the bottom surface of the top layer of the beam and σ_{y2} and τ_{xy2} are stresses acting on the top surface of the bottom layer of the beam. The continuity of displacements across the line of division consists of

$$u_1 = u_2, \quad w_1 = w_2 \quad \text{on } y = 0 \quad (9)$$

Applying boundary conditions (7) and (8) yields

$$A_{ni} \cosh \gamma d_i + B_{ni} \gamma d_i \sinh \gamma d_i + C_{ni} \sinh \gamma d_i + D_{ni} \gamma d_i \cosh \gamma d_i = 0 \quad (10)$$

$$A_{n2} = A_{n1} \quad (11)$$

$$(A_{ni} + B_{ni} + D_{ni} \gamma d_i) \sinh \gamma d_i + (C_{ni} + D_{ni} + B_{ni} \gamma d_i) \cosh \gamma d_i = 0 \quad (12)$$

$$C_{n2} + D_{n2} = C_{n1} + D_{n1} \quad (13)$$

From equations (1) and (3) we obtain

$$\frac{\partial u_i}{\partial x} = \frac{1}{E_i} \left(\frac{\partial^2 \phi_i}{\partial y^2} - v_i \frac{\partial^2 \phi_i}{\partial x^2} \right) + \alpha_i t, \quad \frac{\partial w_i}{\partial y} = \frac{1}{E_i} \left(\frac{\partial^2 \phi_i}{\partial x^2} - v_i \frac{\partial^2 \phi_i}{\partial y^2} \right) + \alpha_i t \quad (14)$$

Integrating the two equations of (14) and expressing x as a Fourier cosine series

$$x = \frac{\ell}{2} - \frac{4\ell}{\pi^2} \sum_{n=1}^{\infty} \frac{1}{n^2} \cos \gamma x \quad (0 \leq x \leq \ell, n = 1, 3, 5, \dots) \quad (15)$$

we obtain

$$u_i = \frac{-1}{E_i} \sum_{n=1}^{\infty} \gamma \{ [(1+v_i)A_{ni} + 2B_{ni}] \cosh \gamma y + (1+v_i)B_{ni} \gamma y \sinh \gamma y + [(1+v_i)C_{ni} + 2D_{ni}] \sinh \gamma y + (1+v_i)D_{ni} \gamma y \cosh \gamma y + \frac{4t\alpha_i E_i}{\ell \gamma^3} \} \cos \gamma x + u_i^0 \quad (n = 1, 3, 5, \dots) \quad (16)$$

$$\begin{aligned}
w_i = & \frac{-1}{E_i} \sum_{n=1}^{\infty} \gamma \{ (1+v_i) A_{ni} \sinh \gamma y + B_{ni} [(1+v_i) \gamma y \cosh \gamma y - (1-v_i) \sinh \gamma y] \\
& + (1+v_i) C_{ni} \cosh \gamma y + D_{ni} [(1+v_i) \gamma y \sinh \gamma y - (1-v_i) \cosh \gamma y] \} \sin \gamma x \\
& + \alpha_i t y + w_i^0 \quad (n = 1, 3, 5, \dots)
\end{aligned} \tag{17}$$

Taking the point $x = 0$ and $y = 0$ as rigidly fixed so that $u_i = 0$, $w_i = 0$ at that point, from equations (16) and (17) we find

$$u_i^0 = \frac{1}{E_i} \sum_{n=1,3,\dots}^{\infty} \gamma \left[\frac{4t\alpha_i E_i}{\ell \gamma^3} + (1+v_i) A_{ni} + 2B_{ni} \right], \quad w_i^0 = 0 \tag{18}$$

Applying the continuity conditions (9) $u_1 = u_2$ and $w_1 = w_2$ along the line of division $y = 0$ yields

$$\frac{E_2}{E_1} [(1+v_1) A_{n1} + 2B_{n1}] - (1+v_2) A_{n2} - 2B_{n2} = \frac{4E_2 t}{\ell \gamma^3} (\alpha_2 - \alpha_1) \tag{19}$$

$$\frac{E_2}{E_1} [(1+v_1) C_{n1} - (1-v_1) D_{n1}] - (1+v_2) C_{n2} + (1-v_2) D_{n2} = 0 \tag{20}$$

and

$$u_2^0 = u_1^0 \tag{21}$$

Solving the eight linear algebraic equations (10) to (13), (19), and (20), the eight unknown coefficients A_{ni} , B_{ni} , C_{ni} , and D_{ni} can be expressed in closed forms in terms of elastic moduli, thermal linear strain, and beam dimensions. For numerical solutions these eight linear algebraic equations can be easily solved in each particular case and, once the eight coefficients are determined, the following stresses can be obtained.

$$\begin{aligned}
\sigma_{yi} &= - \sum_{n=1,3,\dots}^{\infty} \gamma^2 (A_{ni} \cosh \gamma y + B_{ni} \gamma \sinh \gamma y + C_{ni} \sinh \gamma y \\
&\quad + D_{ni} \gamma \cosh \gamma y) \sin \gamma x \\
\tau_{xyi} &= - \sum_{n=1,3,\dots}^{\infty} \gamma^2 [A_{ni} \sinh \gamma y + B_{ni} (\sinh \gamma y + \gamma y \cosh \gamma y) \\
&\quad + C_{ni} \cosh \gamma y + D_{ni} (\cosh \gamma y + \gamma y \sinh \gamma y)] \cos \gamma x \\
\sigma_{xi} &= \sum_{n=1,3,\dots}^{\infty} \gamma^2 [A_{ni} \cosh \gamma y + B_{ni} (2 \cosh \gamma y + \gamma y \sinh \gamma y) \\
&\quad + C_{ni} \sinh \gamma y + D_{ni} (2 \sinh \gamma y + \gamma y \cosh \gamma y)] \sin \gamma x
\end{aligned} \tag{22}$$

From equation (22) it is seen that one of the traction-free end conditions $\tau_{xi} = 0$ at $x = 0$ and $x = \ell$ is already satisfied. The second condition, $\tau_{xyi} = 0$ at both ends of the beam, cannot be satisfied exactly without superimposing additional solutions. However, this condition is satisfied in the sense of Saint Venant's principle. With this principle we may replace the second condition by its statically equivalent condition:

$$\int_{-h_2}^{h_1} \tau_{xyi} dy = 0, \quad \text{at } x = 0 \text{ and } x = \ell \tag{23}$$

To show that this condition is satisfied, we proceed as follows. Since

$$\sigma_{yi} = \frac{\partial^2 \phi_i}{\partial x^2} \quad \text{and} \quad \tau_{xyi} = - \frac{\partial^2 \phi_i}{\partial x \partial y},$$

the satisfaction of the boundary

conditions $\sigma_{yi} = 0$, $\tau_{xyi} = 0$, on $y = d_i$ and $\sigma_{y1} = \sigma_{y2}$, $\tau_{xy1} = \tau_{xy2}$, on $y = 0$ implies

$$\phi_i = 0, \quad \frac{\partial \phi_i}{\partial y} = 0 \quad \text{on } y = d_i \quad (24)$$

and $\phi_1 = \phi_2, \quad \frac{\partial \phi_1}{\partial y} = \frac{\partial \phi_2}{\partial y} \quad \text{on } y = 0$

respectively. But the conditions (24) imply zero resultant shearing force on any section, $x = \text{constant}$, of the beam. This follows from

$$\begin{aligned} \int_{-h_2}^{h_1} \tau_{xy} dy &= - \frac{\partial}{\partial x} \left[\int_0^{h_1} \frac{\partial \phi_1}{\partial y} dy + \int_{-h_2}^0 \frac{\partial \phi_2}{\partial y} dy \right] \\ &= \frac{\partial}{\partial x} (\phi_1 - \phi_2) \Big|_{y=0} = 0 \end{aligned} \quad (25)$$

Thus, the load on each end of the laminated beam is self-equilibrating. The problem of exact satisfaction of end conditions of an elastic strip has been treated by authors [11,12] in the literature, It is beyond the scope of this article, however, to modify the solution by satisfying more precisely the free end conditions of the beam.

LAMINATED BEAMS OF ORTHOTROPIC MATERIALS

In the case of plane stress distribution, the stress-strain and strain-displacement relations for orthotropic materials are [10,13]

$$\varepsilon_x = \frac{\sigma_x}{E_x} - \frac{\nu_{yx}}{E_y} \sigma_y + \alpha_x T = \frac{\partial u}{\partial x}, \quad \varepsilon_y = \frac{\sigma_y}{E_y} - \frac{\nu_{xy}}{E_x} \sigma_x + \alpha_y T = \frac{\partial w}{\partial y} \quad (26)$$

$$\gamma_{xy} = \frac{1}{G} \tau_{xy} = \frac{\partial u}{\partial y} + \frac{\partial w}{\partial x}$$

where E_x, E_y are the Young's moduli along the principal directions x and y ; $G = G_{xy}$, the shear modulus which characterizes the change of angles between principal directions x and y , and ν_{yx} , the Poisson's ratio.

This ratio characterizes the decrease in direction x due to tension in direction y with a similar meaning for the expression V_{xy} , related to v_{xy} by

$$E_y v_{xy} = E_x v_{yx} \quad (27)$$

u, w are the displacements along the x and y directions, respectively, and α_x, α_y represent the coefficients of thermal expansion in principal directions x and y. Solving equations (26) for stresses yields

$$\begin{aligned} \sigma_x &= \frac{E_x}{1 - v_{xy} v_{yx}} [\epsilon_x + v_{yx} \epsilon_y - (\alpha_x + v_{yx} \alpha_y) T(x)] \\ \sigma_y &= \frac{E_y}{1 - v_{xy} v_{yx}} [\epsilon_y + \frac{E_x}{E_y} v_{yx} \epsilon_x - (\alpha_y + \frac{E_x}{E_y} v_{yx} \alpha_x) T(x)] \end{aligned} \quad (28)$$

$$\tau_{xy} = G \gamma_{xy}$$

The equations of equilibrium are identically satisfied by introducing Airy stress function ϕ as

$$\sigma_x = \frac{\partial^2 \phi}{\partial y^2} \quad \sigma_y = \frac{\partial^2 \phi}{\partial x^2} \quad \tau_{xy} = - \frac{\partial^2 \phi}{\partial x \partial y} \quad (29)$$

The substitution of equations (26) and (29) into the compatibility relation yields the following governing differential equation for orthotropic materials

$$\frac{1}{E_y} \frac{\partial^4 \phi}{\partial x^4} + \left(\frac{1}{G} - \frac{2v_{xy}}{E_x} \right) \frac{\partial^4 \phi}{\partial x^2 \partial y^2} + \frac{1}{E_x} \frac{\partial^4 \phi}{\partial y^4} = - (\alpha_y \frac{\partial^2 T}{\partial x^2} + \alpha_x \frac{\partial^2 T}{\partial y^2}) \quad (30)$$

In the general case a complete Fourier series expansion for $T_i(x)$ is required. However, the method of solution remains the same whether

$T_i(x)$ is expanded in a complete Fourier Series or a sine or a cosine series. Thus, in the following, only a sine-series expansion is considered, i.e.,

$$T_i(x) = \sum_n^{\infty} t_{ni} \sin \frac{n\pi x}{\ell} \quad (31)$$

in which

$$t_{ni} = \frac{2}{\ell} \int_0^{\ell} T_i(x) \sin \frac{n\pi x}{\ell} dx \quad (32)$$

Corresponding to the expansion (31), the particular and homogeneous solutions for the i th layer, which satisfy equation (30), may be expressed, respectively, as

$$\phi_{pi} = \sum_{n=1}^{\infty} E_{yi} \alpha_{yi} t_{ni} \left(\frac{\ell}{n\pi}\right)^2 \sin \frac{n\pi x}{\ell} \quad (33)$$

$$\begin{aligned} \phi_{hi} = \sum_{n=1}^{\infty} (A_{ni} \cosh n\lambda_i y + B_{ni} \sinh n\lambda_i y + \\ + C_{ni} \cosh n\mu_i y + D_{ni} \sinh n\mu_i y) \sin \frac{n\pi x}{\ell} \end{aligned} \quad (34)$$

in which A_{ni} , B_{ni} , C_{ni} , D_{ni} are arbitrary constants and

$$\lambda_i = \frac{\pi}{\ell} \sqrt[4]{\frac{E_{xi}}{E_{yi}}} \sqrt{K_i + \sqrt{K_i^2 - 1}} \quad (35)$$

$$\mu_i = \frac{\pi}{\ell} \sqrt{\frac{E_{xi}}{E_{yi}}} \sqrt{K_i - \sqrt{K_i^2 - 1}}$$

where
$$K_i = \frac{1}{2} \sqrt{E_{xi} E_{yi}} \left(\frac{1}{G_i} - \frac{2v_{xyi}}{E_{xi}} \right)$$

The general integral of equation (30) is the sum of the solutions (33) and (34), i.e.,

$$\phi_i = \phi_{pi} + \phi_{hi} \quad (36)$$

In the special case where the two roots A_1 and μ_i of equation (35) are real and equal (i.e., $K_i = 1$), the following solution of equation (30) should be used instead of the solution (34).

$$\phi_i = \sum_{n=1}^{\infty} [A_{ni} + B_{ni}y] \cosh n\lambda_i y + (C_{ni} + D_{ni}y) \sinh n\lambda_i y \sin \frac{n\pi x}{\ell} \quad (37)$$

Having obtained solution (36) we will be able to show that all the boundary conditions (7), (8), and (9) can be satisfied. Applying the boundary conditions (7) and (8) yields, respectively,

$$\begin{aligned} & A_{ni} \cosh n\lambda_i d_i + B_{ni} \sinh n\lambda_i d_i + C_{ni} \cosh n\mu_i d_i \\ & + D_{ni} \sinh n\mu_i d_i = - E_{yi} \alpha_{yi} t_{ni} \left(\frac{\ell}{n\pi} \right)^2, \quad (n = 1, 2, \dots) \end{aligned} \quad (38)$$

$$\begin{aligned} & \lambda_i (A_{ni} \sinh n\lambda_i d_i + B_{ni} \cosh n\lambda_i d_i) \\ & + \mu_i (C_{ni} \sinh n\mu_i d_i + D_{ni} \cosh n\mu_i d_i) = 0, \quad (n = 1, 2, \dots) \end{aligned} \quad (39)$$

$$A_{n1} + C_{n1} - A_{n2} - C_{n2} = (E_{y2}\alpha_{y2}t_{n2} - E_{y1}\alpha_{y1}t_{n1})\left(\frac{\ell}{n\pi}\right)^2, \quad (n = 1, 2, \dots) \quad (40)$$

$$\lambda_2 B_{n2} + \mu_2 D_{n2} = \lambda_1 B_{n1} + \mu_1 D_{n1}, \quad (n = 1, 2, \dots) \quad (41)$$

From equations (26) and (29) we obtain

$$\begin{aligned} \frac{\partial u_i}{\partial x} &= \frac{1}{E_{xi}} \left(\frac{\partial^2 \phi_i}{\partial y^2} - v_{xyi} \frac{\partial^2 \phi_i}{\partial x^2} \right) + \alpha_{xi} T_i, \\ \frac{\partial w_i}{\partial y} &= \frac{1}{E_{yi}} \left(\frac{\partial^2 \phi_i}{\partial x^2} - v_{yxi} \frac{\partial^2 \phi_i}{\partial y^2} \right) + \alpha_{yi} T_i \end{aligned} \quad (42)$$

Integrating the two equations of (42) yields

$$\begin{aligned} u_i &= \frac{\ell}{\pi E_{xi}} \sum_{n=1}^{\infty} n \left[(\lambda_i^2 + v_{xyi} \frac{\pi^2}{\ell^2}) (A_{ni} \cosh n\lambda_i y + B_{ni} \sinh n\lambda_i y) \right. \\ &\quad \left. + (\mu_i^2 + v_{yxi} \frac{\pi^2}{\ell^2}) (C_{ni} \cosh n\mu_i y + D_{ni} \sinh n\mu_i y) \right. \\ &\quad \left. - \frac{t_{ni}}{n^2} (E_{xi}\alpha_{xi} + v_{xyi}E_{yi}\alpha_{yi}) \right] \cos \frac{n\pi x}{\ell} + u_i^o \\ w_i &= \frac{-1}{E_{yi}} \sum_{n=1}^{\infty} n \left\{ \frac{1}{\lambda_i} (v_{yxi}\lambda_i^2 + \frac{\pi^2}{\ell^2}) (A_{ni} \sinh n\lambda_i y + B_{ni} \cosh n\lambda_i y) \right. \\ &\quad \left. + \frac{1}{\mu_i} (v_{yxi}\mu_i^2 + \frac{\pi^2}{\ell^2}) (C_{ni} \sinh n\mu_i y + D_{ni} \cosh n\mu_i y) \right\} \sin \frac{n\pi x}{\ell} \quad (44) \end{aligned}$$

in which the constant u_i^0 can be determined by taking the point $x = 0$ and $y = 0$ as rigidly fixed in order to have $u_i = 0$, $w_i = 0$ at $x = 0$, $y = 0$. Applying the continuity conditions (9) $u_1 = u_2$ and $w_1 = w_2$ along the axis $y = 0$ yields

$$\begin{aligned} & [(\frac{\lambda_2 \ell}{\pi})^2 + v_{xy2}]A_{n2} + [(\frac{\mu_2 \ell}{\pi})^2 + v_{xy2}]C_{n2} - \frac{E_{x2}}{E_{x1}} \{ [(\frac{\lambda_1 \ell}{\pi})^2 \\ & + v_{xy1}]A_{n1} + [(\frac{\mu_1 \ell}{\pi})^2 + v_{xy1}]C_{n1} + (\frac{\ell}{n\pi})^2 [\frac{E_{x1}}{E_{x2}} t_{n2}(E_{x2}\alpha_{x2} + v_{xy2}E_{y2}\alpha_{y2}) \\ & - t_{n1}(E_{x1}\alpha_{x1} + v_{xy1}E_{y1}\alpha_{y1})] \} = 0 \end{aligned} \quad (45)$$

$$\begin{aligned} & \lambda_2 [(\frac{\pi}{\lambda_2 \ell})^2 + v_{yx2}]B_{n2} + \mu_2 [(\frac{\pi}{\mu_2 \ell})^2 + v_{yx2}]D_{n2} = \frac{E_{y2}}{E_{y1}} \{ \lambda_1 [(\frac{\pi}{\lambda_1 \ell})^2 \\ & + v_{yx1}]B_{n1} + \mu_1 [(\frac{\pi}{\mu_1 \ell})^2 + v_{yx1}]D_{n1} \} \end{aligned} \quad (46)$$

$$\text{and also } u_1^0 = u_2^0. \quad (47)$$

The eight arbitrary coefficients A_{ni} , B_{ni} , C_{ni} , and D_{ni} are determined from the eight linear algebraic equations (38) to (41), (45), and (46) and then the stresses can be obtained from the following expressions:

$$\begin{aligned} \sigma_{yi} = & - \sum_{n=1}^{\infty} \{ (\frac{n\pi}{\ell})^2 [A_{ni} \cosh n\lambda_i y + B_{ni} \sinh n\lambda_i y \\ & + C_{ni} \cosh n\mu_i y + D_{ni} \sinh n\mu_i y] + E_{yi} \alpha_{yi} t_{ni} \} \sin \frac{n\pi x}{\ell} \end{aligned} \quad (48)$$

$$\tau_{xyi} = - \frac{\pi}{\ell} \sum_{n=1}^{\infty} n^2 [\lambda_i (A_{ni} \sinh n\lambda_i y + B_{ni} \cosh n\lambda_i y)$$

$$+ \mu_i (C_{ni} \sinh n\mu_i y + D_{ni} \cosh n\mu_i y) \cos \frac{n\pi x}{\ell} \quad (49)$$

$$\begin{aligned} \sigma_{xi} = & \sum_{n=1}^{\infty} n^2 [\lambda_i^2 (A_{ni} \cosh n\lambda_i y + B_{ni} \sinh n\lambda_i y) \\ & + \mu_i^2 (C_{ni} \cosh n\mu_i y + D_{ni} \sinh n\mu_i y)] \sin \frac{n\pi x}{\ell} \end{aligned} \quad (50)$$

NUMERICAL EXAMPLE AND CONCLUSIONS

CASE A: TWO LAYERED BEAMS HAVING RIGID BOND

The following material constants and beam dimensions are used:

$$\begin{aligned} \ell &= 90 \text{ cm.}, & h_1 &= h_2 = 5 \text{ cm.}, & t_o &= 400^\circ \text{ C} = 752^\circ \text{ F} \\ v_1 &= 0.27 & E_1 &= 20.69 \times 10^6 \text{ N/cm}^2., & \alpha_1 &= 6.5 \times 10^{-6} / \text{F}^\circ, \\ v_2 &= 0.33, & E_2 &= 6.89 \times 10^6 \text{ N/cm}^2., & \alpha_2 &= 13 \times 10^{-6} / \text{F}^\circ, \end{aligned}$$

Figures 1 and 2 show the distributions of interlaminar normal and shear stresses, respectively, along the longitudinal axis of the beam. Both indicate that high stress intensity occurs in the end zones of the beam and that both stresses decay rapidly with increasing distance from the ends. The distribution of the stresses in the end zones within a distance equal to the thickness of each beam, may not be very accurate, especially for the shear stress. However, the results in figures 1 and 2 show that both stresses do increase toward the ends of the beam, starting from a distance greater than the thickness of each layer. Thus, it may be concluded that delamination failure, when it occurs, will start at the ends of the beam.

Figures 3 and 4 show the distribution of axial stress along the bonding surface of the upper layer and the lower layer, respectively. The bonding surface of the upper layer is under tension and that of the lower layer under compression. Within a short distance from the free ends of the beam, the axial stress of the upper layer (Fig. 3) reaches its maximum value and the axial stress of the lower layer (Fig. 4)

reaches a value slightly less than its maximum value. Both then remain constant for the remainder of the beam. Other observations which could be made for Case A will be described later as observations for Case B.

CASE B: TWO LAYERED BEAMS HAVING AN ELASTIC BOND BETWEEN THE LAYERS (FIG. 5)

The material constants and beam dimensions for the two layers are the same as those used in the Case A. The following constants are taken for the thin bonding material between the layers:

$$h_3 = .33 \text{ cm.}, \quad \nu_3 = .33, \quad \alpha_3 = 2.5 \times 10^{-6} / \text{F}^\circ,$$

Three cases of different modulus of elasticity of the bonding material are considered, i.e.,

$$\begin{aligned} (1) \quad E_3 &= 27.58 \times 10^6 \text{ N/cm}^2, & (2) \quad E_3 &= 4 \times 10^6 \text{ N/cm}^2 \\ (3) \quad E_3 &= 10^6 \text{ N/cm}^2 \end{aligned}$$

We note that the constant c appearing in figures 1-4 and the constant C in figures 6 through 13 are related as

$$C = E_2(\alpha_2 - \alpha_1)c = 44.785c.$$

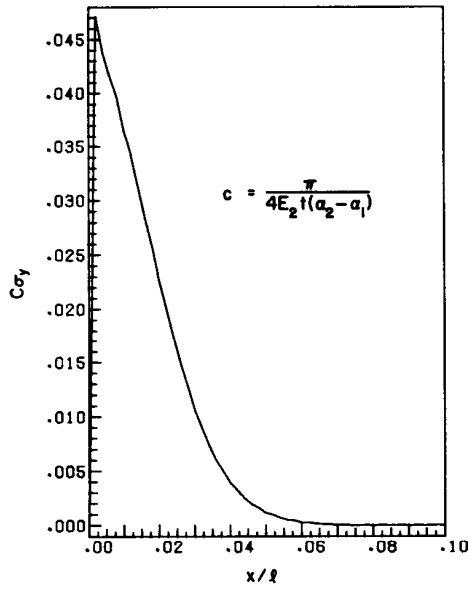
The observations and conclusions made in the previous Case A also hold true for the present case. In addition, other observations can be made from figures 6 through 13. Figures 6 and 8 illustrate that the modulus of elasticity of the bonding material (adhesive) affects the interlaminar normal stress only in the narrow end regions, these regions being of the order of the thickness of the beam. A larger modulus of elasticity in the bonding material yields significantly larger interlaminar normal stress in the end regions. However, the interlaminar normal stress is virtually unaffected by the modulus of elasticity of the adhesive for the remainder of the beam. Thus, if delamination failure should occur, it will occur at the ends of the beam. To prevent such failure, adhesives having smaller modulus of elasticity should be preferred.

It is of interest to note that the modulus of elasticity of the bonding material has only a slight effect on the distribution of the interlaminar

shear stress as seen from figures 7 and 9, except perhaps in a very short distance from the ends of the beam. However, in this region the distribution of the shear stress is not accurate, as previously pointed out. An examination of figures 3, 4, 10 and 11 indicates that the modulus of elasticity of the bonding material has only slight effect on the axial stress in the upper and lower layers of the beam. On the other hand, the modulus of elasticity of the bonding material has significant effect on the axial stress in the bonding material as seen from figures 12 and 13; a greater modulus yields significantly larger axial stress. We further note that the maximum axial stress in the upper and lower layers of the beam is greater than that of the interlaminar normal stress. However, delamination failure of the bond caused by the interlaminar normal stress may still occur, as is often the case in practice, if the bond cannot sustain the required interlaminar normal stress.

The computer program developed for the numerical calculations is presented in the Appendix.

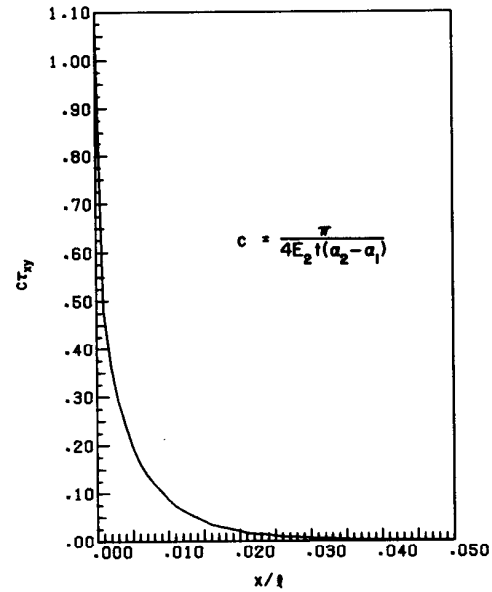
NORMAL STRESS



M 148 336

Figure 1. - Interlaminar normal stress distribution.

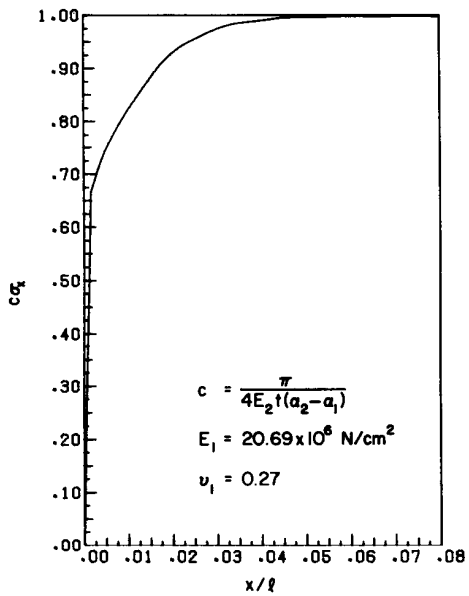
SHEAR STRESS



M 148 337

Figure 2. - Interlaminar shear stress distribution.

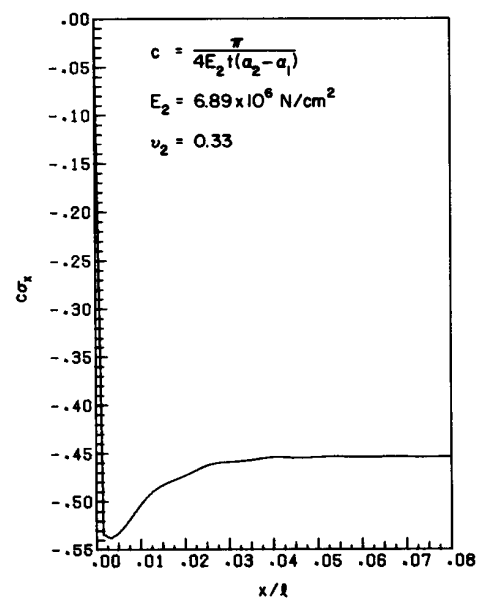
AXIAL STRESS



M 148 778

Figure 3. - Axial stress distribution along the bonding surface of the upper layer of a two-layered beam.

AXIAL STRESS



M 148 779

Figure 4. - Axial stress distribution along the bonding surface of the lower layer of a two-layered beam.

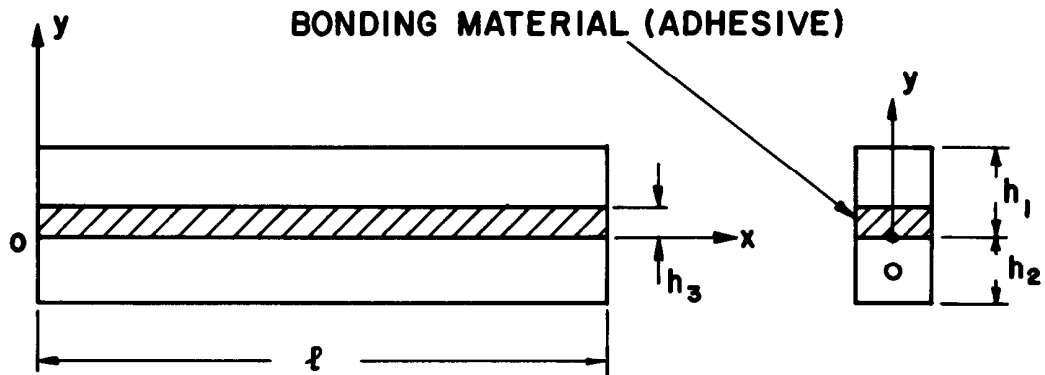
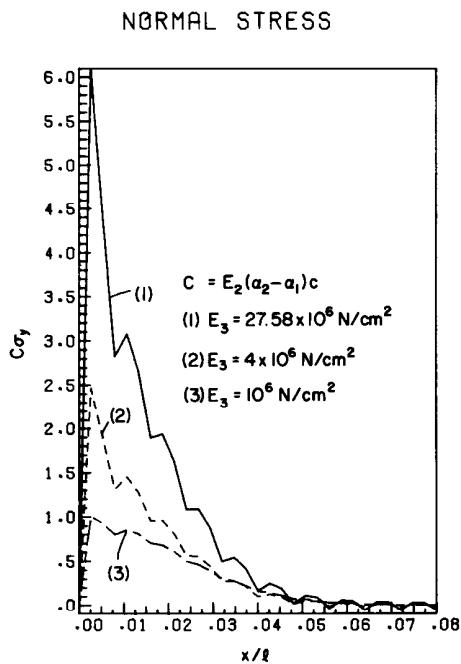
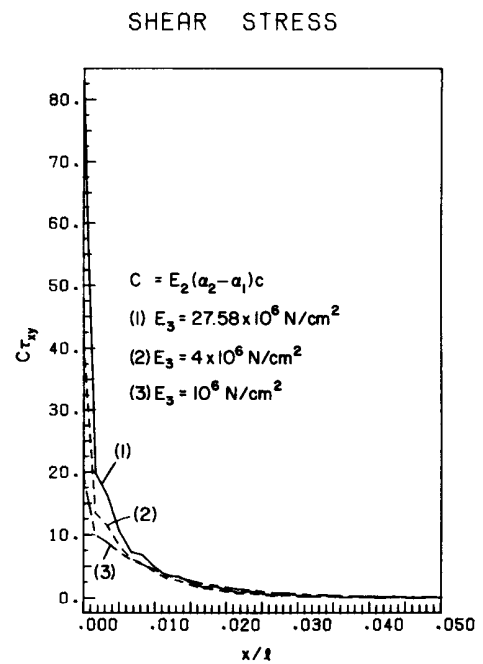


Figure 5. - Two-layered beam having an elastic bond between the layers.



M 148 780

Figure 6. - Interlaminar normal stress distribution along the bonding surface of the lower layer and the middle layer (bonding material).



M 148 781

Figure 7. - Interlaminar shear stress distribution along the bonding surface of the lower layer and the middle layer (bonding material).

NORMAL STRESS

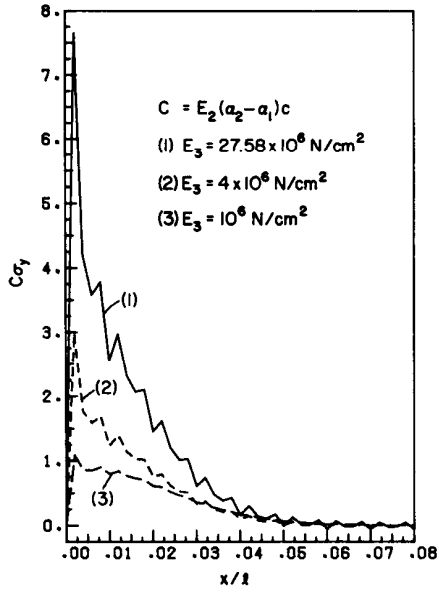


Figure 8. - Interlaminar normal stress distribution along the bonding surface of the upper layer and the middle layer (bonding material).

SHEAR STRESS

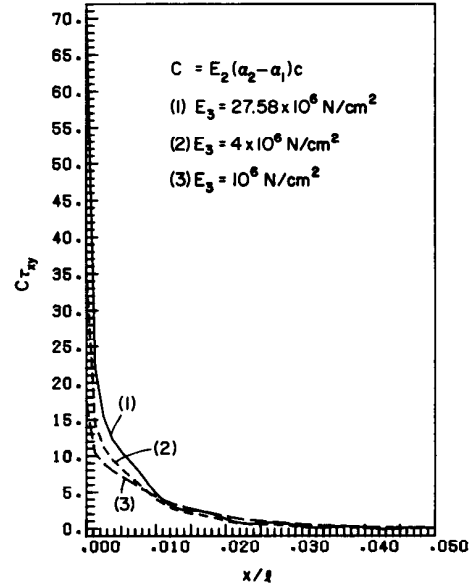


Figure 9. - Interlaminar shear stress distribution along the bonding surface of the upper layer and the middle layer (bonding material).

AXIAL STRESS

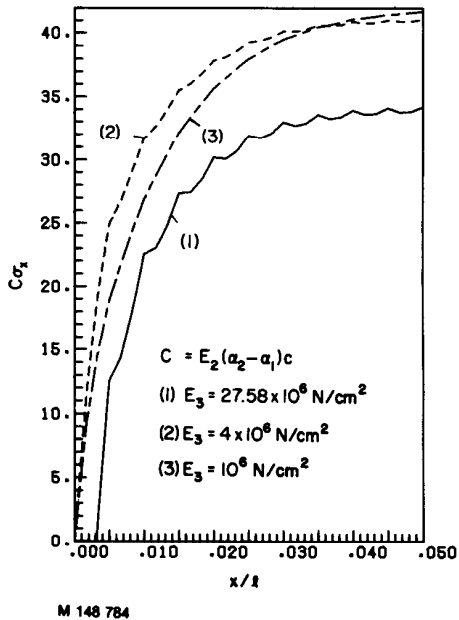


Figure 10. - Axial stress distribution along the bonding surface of the upper layer.

AXIAL STRESS

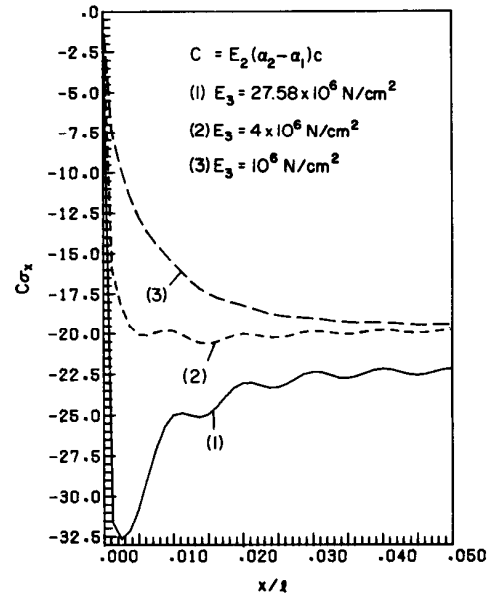


Figure 11. - Axial stress distribution along the bonding surface of the lower layer.

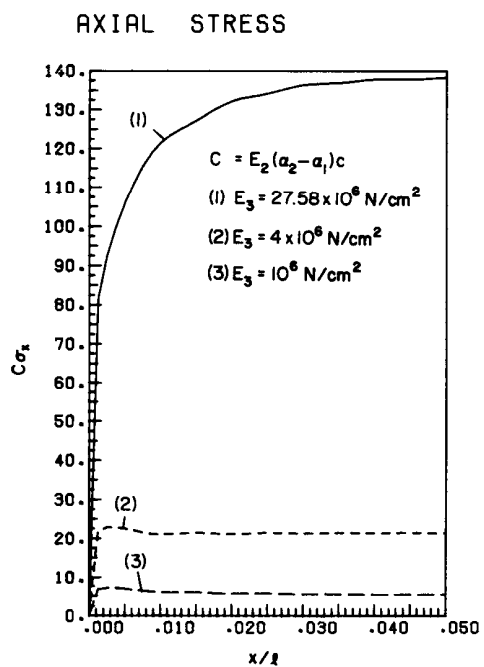


Figure 12. - Axial stress distribution along the lower surface of the bonding material.

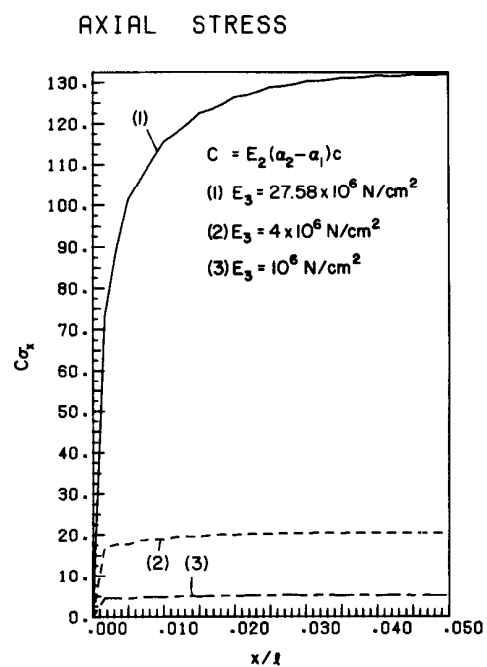


Figure 13. - Axial stress distribution along the upper surface of the bonding material.

APPENDIX--THERMAL STRESSES ALONG BEARING SURFACES OF A TWO-LAYER

ISOTROPIC BEAM--A COMPUTER PROGRAM

```

#GSP,P
PLOTTER PEN/LIQ
#FOR,IS MAIN
    IMPLICIT DOUBLE PRECISION (A-H,O-Z)
    COMMON/BLK2/ N
    COMMON/BLK3/ NUMLAY
    COMMON/BLK7/NTERMS
    COMMON/BLK12/X(301),Y(301)
    COMMON/BLK13/CC(8,1)
    COMMON/BLK20/ AA(8,8),BB(8,1)
    CALL MTAMDF (AA,8,8,'D',8,8,'GEN')
    CALL MTAMDF (BB,8,1,'D',8,1,'GEN')
    CALL MTADEF (CC,8,1,'D')

C
C THIS PROGRAM CALCULATES THE STRESSES AT THE INTERFACE OF THE TWO
C ISOTROPIC LAYERS (YC = 0.). THE NORMAL, SHEAR, AND AXIAL STRESSES
C ARE PLOTTED FOR THE UNIFORM TEMPERATURE CASE.
C
C READ THE LAST FOURIER TERM TO BE USED. (SHOULD BE ODD) (I4)
    READ (5,201) NTERMS
    CALL INPUT
    WRITE (6,290) NTERMS
    I = 1
    YC = .0
    DO 210 N = 1,NTERMS,2
    CALL DETMAT
    CALL SOLVE
    CALL COEF(I,YC)
210 CONTINUE
    NPOINT = 51
    XSTART = 0.
    XEND = .1
    CALL STRESY (1,I,NPOINT,YC,XSTART,XEND)
    CALL GRAPH (X,+2,Y,+2,NPOINT,'NONE','SOLID',' X/L $$','  $$'
1,'NORMAL','NORMAL STRESS$$','FULL','BIND')
    XEND = .05
    CALL STRESY (2,1,NPOINT,YC,XSTART,XEND)
    CALL GRAPH (X,+2,Y,+2,NPOINT,'NONE','SOLID',' X/L $$','  $$'
1,'NORMAL','SHEAR STRESS$$','FULL','BIND')
    XEND = .08
    CALL STRESY (3,1,NPOINT,YC,XSTART,XEND)
    CALL GRAPH (X,+2,Y,+2,NPOINT,'NONE','SOLID',' X/L $$','  $$'
1,'NORMAL','AXIAL STRESS$$','FULL','BIND')
    CALL STRESY (3,2,NPOINT,YC,XSTART,XEND)
    CALL GRAPH (X,+2,Y,+2,NPOINT,'NONE','SOLID',' X/L $$','  $$'
1,'NORMAL','AXIAL STRESS$$','FULL','BIND')
299 STOP
201 FORMAT (I4)
290 FORMAT (1X,/,30X,'THE LAST FOURIER TERM = ',I4)
END

```



```

*FOR,IS INPUT
  SUBROUTINE INPUT
    IMPLICIT DOUBLE PRECISION (A-H,O-Z)
C   THIS ROUTINE READS THE MATERIAL PROPERTIES FOR THE 2 OR 3 ISOTROPIC
C   LAYERS WHICH ARE UNDER A UNIFORM TEMPERATURE DISTRIBUTION.
C
    COMMON/BLK1/E1,E2,E3,XNU1,XNU2,XNU3,ALPHA1,ALPHA2,ALPHA3,BETA1,
1BETA2,BETA3
    COMMON/BLK3/ NUMLAY
    COMMON/BLK8/XLEN
    COMMON/BLK10/DELTA(3)
C
C   IF ONLY TWO LAYERS ARE TO BE CONSIDERED, THE CONSTANTS FOR LAYER
C   NUMBER 3 SHOULD BE SET EQUAL TO ZERO.
C   READ THE NUMBER OF LAYERS. (FIRST COLUMN IN INPUT CARD)
    READ (5,100) NUMLAY
    WRITE (6,180)
    WRITE (6,190) NUMLAY
C
C   READ THE MATERIAL PROPERTIES. (3F20.8)
    WRITE (6,181)
    READ (5,101) E1,E2,E3
    WRITE (6,191) E1,E2,E3
    READ (5,101) XNU1,XNU2,XNU3
    WRITE (6,192) XNU1,XNU2,XNU3
    READ (5,101) ALPHA1,ALPHA2,ALPHA3
    WRITE (6,193) ALPHA1,ALPHA2,ALPHA3
C
C   READ THE THICKNESS OF EACH LAYER.
    READ (5,101) (DELTA(I),I=1,3)
    WRITE (6,194) (DELTA(I),I=1,3)
C   XLEN = THE LENGTH OF THE BEAM/PI.
    READ (5,101) XLEN
    WRITE (6,195) XLEN
    BETA1 = DELTA(1)/XLEN
    BETA2 = DELTA(2)/XLEN
    BETA3 = DELTA(3)/XLEN
    RETURN
100 FORMAT (I1)
101 FORMAT (3F20.8)
180 FORMAT ('1',37X,'THERMAL STRESSES IN LAMINATED BEAMS OF ISOTROPIC
1MATERIALS',//)
181 FORMAT (1X,/,45X,'LAYER 1',13X,'LAYER 2',13X,'LAYER 3')
190 FORMAT (1X,/,45X,'NUMBER OF ISOTROPIC LAYERS = ',I1,//)
191 FORMAT (1X,/,10X,'YOUNG S MODULOUS',12X,E15.5,2(5X,E15.5))
192 FORMAT (1X,/,10X,'POISSON S RATIO',13X,E15.5,2(5X,E15.5))
193 FORMAT (1X,/,10X,'THERMAL EXPANSION COEF',6X,E15.5,2(5X,E15.5))
194 FORMAT (1X,/,10X,'THICKNESS',8X,E15.5,2(5X,E15.5))
195 FORMAT (1X,/,30X,'(BEAM LENGTH) / PI =',E15.5)
END

```

```

*FOR,IS DETMAT
  SUBROUTINE DETMAT
    IMPLICIT DOUBLE PRECISION (A-H,O-Z)
C   THIS SUBROUTINE DETERMINES THE MATRICIES FOR THE TWO LAYER ISOTROPIC
C   AND UNIFORM TEMPERATURE CASE
C
    COMMON/BLK1/E1,E2,E3,XNU1,XNU2,XNU3,ALPHA1,ALPHA2,ALPHA3,BETA1,
1  BETA2,BETA3
    COMMON/BLK2/ N
    COMMON/BLK4/A(8,8),B(8)
C
    IF (N.GT.1) GO TO 10
    D1 = E2/E1
    D2 = 1. + XNU1
    D3 = 1. + XNU2
    DO 1 I = 1,8
    B(I) = 0.
    DO 2 J = 1,8
    A(I,J) = 0.
2  CONTINUE
1  CONTINUE
    A(1,1) = 1.
    A(2,5) = 1.
    A(3,3) = 1.
    A(4,7) = -1.
    A(5,1) = 1.
    A(5,5) = -1.
    A(6,3) = 1.
    A(6,4) = 1.
    A(6,7) = -1.
    A(6,8) = -1.
    A(8,3) = D1*D2
    A(7,1) = A(8,3)
    A(7,2) = 2.*D1
    A(7,5) = -1.*D3
    A(7,6) = -2.
    A(8,4) = -1.*D1*(1. - XNU1)
    A(8,7) = A(7,5)
    A(8,8) = 1. - XNU2
10  XN = N
    B1N = XN*BETA1
    B2N = XN*BETA2
    TH1 =DTANH(B1N)
    TH2 =DTANH(B2N)
    A(1,2) = B1N*TH1
    A(1,3) = TH1
    A(1,4) = B1N
    A(3,1) = TH1
    A(3,2) = B1N + TH1
    A(3,4) = 1. + A(1,2)
    A(2,6) = B2N*TH2
    A(2,7) = -1.*TH2
    A(2,8) = -1.*B2N
    A(4,5) = TH2

```

```

A(4,6) = B2N + TH2
A(4,8) = -1. - (B2N*TH2)
B(7) = 1./XN
RETURN
END

```

*FOR,IS SOLVE

```

SUBROUTINE SOLVE
IMPLICIT DOUBLE PRECISION (A-H,O-Z)
C THIS SUBROUTINE SOLVES THE MATRIX EQUATIONS FOR A GIVEN N FOR THE
C TWO OR THREE LAYER CASE. THE REQUIRED INPUT IS THE FOURIER NUMBER N,
C THE NUMBER OF LAYERS NUMLAY, AND THE MATRIX A AND VECTOR B FROM
C SUBROUTINE DETMAT. THE MACC SUBROUTINE MTSOLV IS UTILIZED.
C THE SOLUTION VECTOR IS CC(K,1).
COMMON/BLK2/ N
COMMON/BLK3/ NUMLAY
COMMON/BLK4/AMAT(8,8),BVEC(8)
COMMON/BLK13/CC(8,1)
COMMON/BLK20/ AA(8,8),BB(8,1)
MATSIZ = 4*NUMLAY
DO 410 I = 1,MATSIZ
BB(I,1) = BVEC(I)
DO 420 J = 1,MATSIZ
AA(I,J) = AMAT(I,J)
420 CONTINUE
410 CONTINUE
CALL MTSLVD (AA,BB,CC,IRET)
299 RETURN
END

```

*FOR,IS COEF

```

SUBROUTINE COEF(I,YC)
IMPLICIT DOUBLE PRECISION (A-H,O-Z)
C THIS SUBROUTINE DETERMINES THE FOURIER COEFFICIENTS FOR THE STRESSES
C IN THE I TH LAYER ON Y = 0.
COMMON/BLK2/ N
COMMON/BLK8/XLEN
COMMON/BLK13/BB(8,1)
COMMON/BLK14/COEFN(1800),COEFS(1800),COEFA(1800,2)
ZN = N
M = (N + 1)/2
COEFN(M) = BB(1,1)
COEFS(M) = BB(3,1) + BB(4,1)
COEFA (M,1) = BB(1,1) + 2.*BB(2,1)
COEFA (M,2) = BB(5,1) + 2.*BB(6,1)
RETURN
END

```

*FOR,IS STRESY

```

SUBROUTINE STRESY (ITYPE,I,NPOINT,YC,XSTART,XEND)
IMPLICIT DOUBLE PRECISION (A-H,O-Z)

```

```

C   THIS SUBROUTINE COMPUTES THE NORMAL, SHEAR, OR AXIAL STRESSES ON
C   THE LINE Y = 0, AT NPOINT NUMBER OF POINTS FROM X = XSTART TO X =
C   XEND. (X = 0 TO 1 IS THE ENTIRE LENGTH OF THE BEAM) IF ITYPE = 1,
C   THE NORMAL STRESS IS COMPUTED, IF ITYPE = 2, THE SHEAR STRESS IS
C   COMPUTED, OR IF ITYPE = 3, THE AXIAL STRESS IS COMPUTED. THE (X,Y)
C   COORDINATES ARE RETURNED IN THE VECTORS X AND Y. THESE STRESSES
C   ARE CALCULATED FOR THE I TH LAYER.
      COMMON/BLK7/NTERMS
      COMMON/BLK8/XLEN
      COMMON/BLK12/X(301),Y(301)
      COMMON/BLK14/COEFN(1800),COEFS(1800),COEFA(1800,2)
      IF (ITYPE.EQ.2) GO TO 600
      IF (ITYPE.EQ.3) GO TO 611
      WRITE (6,691) YC
      GO TO 601
600  WRITE (6,692) YC
      GO TO 601
611  WRITE (6,671) YC
601  WRITE (6,693) I,NPOINT
      WRITE (6,694) XSTART,XEND
C   THIS IS DONE SO BOTH END POINTS WILL BE PLOTTED . HOWEVER, IF NPOINT
C   = 1, THIS ROUTINE WILL FAIL.
      XPOINT = NPOINT - 1
C
      PI = 3.1415926536
      IF (ITYPE.EQ.2) GO TO 755
      IF (ITYPE.EQ.3) GO TO 756
      DO 700 J = 1,NPOINT
      ZJ = J - 1
      X(J) = ZJ*(XEND - XSTART)/XPOINT + XSTART
      ARG = X(J)*PI
      XSUM = 0.
      DO 720 N = 1,NTERMS,2
      M = (N + 1)/2
      XN = N
      XSUM = XSUM + COEFN(M)*DSIN(ARG*XN)
720  CONTINUE
      Y(J) = -1.*XSUM
700  CONTINUE
      GO TO 770
756  DO 701 J = 1,NPOINT
      ZJ = J - 1
      X(J) = ZJ*(XEND - XSTART)/XPOINT + XSTART
      ARG = X(J)*PI
      XSUM = 0.
      DO 721 N = 1,NTERMS,2
      M = (N + 1)/2
      XN = N
      XSUM = XSUM + COEFA(M,I)*DSIN(ARG*XN)
721  CONTINUE
      Y(J) = XSUM
701  CONTINUE
      GO TO 770
755  DO 710 J = 1,NPOINT

```

```

ZJ = J - 1
X(J) = ZJ*(XEND - XSTART)/XPOINT + XSTART
ARG = X(J)*PI
XSUM = 0.
DO 730 N = 1, NTERMS, 2
  M = (N + 1)/2
  XN = N
  XSUM = XSUM + COEFS(M)*DCOS(ARG*XN)
730 CONTINUE
Y(J) = -1.*XSUM
710 CONTINUE
770 RETURN
694 FORMAT (1X, ///, 20X, 'THIS STRESS IS TO BE PLOTTED FROM X = ', F6.3, '
1 TO ', F6.3)
691 FORMAT (1X, ///, 20X, 'THE NORMAL STRESS ON THE LINE Y = ', F6.3,
1' IS PLOTTED ')
692 FORMAT (1X, ///, 20X, 'THE SHEAR STRESS ON THE LINE Y = ', F6.3,
1' IS PLOTTED ')
671 FORMAT (1X, ///, 20X, 'THE AXIAL STRESS ON THE LINE Y = ', F6.3,
1' IS PLOTTED ')
693 FORMAT (20X, 'FOR LAYER NUMBER ', I1, ' . THE NUMBER OF POINTS PLOT
1TED IS ', I4, ' .')
780 FORMAT (I3)
781 FORMAT (F10.5)
790 FORMAT (1X, 40X, 'NUMBER OF POINTS TO BE PLOTTED = ', I3)
791 FORMAT (1X, 40X, ' LENGTH OF BEAM TO BE PLOTTED = ', F6.3)
END

```

1.5-27-6/80

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27 p. (USDA For. Serv. Res. Pap. FPL 375).

Two-dimensional elasticity theory is used so that beams composed of layers of orthotropic materials like wood can be studied. Stress intensity along the bearing surfaces of the layers is responsible for delamination failures. Distributions of interlaminar normal and shear stresses measured along the beam's longitudinal axis indicate that delamination failure, when it occurs, will start at the beam's ends.