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POINT-SAMPLING AND LINE-SAMPLING: PROBABILITY THEORY, GEOMETRIC IMPLICATIONS, SYNTHESIS

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POINT-SAMPLING AND LINE-SAMPLING: PROBABILITY THEORY, GEOMETRIC IMPLICATIONS, SYNTHESIS

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Foresters concerned with measuring tree populations on definite areas have long employed two well-known methods of representative sampling. In list or enumerative sampling the entire tree population is tallied with a known proportion being randomly selected and measured for volume or other variables. In area sampling all trees on randomly located plots or strips comprising a known proportion of the total area are selected and measured for volume or other variables. List or enumerative sampling is commonly used in timber sales employing sample-tree measurement, and area sampling in timber reconnaissance. Each method, in its simplest valid form, operates to give every tree in the studied population an equal chance of being selected. A class of trees, therefore, can expect to be sampled in proportion to the frequency of trees in that class, and the frequency of a single tree is one.

Modifications of the above two techniques are usually designed to avoid sampling an identical proportion of trees in every class, since many classes are of slight interest but great frequency. These modifications involve the use either of different sampling fractions for different tree classes, or--what amounts to the same thing--different plot or strip sizes for different tree classes. Such stratification and use of several different sampling fractions or plot sizes is the simplest valid case of sampling where individual trees are not given an equal opportunity of being selected. Within strata the sampling fraction or plot size is constant, and most foresters are familiar with the calculation of appropriate blow-up factors for different sampling fractions or plot sizes.

Few foresters, however, appear to be acquainted with the underlying concept of "probability sampling", especially p. p. s. sampling (6)[1] (p. p. s. denotes probability proportional to size). Familiarity with this concept is necessary to comprehend all the implications and potentialities of point-sampling and line-sampling. It may be worth while to give a brief explanation of probability sampling and to show how point- and line-sampling of trees are types of probability sampling.

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[1] Underscored numbers in parentheses refer to Literature Cited, p. 34.

PROBABILITY THEORY

GENERAL

Suppose that a tree population on a tract of land whose area is (A) is comprised of (M) trees, and that each has a different probability (Pi) of being selected by a single random sample. Suppose further that (n) such random samples have selected (m) sample trees (with replacement) and that a dimension or quantity (Yi) associated with each sample tree will be measured (as will its P_i). The variable (Yi) might be frequency (in which case each Yi = 1), diameter, basal area, volume, height, value, growth, or some other quantity. An unbiased sample-based estimate of the total value of (Y) for the population (i. e., $\sum^{M} Y_i$) would be $\sum^{m} \frac{Y_i}{nP_i}$ if (m) sample trees were selected by (n) equivalent samples and measured for (Yi) (with replacement). It is apparent that nP_i is merely the expected number of times that the i^{th} tree will be drawn by n equivalent samples.

There are two special cases of this theory which are simple and familiar to all foresters. If each of the (M) trees in the population has an equal chance of being included in a sample of (m) trees, then each $(nP_i) = \frac{m}{M}$, and the estimate simplifies to $\frac{M}{m} \sum^{m} Y_i$. Similarly, if a tract of area (A) contains (Na) acres, and if all (m) trees occupying (n) randomly located plots or strips, each of identical acreage (a), were measured for (Yij, then each tree on the area (A) would have an equal chance $\frac{a}{Na}$ of being selected in any given sample plot, and each $(nP_i) = \frac{na}{Na} = \frac{n}{N}$. The estimate of total (Y) on the area then becomes $\sum^{m} \frac{Y_i}{nP_i} = \frac{N}{n} \sum^{m} Y_i$. If the estimate of total (Y) is divided by (Na) to place it on a per-acre basis, it becomes $\frac{1}{na} \sum^{m} Y_i$. The same result can be demonstrated by regarding the center of each tree as surrounded by a circle or rectangle of constant size, with the tree being selected as a sample tree whenever a random sample point or line falls inside the circle or rectangle belonging to the tree. The "blow-up factors" $\frac{M}{m}$, $\frac{N}{n}$, and $\frac{1}{na}$ appropriate to a constant sampling fraction or a constant plot size are well known.

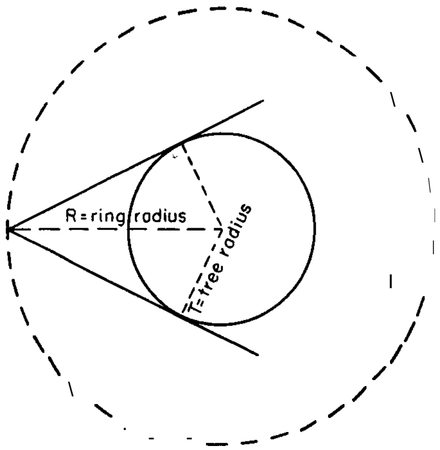
POINT-SAMPLING

Bitterlich (2) first employed the horizontal angle-gauge for estimating basal area density of trees per unit of land area by counting those trees whose d. b. h. subtended angles appearing larger than the horizontal angle-gauge; he did not visualize the implications of probability sampling which permitted sampling tree variables such as frequency, volume, height, and growth. Hirata (7) first employed the vertical angle-gauge and counted qualifying trees to estimate mean squared height.

Grosenbaugh (3, 4, 5) first recognized that any angle-gauge is actually a tool for selecting sample trees with probability proportional to some element of size, and postulated the theory of point-sampling to obtain unbiased estimates of frequency, volume, growth, value, height, etc. per acre from measurements of such p. p. s. sample trees.

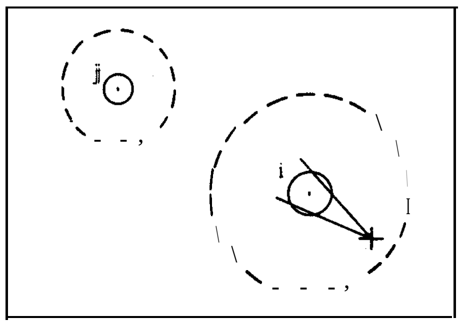
Horizontal point-sampling theory postulates that the vertex of a constant angle whose sides are exactly tangent to a circular tree cross section will generate a huge imaginary ring on a level plane around the tree if the vertex is pivoted about tree center (cf. fig. 1A). Vertical point-sampling postulates that the vertex of a constant vertical angle pivoted about a vertical segment of a tree will generate a huge imaginary ring on a level plane around the tree (cf. fig. 2A).

A. Circular geometry of horizontal angle-gouge



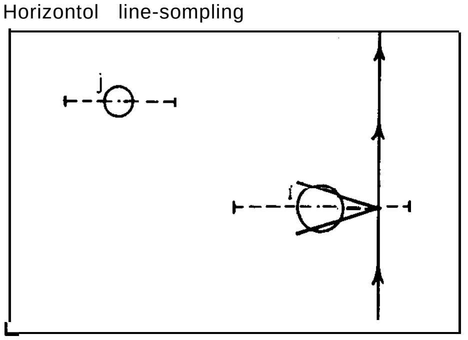
Imoginory ring generoted by vertex of horizontol ongle-gauge pivoting oround o circular tree cross section. Ring radius is always equal to tree rodius times a constant. The constant ($K = \frac{R}{r}$) depends on the size of the gouge-ongle.

B. Horizontol point-sompling



A somple point (+) rondomly locoted within o level, rectangular troct of land hos selected the $i^{th.}$ tree os o somple tree for meosurement, with probobility of selection (P_i) equol to $\left(\frac{K^2 \text{(tree basal area)}}{\text{troct area}}\right)$. Horizontal ongle-gouge with vertex at sample point (+) tells observer point lies inside imaginary ring of $i^{th.}$ tree (becouse tree d.b.h. oppeors to more than subtend angle-gouge), ond outside the imaginory ring of the $j^{th.}$ tree.

C. Horizontol line-sompling



A somple line (f) rondomly located within ond porollel to o side of o level rectongular troct of lond hos selected the $i^{th.}$ tree os o somple tree for meosurement, with probability of selection (P_i) equol to $\left(\frac{K \text{ (tree diameter)(line length)}}{\text{tract area}}\right)$. Horizontol ongle-gouge with vertex on somple line ond with bisector perpendiculor to line and passing through center of $i^{th.}$ tree tells observer thot projection of $i^{th.}$ tree rodius is intersected by line (because tree d.b. h. oppeors to more than subtend ongle-gouge). The line does not somple the projection of rodius of the $j^{th.}$ tree.

Figure 1. --*Horizontal point-sampling and line-sampling.*

Ring radius will be a gauge-determined constant (K) multiplied by tree radius if a horizontal gauge is used, or it will be a gauge-determined constant (Q) multiplied by tree height if a vertical gauge is used. Laying out these imaginary rings on the ground with an angle-gauge (as in fig. 1B and 2B) is not necessary, however, since the vertex of the angle-gauge at the sample point is known to be inside the ring whenever the tree diameter or height appears larger than the gauge, and outside whenever it appears smaller. Hence, an angle-gauge with its vertex at a sample point can be used to identify all sample trees within whose rings the sample point lies. As can be seen from figures 1B and 2B, trees of different sizes have different chances of having a sampling point fall within their rings. With the horizontal gauge, the probability is proportional to the square of d. b. h. (i.e., to tree basal area); with the vertical gauge, to the square of tree height.

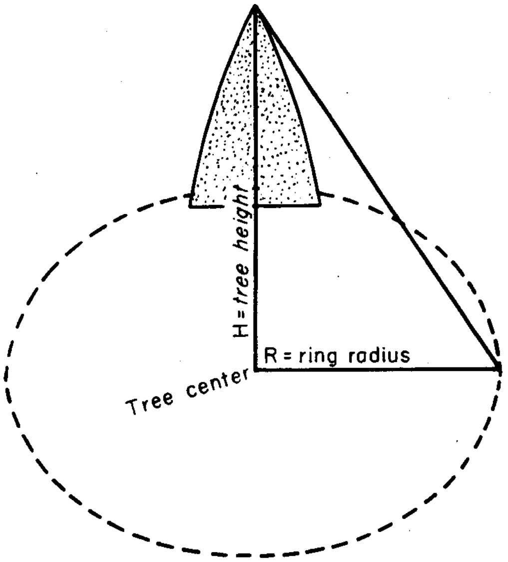
If Di = d. b. h. is the horizontally gauged tree dimension, the expected number of times that the i^{th} tree inside a tract of area (A) will selected by (n) random point-samples (with replacement if selected) is $nP_i = \left(\frac{nK^2D_i^2}{A}\right)\left(\frac{\pi}{4}\right) = \frac{nK^2B_i}{A}$, where $B_i = \frac{\pi}{4} D_i^2$ = tree basal area measured in the same units as land area (A), and where K = csc ½ horizontal gauge-angle. Where (Yi) is any desired variable associated with an individual tree (such as frequency, diameter, height, basal area, volume, value, growth, or other quantity), the unbiased estimate of $\sum^{M} Y_i$ (or the total (Y) for all M trees on the tract) is given by $\sum^{m} \frac{Y_i}{nP_i} = \frac{A}{nK^2}\sum^{m} \frac{Y_i}{B_i}$ where (m) sample trees have been selected by (n) point-samples (with replacement) and where each sample tree has been measured as to its (Bi) and (Y_i). Dividing through by tract area (A) reduces the expression to the estimate of Y per unit of land area, thus: $\frac{1}{nK^2}\sum^{m}\frac{Y_i}{B_i} = \frac{4}{n\pi K^2}\sum^{m}\frac{Y_i}{D_i^2}$. No measurement of (Yi) or (Bi) is necessary when (Yi) is chosen identical with present basal area (B_i), since then $\sum^{m}\frac{Y_i}{B_i}$ = m, the fundamental count on which Bitterlich published.

If Hi = total tree height is the vertically gauged, point-sampled tree dimension, then Q^2 = $\cot^2$ vertical gauge-angle is used instead of $K^2 = \csc^2$ ½ horizontal gauge-angle, and πH_i^2 replaces $B_i = \frac{\pi}{4} D_i^2$, so that the estimate of Y per unit of land area is $\frac{1}{n\pi Q^2}\sum^{m}\frac{Y_i}{H_i^2}$, with (Yi) being any desired variable, as in horizontal point-sampling. Ordinarily, point-sampling with a vertical angle-gauge will be much less efficient than point-sampling with a horizontal angle-gauge, since tops are frequently invisible and since checking doubtful trees is very expensive.

LINE-SAMPLING

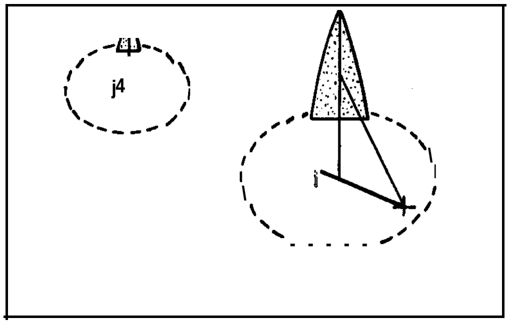
Line-sampling is an extension of point-sampling theory. Strand (8) first published on the use of line-sampling in forest inventory. It employs random-sample line- segments instead of sample points, and the probability per unit-length of line that a particular tree will be selected is proportional to tree diameter (horizontal gauging) or to tree height (vertical gauging) instead of proportional to the square of these dimensions as in point-sampling. The angle-gauge is used to select sample trees on both sides of sample line- segments located at random on the tract. Often continuous and parallel lines will be used; they are analogous to strips. At other times short discontinuous segments, analogous to rectangular plots, will be preferred.

A. Circulor geometry of vertical angle-gouge



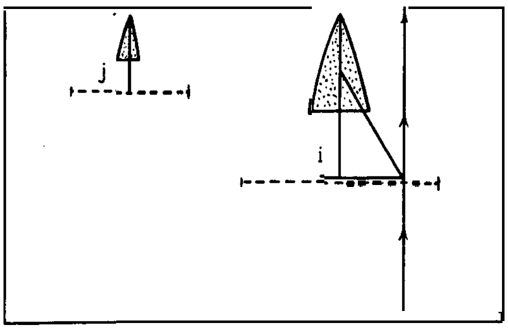
Imaginary ring (in perspective) generated by vertex of vertical ongle-gouge pivoting oround o vertical tree height. Ring rodius is olwoys equal to tree height times o constont. The constont $(Q = \frac{R}{H})$ depends an the size of the gouge-angle.

B. Vertical point-sompling



A sample point(+) rondomly located within a level, rectongulor troct of lond has selected the i^{th} tree os o somple tree for meosurement, with probobility of selection (P_i) equol to $\left(\frac{\pi Q^2 (\text{tree height})^2}{\text{troct oreo}}\right)$. Verticol ongle-gouge with vertex at sample point (+) tells observer that point lies inside the imoginory ring of the i^{th} tree (because tree height more than subtends ongle-gouge), and outside the imoginary rinq of the j^{th} tree.

C. Vertical line-sampling



A sample line (↕) rondomly locoted within and porallel to o side of a level rectongulor troct of land hos selected the i^{th} tree os o somple tree for measurement, with probability of selection (P_i) equal to $\left(\frac{2\,Q(\text{tree height})(\text{line length})}{\text{tract oreo}}\right)$. Verticol angle-gouge with vertex on somple line (↕) held in plane narmol to somple line with base level tells observer that i^{th} tree height projected ot right ongles to sample line will be intersected by line (becouse tree height appears to more than subtend angle-gouge), and that the j^{th} tree height will not be intersected.

Figure 2. --*Vertical point-sampling and line-sampling.*

Any tree with a radius (or height) projection at right angles to and intersecting the sample line-segment is a sample tree (cf. figures 1C and 2C). Critical gauging of doubtful trees or exact comparisons of distance with tree dimension must be made perpendicular to the sample line.

If D_i = d. b. h. is the horizontally gauged tree dimension, the expected number of times that the i^{th} tree inside a tract of area (A) will be selected by n equivalent randomly located sample line-segments of aggregate length (L) is $nP_i = \frac{LKD_i}{A}$, where K = csc ½ horizontal gauge-angle, and where (A) is measured in the square of the units in terms of which both (L) and (Di) are measured. The unbiased estimate of $\sum^{M} Y_i$ (or the total (Y) for all (M) trees on the tract) is given by $\sum^{m} \frac{Y_i}{nP_i} = \frac{A}{LK} \sum^{m} \frac{Y_i}{D_i}$ where (m) sample trees have been selected and each has been measured as to its (Di) and (Yi), which last might be any desired variable associated with the i^{th} tree--e.g., frequency, diameter, height, basal area, volume, value, growth. Dividing through by tract area (A) reduces the expression to the estimate of Y per unit of land area, thus: $\frac{1}{LK} \sum^{m} \frac{Y_i}{D_i}$. No measurement of (Y_i) or (Di) is necessary when (Yi) is chosen identical with present diameter (D_i), since then $\frac{1}{LK} \sum^{m} \frac{Y_i}{D_i} = \frac{m}{LK}$ where (m) is merely a tree count (by class if desired). Such line-sampling might be useful in estimating sum of cull tree diameters per acre to be girdled or poisoned.

If H_i (or length of stem on the i^{th} tree from breast height to some specified point such as tree top, pole top, merchantable top) is the vertically gauged tree dimension in line-sampling, then Q = cot vertical gauge-angle is used (instead of K = csc ½ horizontal gauge-angle)and (2Hi) replaces (D_i); the estimate of Y per unit of land area becomes $\frac{1}{2LQ} \sum^{m} \frac{Y_i}{H_i}$. Then if (Yi) be chosen identical with (Hi), total length (above breast height) of tree stems per unit of land area can be estimated simply as $\frac{m}{2LQ}$, where (m) is merely a sample-tree count (by class desired). Such line-sampling might be useful in estimating lineal feet of poles (above breast height) per unit of land area.

Of course, where sampling is restricted to only one side of the line, all preceding line-sampling formulae must be doubled.

The foregoing discussion of both point- and line- sampling assumes uniform units of measure, but scale factors involving different units such as inches, feet, chains, and acres can readily be introduced.

Geometric assumptions inplicit in the preceding discussion are:

(1) Enlarged tree rings or projected radii or heights never extend beyond sampling universe boundaries.
(2) Effective size of angle-gauge is known, and outcome of comparing angle-gauge with tree dimension is unambiguous and consistent.
(3) Terrain is level.
(4) Trees are truly vertical.
(5) Sample trees are visible from points or lines which select them,
(6) Tree cross sections are truly circular.

The next sections will discuss how contradicting each of these assumptions affects the probability of a given tree's being selected, and how the changed situation in turn must be met either by modifying the size, orientation, or sector swept by the angle-gauge, by adjusting the blow-up factors of individual trees, or by using auxiliary methods. Failure to take the appropriate action will inject a bias into the estimate. That appropriate procedures will prevent appreciable bias even when point-sampling is conducted in a routine manner by cruisers under field conditions has been established by Grosenbaugh and Stover (5). Relative efficiencies of various sampling methods and angle-gauges will depend on local investigations of relative variance and relative costs; indications are that in many situations point-sampling will be more efficient than plot- sampling or line- sampling.

GEOMETRIC IMPLICATIONS

ENLARGED TREE RINGS OR DIMENSIONS PROJECTING BEYOND SAMPLING UNIVERSE BOUNDARIES ("SLOPOVER")

The situation where rings or radii do not overlap (illustrated in figures 1 and 2) is rarely encountered. Usually enlarged tree rings or projected dimensions, be they vertically or horizontally generated, overlap one another. This does absolutely no harm, nor does it affect procedure.

Besides this, two different sample points or lines (located in an unbiased fashion) may each sample the same tree. This again creates no bias. In fact, sampling without replacement of the sampled tree in the tree population (where it may be sampled again) would lead to bias unless unusual procedures were followed.

Enlarged tree rings or dimensions may project beyond the boundary of the tract. With large tracts, this slopover is inconsequential, but theoretically it injects a bias (which may be very large on very small tracts of land) unless special precautions are taken.

The bias arises because random or systematic location of sample points or lines is limited by the tract boundaries, so that the trees with enlarged rings or dimensions projecting beyond the boundary have less chance of being sampled than their size (unadjusted for slopover) would indicate.

If sample points or lines are arbitrarily restrained from falling in the peripheral zone where slopover occurs, an edge-effect bias will result because peripheral trees will not be represented in the sample as heavily as their occurrence in the population warrants.

The best way of eliminating slopover bias is to specify peripheral zones in advance of sampling. Trees with centers in the interior zone will generate whole circles or will project heights or radii on both sides of the tree. Point- sampled trees with centers in peripheral zones will be allowed to generate only half- or quarter-circles away from the outside boundary; line-sampled trees with centers in peripheral zones will be allowed to project heights or radii in only one direction away from the outside boundary. All point-sampled trees are weighted 1, 2, or 4 depending on whether they were allowed to generate whole, half, or quarter circles. All line-sampled trees are weighted 1 or 2 depending on whether they were allowed to project height or radius in 2 directions or only 1 direction.

The practical effect of these geometric limitations on point-sampling is that when sample points fall within these peripheral zones, normally qualified trees with centers in these zones can qualify for tally only if they occur within a 90° *or* 180° outwards sweep of an angle-gauge and have rings overlapping the sample point. Regardless of whether a sample point is in the interior zone or in a peripheral zone when it selects a sample tree, the sample tree must be given a weight appropriate to its own zone (1, 2, 4). The practical effect of these geometric limitations on line-sampling is that when sample lines extend into these peripheral zones, normally qualified trees with centers in these zones can qualify for tally only if they occur on the side of the line nearest the outer boundary. Regardless of whether a sample line is in the interior or in a peripheral zone when it selects a sample tree, the sample tree must be given a weight appropriate to its own zone (1, 2).

Figure 3A illustrates, for point-sampling, how a rectangular tract should be divided into 8 peripheral zones (4 corner zones, 4 side zones) and an interior zone. Peripheral zone width should be a little wider than the radius of the maximum tree ring expected. Tree A, if selected by a sample point in the interior zone,would be given weight 1. Tree B might be selected by a sample point in the interior zone, but it could also be selected by points in a corner zone or two side zones; in any case, it would be given weight 1. Tree C could only be tallied from a sample point in the side zone, from which an outward 180° sweep with the angle-gauge (commencing and ending parallel to the outer boundary) would tally the tree, and it would be given weight 2. Tree D could be tallied from sample points in the interior zone, a corner zone, or either of 2 side zones, but in any case it would be given weight 2.

Tree E in Figure 3A could be tallied only from a point in the corner zone, from which an outward 90° sweep with the angle-gauge (commencing parallel to one outside boundary and ending parallel to the other) would tally the tree; it would be given weight 4. Tree F might be similarly selected from a point in the corner zone, but it could also be selected from points in two side zones and the interior zone; in all cases, it would be given weight 4.

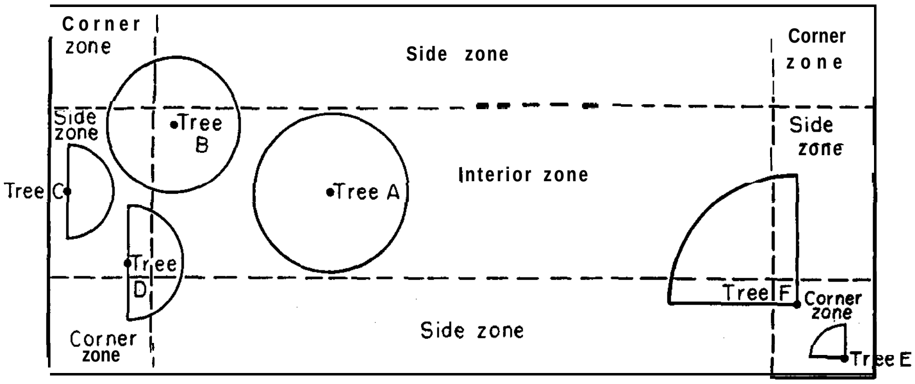
Trees B and D are illustrations of the fact that sometimes sample points falling in corner zones will tally some nearby trees with centers in other zones, with sweep and weights appropriate to the other zones. Trees B and F indicate that a similar phenomenon is possible in the side zones, and trees F and D illustrate that a similar situation can exist near the edge of the interior zone. In the interior zone, angle-gauge sweep is unlimited, but near the margin of the interior zone some selected trees may lie in a peripheral zone, and will be weighted accordingly.

It is apparent that non-rectangular, obtuse-, or acute-angled tracts may be handled by an extension of this technique (with more or fewer peripheral zones, with fractional circles and sweeps involving 30°, 60°, or 120°, etc., and weights such as 12, 6, 3, etc.).

Figure 3B demonstrates that the problem of slopover is much simpler in line-sampling rectangular tracts than in point-sampling them. Sample lines must be run parallel to 2 sides, and only two peripheral zones are needed (both are side zones--there are no corner zones). These side zones are bounded outwardly by a side paralleling the sample line. They should be somewhat wider than the largest projected tree height or radius expected, as in point-sampling. Also as in point-sampling, qualified trees with centers in the interior zone are tallied regardless of whether the line sampling them lies in the interior zone or a side zone, and they are

8 - 9

A. Technique when point-sompling o rectongulor tract; tracts of other shapes ore similorly hondled, with various froctionol circles being used in corner zones.



B. Technique when line-sompling o rectangular tract porollel to sides.

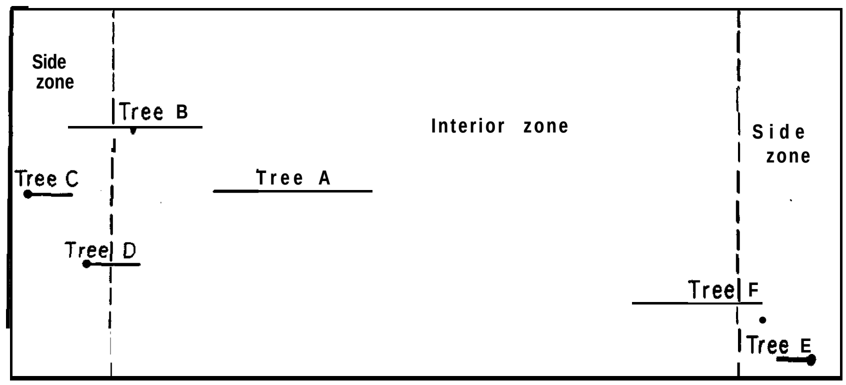


Figure 3. *--Eliminating slopover bias in point-sampling and line-sampling.*

always given weight 1 (fig. 3B, trees A and B). Trees with centers in the side zones which are otherwise qualified are tallied only if they lie on the outer side of the sample line, and they are always given weight 2, regardless of whether the line sampling them is in the side zone or the interior zone (fig. 3B, trees C, D, E, F). It will be noted that interior-zone tree B can be tallied from a point in the side zone even though it lies inwards from a sample line in the side zone, and it will always be given weight 1. Also, trees D and F can be tallied from either interior or side zone, and they will still be given weight 2.

Unfortunately, when non-rectangular tracts are to be line-sampled, the slopover problem becomes much more complex. The simplest way to handle such situations is to subdivide irregularly shaped tracts into smaller rectangular units that can be line-sampled by judicious delineation of the usual side zones, with line-samples running parallel to such zones. Peripheral zones inside of the tract but outside of the area included in the various rectangular subdivisions will have to be plot-sampled. This solution, while quite feasible on large tracts, becomes awkward on small tracts or tracts where interior angles do not lie between 90 and 180 degrees. Because of slopover complications, therefore, line-sampling will probably not be very useful on small non-rectangular tracts. Point-sampling, however, by use of partial sweeps, is well adapted to tracts of any shape.

The above solutions to the slopover problem were first devised by Grosenbaugh (4). Precisely the same slopover bias has long gone unrecognized in circular plot-sampling where random or systematic plot-center location is treated as a point that moves as a continuous variable. Here each tree can be visualized as surrounded by a circle of constant size (independent of tree size), and the so-called plot center is merely a sampling-point which selects any trees within whose rings it falls. The solution is the same as outlined above; translated into plot terms, it is equivalent to the use of half- and quarter-plots with straight sides adjacent to and parallel to outside boundaries when in peripheral zones (zones should be equal in width to so-called plot radius). Even with triangular or rectangular plots, the same slopover occurs if plot-center location is treated as a continuous variable. Here, trees can be regarded as surrounded by triangles or rectangles of constant size instead of by rings. The solution is still the same-- use of half- and quarter-plots in the peripheral zone, with straight sides adjacent to and parallel to outside boundaries.

As has been said, on large areas slopover bias is of small magnitude and has been ignored in conventional plot- sampling. However, the solution is relatively simple for those who care to use it. Only in line-sampling of small, non-rectangular tracts could it become a troublesome procedure.

One last point needs to be mentioned. If tree population outside a tract is exactly the same as that inside a tract, no slopover problem exists. Points in the peripheral zones merely tally all qualifying trees in a 360° sweep, whether inside the tract or outside, and give all trees equal weight. However, equating off-tract trees to on-tract trees is often an unwarranted assumption, and trees outside the tract should ordinarily always be excluded from an estimate.

DETERMINING EFFECTIVE ANGLE-GAUGE SIZE, CHECKING DOUBTFUL TREES, CALCULATING CONVENIENT GAUGE CONSTANTS

Regardless of the angle supposedly represented by an angle-gauge, each prospective user should carefully ascertain for his own eyes the distance from the angle vertex to a target of known width when the target exactly coincides with the optical projection of the angle--a process called "calibration". The ratio of this distance (when coincidence is deemed perfect) to the width of target is called the calibration distance factor (X) if the target is perpendicular to the bisector of the angle along which distance is measured. It is called the vertical distance factor (Q) if the target is perpendicular to one side of the angle along which distance is measured. Calibration factors for a given angle-gauge may vary slightly with individuals, because of physiological and psychological differences.

Figures 4A and B illustrate the two types of calibration. The first, which ascertains X, is convenient for calibrating horizontal angle-gauges. The second, which ascertains Q, is convenient for vertical angle-gauges. Q may be directly measured in a vertical or a horizontal plane or it may be indirectly calculated from X, since $Q = X - \frac{1}{4X}$. If the horizontal gauge-angle is called θ, then $X = \frac{1}{2}\cot\frac{\theta}{2}$; if the vertical gauge-angle is called ϕ, then $Q = \cot\phi$.

There is still a third factor (K) which was used earlier in figure 1A to explain horizontal point-sampling and line-sampling. It is the radial enlargement factor, the ratio of imaginary ring radius to tree radius, $K = \csc\frac{\theta}{2}$. These three basic ratios, undistorted by scale differences, are related to each other and to gauge-angle in the following way: $K^2 = 4X^2 + 1 = (Q + \sqrt{Q^2+1})^2 + 1 = \frac{1}{\text{haversin }\theta}$.

These scale-free calibration values of X (converted to K or Q), coupled with the basic formulae given on pages 3-6 are sufficient to allow estimation of Y per unit of land area (where Y is tree frequency, diameter, height, basal area, volume, value, growth, etc.) from horizontal or vertical point-sampled or line-sampled tree measurements. Whenever the "tally" or "non-tally" status of a possible sample tree is doubtful even after optical gauging, a check should be made of the distance from angle vertex to heart center of tree (in a level plane). All "tally" trees must be closer to the angle vertex than K times tree radius in horizontal gauging, or than Q times tree height above breast height in vertical gauging.

A. Technique for calibrating horizontol gouge-ongle θ in terms of calibration distance required to find X.

B. Technique for calibrating vertical gouge-angle ϕ in terms of calibration distance required to find Q.

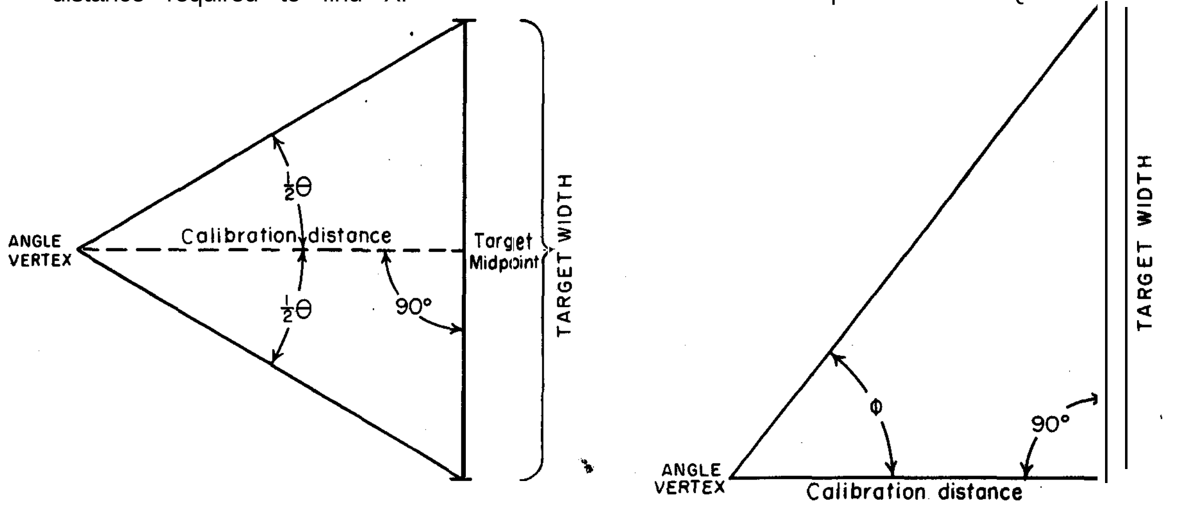


Figure 4.--*Angle-gauge calibration.*

Two instrumental peculiarities must be taken into account in calibration or doubtful tree check. If a magnifying stadia-type angle-gauge is used, the vertex of the angle will occur one focal length in front of the objective lens, and this is the point from which distances should be measured in calibration or doubtful tree check; it is also the point which should be kept above the sample-point or sample line.

If a hand-held wedge-prism is used as a horizontal angle-gauge, it should be positioned so that the knife-edge of the prism is vertical and so that the plane bisecting the prism-wed'ge is parallel to a vertically edged target. When one target edge viewed directly over the top of the prism appears to coincide exactly with the deflected ray from the other target edge (with prism positioned as above), the exit ray of the deflected beam makes the same angle leaving the rear glass surface as it did when entering the front glass surface. This horizontal deflection is the effective gauge-angle, with vertex at the intersection of the prism-wedge bisector and the perpendicular bisector of the target. If the prism is thus positioned, any rotation of the prism in the plane of the prism-wedge bisector will reduce the horizontal component of target-edge deflection; any rotation of the prism in either the plane of the target bisector or a level plane will increase the horizontal component. Taking advantage of this (an action analogous to "swinging" a hand-held sextant so as to ensure measuring the minimum angle between horizon and lower limb of sun) affords a check on proper hand-held positioning of prism.

It should be noted that deflections in prism-diopters, as usually measured by manufacturers of prisms, assume that one prism surface (rather than prism bisector). will be parallel to target and that the exit ray of the. deflected beam will be perpendicular to one or the other glass surface. This means that so-called normal deflections will always be slightly greater than minimum deflections determined as in the preceding paragraph. Figures 5A and B illustrate the different positioning of the prism for calibration in terms of minimum deflection (exit and entrance rays making equal angles with prism surface) as compared with normal deflection (exit ray perpendicular to one or the other prism surface). Hand-held prisms can be consistently used only if calibrated in terms of minimum deflection, but telescopic instruments should utilize normally oriented and calibrated prisms in their optics. The relationship between minimum and normal deflections of a wedge-prism is a complex one, depending on prism-angle, type of glass, and Snell's Law of Refraction:

Let d = minimum deflection-angle.

D' = deflection-angle with exit ray normal to glass surface nearest target.

D = deflection-angle with exit ray normal to glass surface nearest eye.

P = wedge-angle of prism (i.e., angle between glass surfaces)

G = ratio of $\frac{\text{refractive index of glass}}{\text{refractive index of air}}$ (commonly, 1.523 for crown glass and sodium light).

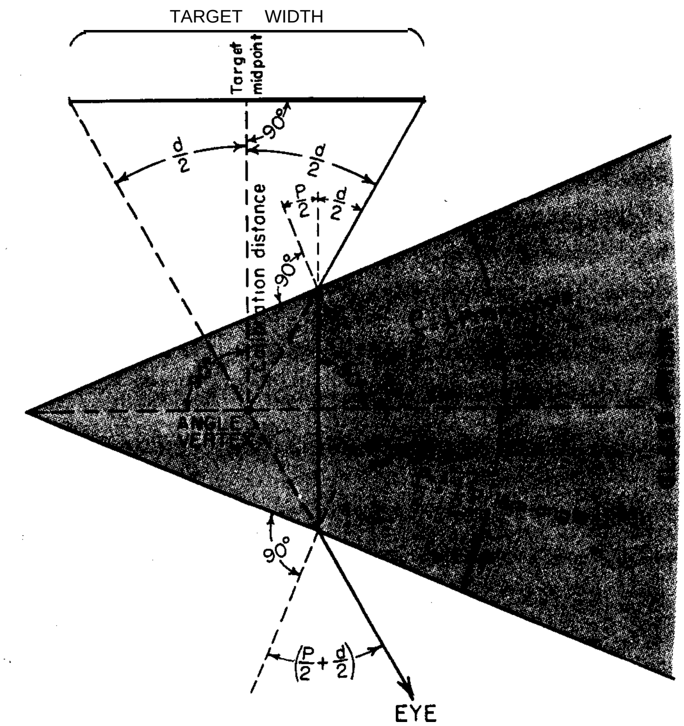
Then $d = \left[2 \arcsin \left(G \sin \frac{P}{2}\right)\right] - P$

$D' = \arcsin \left[G \sin\left(P - \arcsin \frac{\sin P}{G}\right)\right]$

$D = \left[\arcsin (G \sin P)\right] - P$

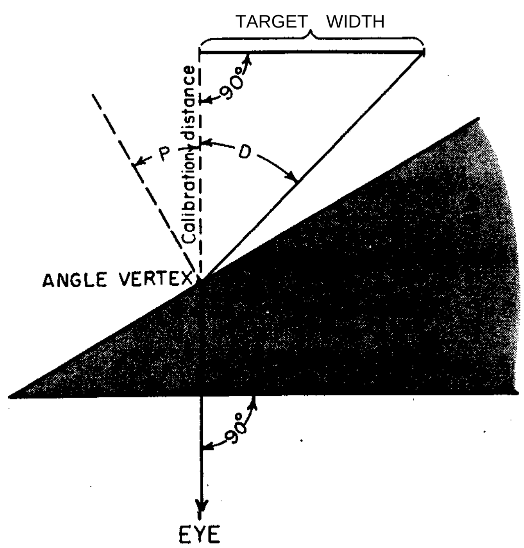
When normal calibration is employed, the ratio of target width to calibration distance that is found directly is Q(not X), and when prisms so calibrated are used as horizontal angle-gaug'es, Q must be converted to X or K, or else used in appropriate formulae (fig. 6).

A. "Minimum" calibration .far use as hand-held horizontal angle-gauge
(prism oriented to estoblish minimum deflection angle d)

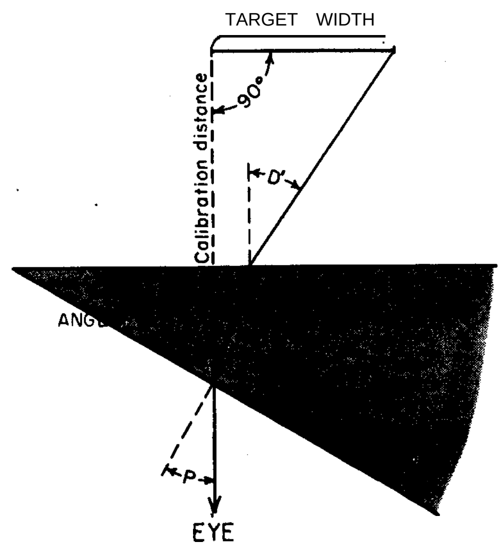


Deflection angle (d) has vertex on prism-bisector, has entrance angle equal to exit angle($\frac{P}{2}+\frac{d}{2}$ in each case), with deflection-angle bisector perpendicular to prism-angle bisector.

B. "Normal" calibration far use in telescopic ongle-gauge
(prism oriented to establish deflection-angle D or D' with exit roy normal to o prism-face;
D > D' > d for o given prism, but D' is only slightly larger thon d)



Deflection·angle (D) hos vertex an prism-face farthest from eye, has entronce angle (P t D), zero exit angle, with exit ray perpendicular to prism-face neorest to eye.



Deflection angle (D') has vertex somewhere in prism interior, hos entrance angle (D'), exit angle (P), with exit ray perpendicular to prism-face forthest from eye.

Figure 5. --Wec-ige-prism calibration.

HORIZONTAL ANGLE-GAUGE

CONSTANT	IDENTITIES
Calibration Distance Factor (X)	$X = \frac{1}{2}\sqrt{K^2-1} = \frac{1}{2}\left(Q+\sqrt{Q^2+1}\right) = \frac{1}{2}\sqrt{(24\,HDF)^2-1} = \frac{1}{2}\sqrt{\frac{7{,}986{,}573}{HPF}-1} = \frac{1}{2}\sqrt{\left(\frac{7920}{HLF}\right)^2-1} = \frac{1}{2}\sqrt{\frac{1+\cos\theta}{1-\cos\theta}} = \frac{1}{2}\cot\frac{\theta}{2}$
Radial Enlargement Factor(K)	$2\sqrt{X^2+\frac{1}{4}} = K = \sqrt{\left(Q+\sqrt{Q^2+1}\right)^2+1} = 24\,HDF = \sqrt{\frac{7{,}986{,}573}{HPF}} = \frac{7920}{HLF} = \sqrt{\frac{1}{hav\,\theta}} = \csc\frac{\theta}{2}$
Gauge-Angle Cotangent (0)	$X-\frac{1}{4X} = \frac{K^2-2}{2\sqrt{K^2-1}} = Q = \frac{(24\,HDF)^2-2}{2\sqrt{(24\,HDF)^2-1}} = \frac{\frac{7{,}986{,}573}{HPF}-2}{2\sqrt{\frac{7{,}986{,}573}{HPF}-1}} = \frac{\left(\frac{7920}{HLF}\right)^2-2}{2\sqrt{\left(\frac{7920}{HLF}\right)^2-1}} = \cot\theta = \frac{\cot\frac{\theta}{2}-\tan\frac{\theta}{2}}{2}$
Horizontal Distance Factor (HDF) (*maximum allowable distance in feet per inch of tree diameter*)	$\frac{\sqrt{X^2+\frac{1}{4}}}{12} = \frac{K}{24} = \frac{\sqrt{\left(Q+\sqrt{Q^2+1}\right)^2+1}}{24} = HDF = \frac{117.7522}{\sqrt{HPF}} = \frac{330}{HLF} = \frac{1}{24}\sqrt{\frac{1}{hav\,\theta}} = \frac{\csc\frac{\theta}{2}}{24}$ = 2.75 $\approx\sqrt{\frac{75.625}{BAF}}$
Horizontal Point Factor (HPF) (Per-acre blowup factor for ratio sum with denominators being squares of tree diameters in square inches)	$\frac{1{,}996{,}643}{X^2+\frac{1}{4}} = \frac{7{,}986{,}573}{K^2} = \frac{7{,}986{,}573}{\left(Q+\sqrt{Q^2+1}\right)^2+1} = \frac{13{,}865.6}{(HDF)^2} = HPF = \frac{(HLF)^2}{7.85398} = 7{,}986{,}573\,hav\,\theta = 7{,}986{,}573\sin^2\frac{\theta}{2}$ HPF = 1833
Horizontal Line Factor (HLF) (per-acre blowup factor for ratio sum with denominators being tree diameters in inches & with line length in chains)	$\frac{3{,}960}{\sqrt{X^2+\frac{1}{4}}} = \frac{7{,}920}{K} = \frac{7{,}920}{\sqrt{\left(Q+\sqrt{Q^2+1}\right)^2+1}} = \frac{330}{HDF} = \frac{\sqrt{HPF}}{.356825} = HLF = 7{,}920\sqrt{hav\,\theta} = 7{,}920\sin\frac{\theta}{2}$
Horirontol Gouge-Angle (θ)	$\text{arccot}\left(X-\frac{1}{4X}\right) = \arccos\frac{K^2-2}{K^2} = \text{arccot}\,Q = \arccos\left[1-\frac{2}{(24\,HDF)^2}\right] = \arccos\left[1-\frac{2\,HPF}{7{,}986{,}573}\right] = \arccos\left[1-2\left(\frac{HLF}{7920}\right)^2\right] = \theta = 2\left(\frac{\theta}{2}\right)$ θ = 104.1
Horizontal Gouge Half-angle $\left(\frac{\theta}{2}\right)$	$\text{arccot}\,2X = \text{arccsc}\,K = \text{arccot}\left(Q+\sqrt{Q^2+1}\right) = \text{arccsc}\,(24\,HDF) = \text{arccsc}\sqrt{\frac{7{,}986{,}573}{HPF}} = \text{arccsc}\frac{7{,}920}{HLF} = \frac{1}{2}\theta = \frac{\theta}{2}$ = 52.05

VERTICAL, ANGLE-GAUGE

CONSTANT	IDENTITIES
Vertiml Distance Factor (Q) (maximum allowable distance in feet per foot of *tree height*)	$X-\frac{1}{4X} = Q = \frac{117.7522}{\sqrt{VPF}} = \frac{330}{VLF} = \cot\phi$
Vertical Point Factor (VPF) (par-ocr. blowup factor for ratio sum with denominators being squares of tree height in square feet)	$\frac{13{,}865.6}{Q^2} = VPF = \frac{(VLF)^2}{7.85398}$
Vertical Line Factor (VLF) (*per-acre* blowup *factor for* ratio sum with denominators being tree heights in feet & rim line length in chains)	$\frac{330}{Q} = \frac{\sqrt{VPF}}{.356825} = VLF$
Vertical Gauge-Angle (ϕ)	$\text{arccot}\,Q = \text{arccot}\frac{117.7522}{\sqrt{VPF}} = \text{arccot}\frac{330}{VLF}$

Figure 6.--*Trigonometric identities for horizontal or vertical angle-gauges.*

117.7522 = radius in feet of an acre. r_f ($r = \sqrt{43{,}560/3.1416}$)
13,865.6 = radius squared in feet of an acre r_f^2 ($r = 43{,}560/3.1416$) ($r^2 = (43{,}560)(144)/3.1416$ where $r^2 = A/\pi$ ...)
1,996,643 = radius squared in inches of an acre r_i^2 ... (144)/3.1416 where $d^2 = 4A/\pi$ ...

csc $\theta = \frac{1}{\sin\theta}$

When it is desired to compare reliability of different angle-gauges, observers, or calibration procedures, the appropriate quantities to compare are coefficients of variation of $\frac{1}{K^2}$ (for horizontal point- sampling), $\frac{1}{K}$ (for horizontal line- sampling), $\frac{1}{Q^2}$ (for vertical point- sampling), or $\frac{1}{Q}$ (for vertical line-sampling). In general, optics with magnification and good light-gathering or light-transmitting capacity will have much lower coefficients of variation than those without. As long as no bias is involved, instrumental variation will be reflected in point-to-point field tally variation, and requires no special consideration.

Although the above discussion covers all that is needful in the matter of calibration theory, for convenience it is usually desirable to derive some additional constants so as to eliminate need for introducing scale factors, π, reciprocals, squares, square roots, and halves.

In addition to X, K, and Q, which are scale-free constants appropriate to a given gauge, it is desirable to have a horizontal distance factor (HDF). When HDF is multiplied by tree diameter in inches, the product establishes the maximum distance in feet allowed between sample point or line and heart center of horizontally gauged questionable sample trees. This constant was formerly called plot radius factor (3) but that term is inappropriate now that lines may also be involved. Doubtful trees should be checked to avoid bias. Distance checks must be perpendicular to the line in line- sampling.

Q may be used directly in vertical gauging as a vertical distance factor which, when multiplied by tree height in feet, sets the maximum distance in feet allowed between sample point (or line) and heart center of vertically gauged questionable sample trees.

Finally,. blowup factors or multipliers are needed to convert horizontal or vertical line- or point-sample sums of ratios to a per-acre basis. The horizontal point factor (HPF) assumes that the denominators of the ratios will be the squares of tree diameters 'in square inches. This constant is called basal area factor (3) when the denominators of the ratios are tree basal areas in square feet. The horizontal line factor (HLF) assumes that the denominators of the ratios will be tree diameters in inches and that line length will be measured in chains (66 feet each). The vertical point factor (VPF) assumes that the denominators of the ratios will be the squares of tree vertical heights in square feet. The vertical line factor (VLF) assumes that the denominators of the ratios will be tree vertical heights in feet, and that line length will be measured in chains.

Formulae for each of these constants in terms of each other are given in Figure 6. Although the introduced scale-factors are for British-American unite of measure, scale factors for metric units could be similarly handled.

People may wish to convert a basal area factor for a given gauge to the HLF appropriate to the same gauge in line-sampling. This can be calculated as $12\sqrt{10 \text{ Basal Area Factor}}$. Thus, a 104.18-minute angle-gauge has a Basal Area Factor of 10 and an HLF of 120 when trees are sampled on both sides of the line, with tree diameters in inches being used to convert volume tables to ratios of volume divided by diameter.

Similarly, it is convenient to know that 183. 346 times Basal Area Factor gives HPF, and that .005454154 times HPF gives Basal Area Factor. HPF is more convenient for machine computations involving slope, elliptical trees, etc. ; Basal Area Factor is more convenient for mental or manual computation that ignores such complications.

SLOPING TERRAIN

All theory previously discussed has assumed that trees were rising vertically from a level plane. When trees rise vertically from an inclined plane, as on hills or mountains, there are two ways of preventing slope from biasing point-sampling or line - sampling estimates.

The first approach is best in horizontal line-sampling or point-sampling where punched cards and automatic data-processing machines are used for compilation. It involves using a constant angle-gauge to project a circular ring (or tree radius in line-sampling) of exactly the same magnitude on the inclined plane as would have been projected on the level plane, and to appropriately modify the probability (Pi) associated with any tree sampled on such inclined plane. This requires measuring slope dihedrals in horizontal point-sampling, or measuring inclination of each line-of-sight perpendicular to sample line in horizontal line-sampling, but it allows use. of a simple and foolproof constant HDF in checking doubtful trees, regardless of slope.

A similar approach can be used in pseudo-vertical angle-gauging only when the height gauged is the imaginary perpendicular dropped from tree top to an inclined plane which passes through breast height on the tree and is parallel to the sloping terrain; the point where this imaginary perpendicular pierces the inclined plane (instead of tree center) is taken as ring center or height origin and the gauge-angle always is kept normal to the inclined plane (rather than vertical or normal to a level plane). These complications in constant pseudo-vertical angle-gauging usually render it less desirable than the slope-adjusted vertical angle-gauging discussed in the next paragraph.

The second approach (and generally the best in vertical line-sampling or point-sampling) is to have a slope-actuated instrumental adjustment of the gauge-angle so that, for any inclined line-of-sight, the probability (Pi) of sampling a tree is the same as would have been given by the unadjusted angle on a level plane. With such instrumental adjustment, the HDF or Q for checking'doubtful trees will vary as the inclination of line-of-sight varies. This is inconvenient, and only for vertical gauging is it deemed preferable to the constant-angle technique discussed first.

Constant gauge-angle techniques will be discussed first.

In horizontal point-sampling with a constant gauge-angle on, slopes exceeding 10 percent, the HPF should be multiplied by the secant of the slope dihedral for the mean plane of the terrain around the point (balancing out hummocks and compound slopes). This generally means measuring the slope perpendicular to the contour through the point. A constant unadjusted HDF is then used for all doubtful trees.

In horizontal line-sampling with a constant gauge-angle on slopes exceeding 10 percent, the HLF should be multiplied by the secant of the slope perpendicular to the given sample line segment. A constant unadjusted HDF is used for all doubtful trees. Constant-angle point-sampling is probably a more efficient technique for precise estimates in hilly country.

In pseudo-vertical line- sampling or point- sampling with a gauge which is normal to and makes a constant angle with the inclined plane, exactly the same procedure is followed as above, except that the gauged height is the imaginary perpendicular to the inclined plane discussed earlier, and except that its foot is used instead of tree center in checking doubtful trees. Of course, if estimates of vertical tree height or squared vertical tree height per acre are desired when the imaginary perpendicular to the inclined plane has actually been gauged, appropriate allowance must be made. In the case of pseudo-vertical point-sampling, the simple count of trees must be multiplied by the cubed secant of the slope dihedral (to correct the square of imaginary height as well as the probability). In the case of pseudo-vertical line-sampling, the simple count must be multiplied by the product of secant of the slope dihedral times secant of slope measured in direction perpendicular to sample line.

One possible alternative in vertical point-sampling would be to use a gauge that is normal to and makes a constant angle with a level plane through the angle vertex (roughly, the eye of the observer). This would be equivalent to surrounding each tree with a constant right cone whose vertex is at tree tip, and whose surface defines an ellipse when cut by an inclined plane parallel to the terrain and passing through breast height of the tree. The area of such an ellipse, as projected on a level plane, can be calculated and compared with the area of a circular cross section of such a cone levelly sectioned through breast height on the tree. The ratio of the latter area to the former would be the appropriate correction factor to apply to the VPF. However, the calculation of this adjusted VPF would be complex, the calculation of adjusted VDF for checking doubtful trees would be very complex, and the critical distance separating "tally" from "non-tally" trees in an uphill direction from the observer would be indeterminate (or infinite) if terrain and cone had nearly the same slope. Such a vertical gauging procedure on sloping terrain would be quite impractical.

Slope-adjusted gauge-angle techniques will be discussed next.

For all horizontal line- sampling or point- sampling with slope-compensated gauges, the gauge-angle adjusted for inclined line-of-sight should be 2 arccsc $[\csc \frac{\theta}{2} \sec S]$, where θ is unadjusted horizontal gauge-angle and S is angle of inclination of line-of-sight to a given tree. This is a simple angular contraction that can be performed mechanically (through rotation), geometrically, or graphically by various types of instruments. The HDF used to check doubtful trees must also be corrected by multiplication by the secant of the angle of inclination of the line-of-sight to the particular tree, which complicates field work. However, HLF and HPF remain constant for a given slope-compensated gauge regardless of slope.

For all vertical line-sampling or point-sampling with slope-compensated gauges, the adjusted, truly vertical gauge-angle (which gauges the angle between tree top and a level plane containing gauge-angle vertex) should be arctan $[\tan \phi \pm \tan S]$. Here ϕ is unadjusted vertical gauge-angle and S is angle of inclination of line-of-sight to breast height on a given tree; plus is used when gauging uphill and minus when gauging downhill. The VDF used to check doubtful trees will be Q for the unadjusted gauge multiplied by the secant of the angle of inclination of line-of-sight to breast height on a given tree. For such a slope-compensated gauge-angle, the VPF or VLF used to convert ratio sums to a per-acre basis remains constant, regardless of slope. Checks of doubtful trees are always expensive with vertical gauging, but the instrumental slope compensation described in'this paragraph is probably simpler than the pseudo-vertical constant-angle technique described earlier.

LEANING TREES

All previous discussion has assumed that trees did not lean (i. e., that they were truly vertical). Lean has little effect on constant-angle horizontal gauging at breast height. The gauge should be slightly rotated about the line-of-sight to compensate for the cross-level component of lean. Doubtful-tree checks will be made from angle vertex to tree heart center at breast height, of course. If a slope-compensating, gravity-actuated instrument is used, it will rarely be feasible to compensate for cross-level, so in addition to doubtful trees, the user should check leaners that appear to barely qualify (using the appropriate Horizontal Distance Factor corrected for inclined line-of-sight). Such checks are intolerably slow or expensive if gauged diameter is not readily accessible, which is one reason that gauging at breast height is commonly adopted. Of course, if a leaning tree has been validly selected by a horizontal angle-gauge and if height is one of the variables to be measured, the usual conversion of vertically assessed height to tree slant height is needed; it can be done by multiplying by secant of angle made by tree axis with a level plane.

Leaning trees cause much greater complications in vertical angle-gauging. Not only is the quantity gauged affected by lean, but the point to which check-distance measurements must be made is translated a considerable distance from heart-center of tree at breast height. With a slope-compensated vertical angle-gauge, the height gauged is the imaginary vertical distance (i. e. , perpendicular to a level plane) from tree tip to intersection with inclined plane parallel to terrain and passing through angle vertex. Any vertically gauged height estimates will be in terms of height of this imaginary vertical above the inclined plane, which excludes stem below breast height. Hence, estimates of true height (above breast height) per acre in vertical line-sampling should use the secant of the angle of lean (from level) instead of counting each tallied tree as 1, and in vertical point-sampling the squared secant should be used instead of 1. The intersection of the imaginary vertical with the inclined plane replaces heart-center of tree as check point or ring-center.

It is impractical to use a constant vertical angle-gauge where trees lean, although the theory involves merely a slight modification of the constant pseudo-vertical angle technique discussed on page 17.

Such non-linear complications as sweep or spiral are handled analogously to lean (i. e., using an imaginary perpendicular). The real problem there is to find some function relating the curved height to the imaginary vertical height (comparable to the secant of angle of lean where straight trees are involved).

INVISIBLE, MASKED, SKIPPED, OR DOUBLE-COUNTED SAMPLE TREES

Any factor that prevents identification of qualifying trees or that encourages confusion of stems will inject a bias into point- or line-sampling estimates.

Dense brush or undergrowth can make trees invisible except at very close range, large stems can mask smaller stems behind them, and very numerous qualifying trees can confuse the observer so that he skips some stems or double-counts others from a single point (the same tree, however, may be validly tallied from two points without bias). The theoretical cure for all of these potential sources of bias is to adopt a gauge-angle large enough to ensure that no tree can qualify farther than it can be detected under woods conditions, and also large enough so that the total number of sample trees qualifying at a point or along a line-segment of given length will be too few to cause frequent confusion in count or close angular juxtaposition of sample trees. Under United States conditions, these requirements are usually met by horizontal gauge-angles within the range 100 to 300 minutes. It is impossible to select vertical gauge-angles that will ensure visibility of tree top in dense stands, especially of hardwoods, but convenient angles in the neighborhood of 60° are probably about the best compromise possible.

After a gauge-angle has been intelligently selected, precautions are still necessary to avoid bias from missing qualified trees. The observer should bodily move from side to side perpendicular to line-of-sight to each nearby tree so as to peer behind each for hidden trees. Any tree thus discovered may be gauged from a point moved from the original sampling point along a line perpendicular to the line-of-sight to the hidden tree (actually, the movement should be along the arc of the circle around the hidden tree through the sample point). In line-sampling, hidden trees often can be detected during progress along the line before or after they become hidden.

Exact gauging of hidden or partially hidden trees at breast height or tree tip is not always possible from a sample point or line, but exact gauging is rarely necessary. If the upper stem qualifies with a horizontal gauge, it is certain that the stem would qualify at breast height unless the tree leans toward the observer. If any height below the vertical gauging point on the upper stem qualifies, the gauging point would qualify. Lastly, if a height or circular cross section qualifies from any point along a sample line, it is sure to qualify when viewed along a perpendicular to the sample line (this is not true of horizontally line-sampled elliptical cross. sections--a case discussed later).

When a tree cannot be gauged by orthodox procedures or by any of the preceding shortcuts, it must be treated as a doubtful tree. Then both the distance to tree heart center and the questionable dimension of the tree must be measured, if the tree is to qualify, its distance must be less than the product of tree dimension times HDF or Q, with a slope correction if appropriate.

The above precautions will eliminate any possible bias from failure to detect sample trees.

ELLIPTICAL TREE CROSS SECTIONS

The foregoing exposition of point-sampling and line-sampling theory has assumed that all trees have circular cross sections. Although plot- sampling and strip-sampling usually make the same assumption, the consequences of non-circularity can be somewhat more serious in point-sampling or line-sampling than in plot-sampling or strip- sampling. As far as is known, this paper for the first time works out the applicable theory, explores the consequences, and proposes remedies.

The discussion will be limited to the implications of elliptical tree cross sections, since most tree departures from circularity approximate that shape. Angle-gauges cannot be employed except on circular or elliptical shapes. Since vertical angle-gauging is only affected by elliptical trees to the same extent as plot-sampling or strip-sampling, subsequent discussion will be concerned only with horizontal angle-gauging.

The obvious implication of elliptical tree cross section on any type of tree sampling is that calculation of individual tree basal area (and tree d. b. h. when defined as $2\sqrt{\frac{\text{tree basal area}}{\pi}}$) will be biased unless tree basal area is calculated as $\frac{\pi Dd}{4}$ and d. b. h. as $\sqrt{Dd}$ where D is the major diameter (i. e., the maximum) and d is the minor diameter (i. e., the minimum) of the elliptical cross section. A number of other expressions (equivalent when D = d) are commonly used to approximate Dd. The quadratic approximation of Dd is $\frac{D^2+d^2}{2}$; the circumferential approximation (when diameter tape is used) is $(\frac{\text{circumference}}{\pi})^2$; the arithmetic approximation (when caliper maximum and minimum are averaged) is $(\frac{D+d}{2})^2$. But the squared geometric mean (Dd) is the only unbiased estimate, and in this day of electronic data-processing machines, there is little excuse for not using it.

A table showing the ratio of calculated ellipse area to true elliptical basal area indicates the magnitude of bias involved in the preceding approximations where $\frac{d}{D} = \frac{10}{10}$(circle), $\frac{9}{10}$, $\frac{5}{10}$, $\frac{0}{10}$ (line):

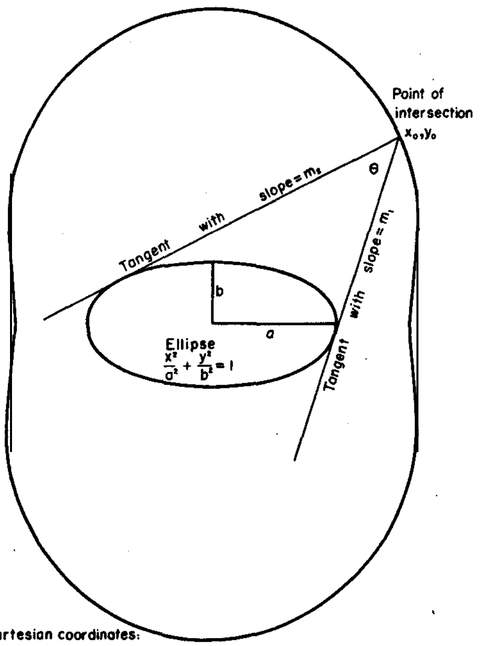
Ratio of $\frac{\text{Calculated ellipse area}}{\text{True ellipse area}}$ for variously flattened ellipses

Calculated area	Elliptical $\frac{d}{D}$			
	$\frac{10}{10}$	$\frac{9}{10}$	$\frac{5}{10}$	$\frac{0}{10}$
	- - - - - Ratios of areas - - - - -			
Quadratic estimate	1.0000	1.0056	1.2500	Infinity
Circumferential estimate	1.0000	1.0052	1.1889 ✓	Infinity
Arithmetic estimate	1.0000	1.0028	1.1250	Infinity
Geometric estimate	1.0000	1.0000	1.0000	1.0000

It is apparent that although bias is negligible in most cases where $\frac{d}{D}$ is nearly 1, it can be quite serious in the most extreme instances likely to be encountered $\left\{\frac{d}{D} = \frac{1}{4}\right\}$. It is infinitely large in the limiting situation $\left\{\frac{d}{D} = 0\right\}$ which could never, of course, be encountered in a tree population. Ellipses flattened so that $\frac{d}{D} = \frac{9}{10}$ are common among tree populations, while those flattened so that $\frac{d}{D} = \frac{5}{10}$ probably are extremely rare.

So much for the way in which elliptical tree cross sections affect individual-tree measurements of diameter and basal area in any type of sampling. In horizontal point-sampling or line- sampling they have an additional effect that can cause bias as large as or larger than the one just described. In practice, both biases tend to act in the same direction and usually cause estimates to be too high. Avoiding both types of bias is simple where field parties use calipers, record both major and minor diameters, and have data processed electronically.

Earlier discussion has assumed that a horizontal angle-gauge pivoted about a tree will generate a circle whose area is always K^2 times tree basal area where K = csc ½ horizontal gauge-angle. Unfortunately, when the tree cross section is elliptical, the shape generated by the pivoted angle-gauge is not elliptical, nor is its area K^2 times the tree's true elliptical basal area.



Locus in Cartesian coordinates:

$$\tan\theta = \frac{2\sqrt{b^2x^2 + ay^2 - a^2b^2}}{x^2 + y^2 - (a^2 + b^2)}$$

Locus in polar coordinates originating at center along ellipse major semidiameter:

$$r^2 = (a^2+b^2)(\csc^2\theta)\left[1-\frac{a^2-b^2}{a^2+b^2}\cos^2\theta\cos 2v+\sqrt{\left(1-\frac{a^2-b^2}{a^2+b^2}\cos^2\theta\cos 2v\right)^2-\sin^2\theta\left[1-\left(\frac{a^2-b^2}{a^2+b^2}\right)^2\cos^2\theta\right]}\right]$$

Where r = variable radius
v = variable angle made with major semidiameter.

Figure 7. *--Locus of vertex of constant angle with sides tangent to an ellipse.*

Consider a tree's cross section to be the ellipse in figure 7, which has the form $\frac{X^2}{a^2} + \frac{Y^2}{b^2} = 1$ in Cartesian coordinates with origin at center. Next consider the constant gauge-angle θ (less than or equal to 90^o, to simplify sign) with variable *vertex* X_o, Y_o, and with sides having slopes m_1 and m_2 each tangent to the ellipse. In analytic geometry,

$$ml = \frac{X_oY_o \text{ t } \overline{b^2X_o^2 + a^2Y_o^2 - a^2b^2}}{X_o^2 - a^2}$$

$$m_2 = \frac{X_oY_o - \sqrt{b^2X_o^2 \text{ t } a^2Y_o^2 - a^2b^2}}{X_o^2 - a^2}$$

$$\tan\theta = \frac{m_1 - m_2}{1 + m_1m_2} = \frac{2\sqrt{b^2X_o^2 + a^2Y_o^2 - a^2b^2}}{X_o^2 + Y_o^2 - (a^2 + b^2)}$$

With subscripts suppressed, the locus of the vertex of a constant angle θ whose sides are tangent to an ellipse whose major and minor semi-diameters are a and b is:

$$\tan\theta = \frac{2 \overline{b^2X^2 \text{ t } a^2Y^2 - a^2b^2}}{X^2 + Y^2 - (a^2 + b^2)}$$

Figure 7 shows that the shape of the locus is definitely not elliptical. It also gives the equation of the locus transformed into polar coordinates, which are desirable in plotting and in calculations of area or mean diameter. From one or the other of the equations, 4 special loci may be easily deduced. When a = b, the locus is a circle. When b = 0, the locus is the external circumference of two intersecting circles (the limiting third case illustrated in figure 8). When θ = 90^o, the locus is a circle (the so-called director circle of a given ellipse). When θ = 0^o or 180^o, the locus is an ellipse (limiting cases approached when θ is very small or very large). Henceforth, the locus of the gauge-angle vertex will be referred to as the "shape."

Figure 8 shows how the ratio of shape area to tree ellipse areaincreases as the tree ellipse flattens. It also indicates that although the algebraic expression for shape area cannot be integrated for the general case, quadrature discloses that it can be closely approximated in all likely cases by $\frac{\pi(a^2+b^2)}{1-\cos\theta} = \frac{K^2\pi\left(\frac{D^2+d^2}{2}\right)}{4}$, where $K^2 = \csc^2\left(\frac{\theta}{2}\right)$.

Table 1.--*Ratio of* $\frac{\textit{true shape area}}{(K^2)(\textit{calculated ellipse area})}$ *for various angle-gauges pivoted about various ellipses*

Type of shape area calculation	Horizontal gauge-angle $= \theta$	Elliptical $\frac{d}{D}$: $\frac{10}{10}$	$\frac{9}{10}$	$\frac{5}{10}$	$\frac{0}{10}$
		. . . . . *Ratios* of *areas*			
Quadratic or $\frac{K^2\pi\left(\frac{D^2+d^2}{2}\right)}{4}$	90°	1.0000		1.0000	*1.0000*
	45°	1.0000	1.0006	1.0199	1.0651
	228.842'	1.0000	1.0000	1.0002	1.0011
	104.142''	1.0000	1.0000	1.0000	1.0002
Circumferential or $\frac{K^2\pi\left(\frac{Circum.}{\pi}\right)^2}{4}$	90°	1.0000	1.0003	1.0514	1.2337
	45°	1.0000	1.0009	1.0723	1.3140
	220.842'	1.0000	1.0003	1.0516	1.2351
	104. 142'	1.0000	1.0003	1.0514 ✓	1.2339
Arithmetic or $\frac{K^2\pi\left(\frac{D+d}{2}\right)^2}{4}$	90°	1.0000	1.0028	1.1111	2.0000
	45°	1.0000	1.0033	1.1333	2.1303
	220. 042'	1.0000	1.0028	1.1114	2.0022
	104. 142'	1.0000	1.0028	1.1112	2.0004
Geometric or $\frac{K^2\pi\ Dd}{4}$	90°	1.0000	1.0056	1.2500	Infinity
	45°	1.0000	1.0061	1.2749	Infinity
	228. 842'	1.0000	1.0056	1.2503	Infinity
	104. 142'	1.0000	1.0056	1.2501	Infinity

Where D = major diameter of tree elliptical cross section
d = minor diameter

$$K^2 = \frac{1}{\text{haversin }\theta} = \frac{2}{1-\cos\theta} = (\cot\theta \text{ t } \sqrt{\cot^2\theta+1})^2 \text{ t } 1 = \csc^2\left(\frac{\theta}{2}\right)$$

θ	cot θ	Basal-area factor	Horizontal point factor
90°	0	21.780.	3,993,286
45°	1	6.379.2	1,169,607
220. 842'	15	48,24	8,844. 3
104.142'	33	9, 99	1,032. 2

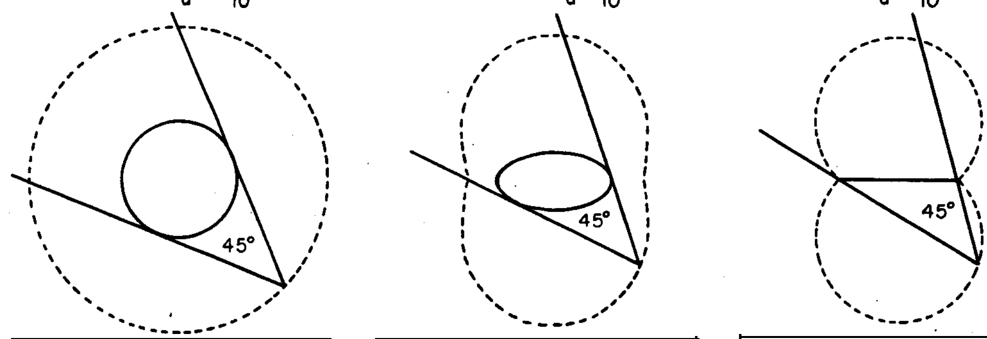
Table 1 employs quadrature to assess magnitude and direction of bias involved in approximating the reciprocal of true shape area by the reciprocal of variously calculated shape areas involving the 4 expressions used earlier to estimate the area of an ellipse. As can be seen, the quadratic approximation given above results in nearly bias-free estimates for shape areas generated by any angle-gauge apt to be used on any tree ellipse apt to be encountered. Assuming that the shape area is K^2 times the true elliptical tree basal area will result in overestimates of frequency, basal area, or voiume. The bias can be as high as 25 percent even with very small gauge-angles when minor axis is only half as large'as major axis.

Line- sampling probability depends on unweighted average shape diameter (roughly a function of the unweighted average of all shape diameters taken at small equiangular intervals through 90°) instead of on shape area (roughly a function of the unweighted average of all squared shape diameters), as in point- sampling.

The unweighted average diameter of an ellipse is $\frac{4}{\pi}\int_0^{\frac{\pi}{2}} r\ dv = D\left(\frac{2\frac{d}{D}}{\pi}\right)\int_0^{\frac{\pi}{2}}\frac{dv}{\sqrt{1-e^2\cos^2 v}}$ where r and v are respectively linear and angular variables in polar coordinates and $e^2 = \frac{a^2-b^2}{a^2}$. This integral has been tabled (often it is called the complete elliptic integral K) and it is better approximated by $\sqrt{Dd}$ than by the square root of any of the other 3 approximations whose bias in estimating elliptical area was compared earlier.

As before, functions which are good estimators of elliptical parameters may be very poor for estimating shape parameters. Figure 9 shows that the ratio of unweighted mean shape diameter to unweighted mean ellipse diameter increases as the ellipse flattens. The appropriate function for calculating unweighted mean shape diameter is also given in figure 9. Although the algebraic expression cannot be integrated for the general case, quadrature discloses that the integral can be closely approximated by $2\sqrt{\frac{a^2+b^2}{1-\cos\theta}} = K\sqrt{\frac{D^2+d^2}{2}}$.

Ellipse with $\frac{b}{a} = \frac{10}{10}$ (circle) | Ellipse with $\frac{b}{a} = \frac{5}{10}$ | Ellipse with $\frac{b}{a} = \frac{0}{10}$ (line)



$\frac{\text{Shape area}}{\text{Ellipse area}} = \frac{6.828\pi a^2}{1.000\pi a^2} = 6.828$ | $\frac{\text{Shape area}}{\text{Ellipse area}} = \frac{4.353\pi a^2}{.500\pi a^2} = 8.706$ | $\frac{\text{Shape area}}{\text{Ellipse areo}} = \frac{3.637\pi a^2}{0} = \infty$

Different shapes are generated by pivoting fixed angle(45°) around various ellipses that range from a circle to a line.

Shape-area $= 2\int_0^{\frac{\pi}{2}} r^2 dv = 2(a^2+b^2)(\csc^2\theta)\int_0^{\frac{\pi}{2}}\left[1-M\cos^2\theta\cos 2v+\sqrt{(1-M\cos^2\theta\cos 2v)^2-\sin^2\theta(1-M^2\cos^2\theta)}\right]dv$

where r-variable radius, v= variable angle } in polar coordinate system originating at center along major semidiameter of ellipse.
θ= fixed horizontal angle (angle illustrated is 45°).
a and b = major and minor semidiameters, respectively, of given ellipse.
$M = \frac{a^2-b^2}{a^2 + b^2}$

The random paint will sample an ellipse only when the paint falls within the shape-area around the ellipse.

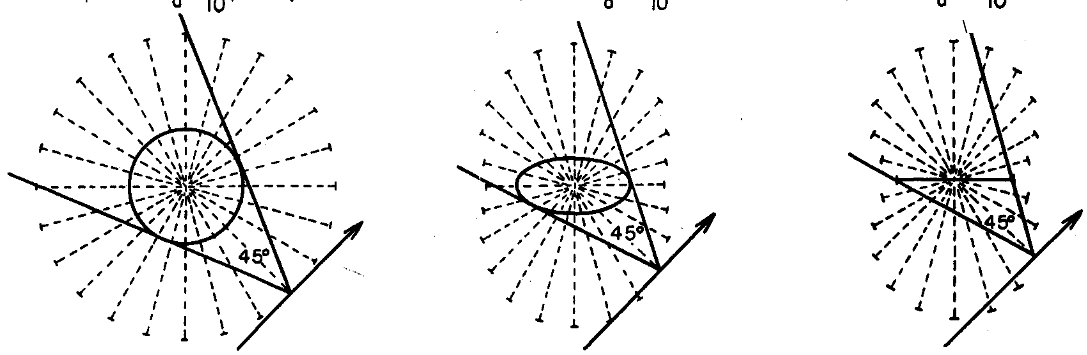
Probability of such sampling is proportional to shape area $= 2\int_0^{\frac{\pi}{2}} r^2 dv \doteq \frac{\pi(a^2+b^2)}{1-\cos\theta}$.

Although this integral cannot be directly evaluated except in special cases, quadrafure indicates that the appraximotian is quite close where $1 \geq \frac{b}{a} \geq \frac{5}{10}$, which includes all tree crass sections usually encountered (see Table 1).

↑ **Figure 8.**--*How elliptical trees affect probability in horizontal point-sampling.*

↓ Figure 9. --*How elliptical trees affect probability in horizontal line-sampling.*

Ellipse with $\frac{b}{a} = \frac{10}{10}$ (circle) | Ellipse with $\frac{b}{a} = \frac{5}{10}$ | Ellipse with $\frac{b}{a} = \frac{0}{10}$ (line)



$\frac{\text{Mean shape-diameter}}{\text{Mean ellipse-diameter}} = \frac{5.226a}{2.0000} = 2.613$ | $\frac{\text{Mean shape-diameter}}{\text{Mean ellipse-diameter}} = \frac{4.126a}{1.373a} = 3.005$ | $\frac{\text{Mean shape-diameter}}{\text{Mean ellipse-diameter}} = \frac{3.7050}{0} = \infty$

Different shapes are generated by pivoting fixed angle(45°) around various ellipses that range from a circle to o line.

Mean shape-diameter $= \frac{4}{\pi}\int_0^{\frac{\pi}{2}} r\,dv = \frac{4}{\pi}\sqrt{(a^2+b^2)(\csc^2\theta)}\int_0^{\frac{\pi}{2}}\sqrt{1-M\cos^2\theta\cos 2v+\sqrt{(1-M\cos^2\theta\cos 2v)^2-\sin^2\theta(1-M^2\cos^2\theta)}}\,dv$

where r = variable radius, v = variable angle } in polar coordinate system originating at center along major semidiameter of ellipse.
θ= fixed horizontal angle (angle illustrated is 45°).
a and b = major and minor semidiameters, respectively, of given ellipse.
$M = \frac{a^2-b^2}{a^2+b^2}$

The randomly directed line will sample an ellipse only when the line intersects a shape-diameter perpendicular to the line.

Probability of such sampling is proportional to mean shape-diameter $= \frac{4}{\pi}\int_0^{\frac{\pi}{2}} r\,dv \doteq 2\sqrt{\frac{a^2+b^2}{1-\cos\theta}}$

Although this integral connot be directly evaluated except in special cases, quadrature indicates that the approximation is quite close where $1 \geq \frac{b}{a} \geq \frac{5}{10}$, which includes all tree crass sections usually encountered (see table 2).

Table 2. --Ratio of $\frac{\text{true unweighted average shape diameter}}{(K)(\text{calculated average ellipse diameter})}$ for various angle-gauges pivoted about various ellipses

Type of shape diameter calculation	Horizontal gauge-angle = θ	Elliptical $\frac{d}{D}$			
		$\frac{10}{10}$	$\frac{9}{10}$	$\frac{5}{10}$	$\frac{0}{10}$
		. . . Ratios of average diameters . . .			
Quadratic or	90°	1.0000	1.0000	1.0000	1.0000
	45°	1.0000	.9999	.9987	1.0026
$K\sqrt{\frac{D^2+d^2}{2}}$	228.842'	1.0000	.9993	.9755	.9043
	104.142'	1.0000	.9993	.9753	.9018
Circumferential or	90°	1.0000	1.0001	1.0254	1.1107
	45°	1.0000	1.0001	1.0240	1.1136
$K(\frac{\text{Circum.}}{\pi})$	228.842'	1.0000	.9995	1.0002	1.0044
	104.142'	1.0000	.9995	1.0000	1.0016
Arithmetic or	90°	1.0000	1.0014	1.0541	1.4142
	45°	1.0000	1.0013	1.0527	1.4180
$K(\frac{D+d}{2})$	228.042'	1.0000	1.0007	1.0282	1.2788
	104.142'	1.0000	1.0007	1.0280	1.2753
Geometric or	90°	1.0000	1.0028	1.1180	Infinity
	45°	1.0000	1.0027	1.1166	Infinity
$K\sqrt{Dd}$	228.842'	1.0000	1.0021	1.0906	Infinity
	104.142'	1.0000	1.0021	1.0904	Infinity

where D = major diameter of tree elliptical cross section
d = minor diameter

$$K = \sqrt{\frac{1}{\text{haversin}\,\theta}} = \sqrt{\frac{2}{1-\cos\theta}} = \sqrt{(\cot\theta + \sqrt{\cot^2\theta + 1})^2 + 1} = \csc\frac{\theta}{2}$$

θ	Cot θ	Horizontal line factor
90°	0	5600.3
45°	1	3030.9
228.842	15	263.56
104.142'	33	119.96

Table 2 uses quadrature to assess the magnitude and direction of bias involved in approximating the reciprocal of true unweighted average shape diameter by the reciprocal of variously calculated shape diameters involving the square root of the 4 approximations used earlier to estimate the area of an ellipse. As can be seen, the quadratic approximation given above results in relatively bias-free estimates for shape diameters projected by any angle-gauge apt to be used on any tree ellipse apt to be encountered. Within the range of the small angle-gauges commonly employed, the circumferential approximation is slightly better, but requires use of a diameter tape, which itself may'involve considerable bias in the direct estimation of elliptical tree basal area. There is a high bias implicit in assuming that unweighted average shape diameter is $K\sqrt{Dd}$. Such an assumption could cause line-sampling to overestimate frequency, basal area, or volume by as much as 10 percent even withverysmall gauge-angles where they sample ellipses whose minor axis is only half of the major axis.

In summary, if a single average measurement of elliptical tree diameter (such as $\sqrt{\frac{D^2+d^2}{2}}$, $\frac{\text{Circumference}}{\pi}$; $\frac{D+d}{2}$, or $\sqrt{Dd}$) is used as though the tree were circular, bias from use of erroneous individual tree probability may be superposed on biae in individual-tree basal area calculation. The resultant relative bias in contribution of the elliptical sample trees to estimates of basal area per unit of land area will be the product of the appropriate entry from the text table on page 20 times the appropriate entry from table 1 if point-sampling.

For example, estimating basal area with a diameter tape where $\frac{d}{D} = \frac{1}{2}$ and the tree has been horizontally point-sampled with a 104-minute angle-gauge would result in erroneously multiplying the correct basal area contribution of that tree by (1. 1889) (1.0514) = 1.25. This means that the basal area contributed by this tree to the estimate would be 25 percent too high.

The remedy where precise estimates are desired is to caliper and record maximum and minimum elliptical diameters. Then electronic machines can calculate basal area as $\frac{\pi Dd}{4}$ and diameter as $\sqrt{Dd}$, with adjusted probability divisor (Pi) being calculated as $\frac{\pi K^2 (\frac{D^2 + d^2}{2})}{4}$ if point-sampling or $LK\sqrt{\frac{D^2+d^2}{2}}$ if line- sampling (where L is length of sample line, and K is Radial Enlargement Factor appropriate to the angle-gauge). Thus it can be seen that elliptical trees horizontally point-sampled should contribute $\left(\frac{2}{\frac{d}{D}+\frac{D}{d}}\right)\left(\frac{1}{K^2}\right)$ instead of $\frac{1}{K^2}$ to basal area estimates per unit of land area.

The elliptical tree in the paragraph above (where $\frac{d}{D}=\frac{1}{2}$) should count only $\left(\frac{2}{\frac{1}{2}+2}\right)\left(\frac{1}{K^2}\right) = \frac{.8}{K^2}$ instead of $\frac{1}{K^2}$. This would remove the 25 percent high bias noted when diameter-tape measurement alone was used. Similarly, elliptical trees horizontally line- sampled should be counted $\sqrt{\frac{2}{\frac{d}{D}+\frac{D}{d}}}$ instead of 1 in estimating diameter per unit area of land. Where volume, etc., rather than basal area is being estimated, it should be that volume appropriate to a tree whose d. b. h. is $\sqrt{Dd}$, and the adjusted probability divisor (Pi) will take care of the rest.

Actually, if $\frac{d}{D}=\frac{9}{10}$, the use of a diameter tape on elliptical trees horizontally point-sampled or line-sampled with a 104-minute gauge will result in a combined high bias of only about ½ of 1 percent, which is quite tolerable in ordinary work, especially when it is considered that only a small part of the sample trees will be much more elliptical. In precise work processed electronically, the bias should be eliminated by recording and properly using D and d.

One last implication in horizontally gauging the cross sections of elliptical trees should be noted. In horizontal point- sampling, the point can fall anywhere inside the shape and select the generating tree, Hence, in checking doubtful trees the distance from point to tree heart center should be compared with the product of HDF times tree diameter (calculated as $\sqrt{\frac{D^2+d^2}{2}}$). This will in effect establish an imaginary circular shape with the same area as the imaginary non-circular shape, and no bias will result. A less desirable alternative is to use the product, of HDF times calipered tree diameter perpendicular to the line-of-sight. This product, however, involves the same bias as assuming that shape area is K^2 times the true basal area of the elliptical tree.

Similarly, in line-sampling elliptical trees, any check of doubtful trees should employ as a criterion the product of HDF times tree diameter (either determined by diameter tape or calculated as $\sqrt{\frac{D^2+d^2}{2}}$). Although this criterion may include some elliptical trees that the point or line would not qualify, it will exclude an equal number which point or line would qualify, so' it is unbiased. A sample line passing through the two bulges of a shape but missing the cordate cleft poses a minor'problem. However, the criterion should be rigorously applied perpendicular to the sample line, regardless of whether or not the tree appears to qualify from some inappropriate point on the sample line.

SYNTHES I S

PRECISE ESTIMATES

From the foregoing, it is apparent that vertical angle-gauging is not adapted to precise work. Tree tips or points on the upper stem are frequently masked, they are inaccessible for doubtful-tree checks, and the complications introduced by lean are intolerable in precise work. Application to less precise work will be considered subsequently.

Line-sampling has much less application in precise work than point-sampling, for the same reasons that plot-sampling superseded strip-sampling years ago. However, horizontal line-sampling can be done precisely, since d. b. h. of and distance to doubtful trees can be cheaply and quickly measured. Procedures for the most part are analogous to those of point-sampling, except that slope corrections depend on measurements perpendicular to the sample line, and slopover bias is somewhat more troublesome on non-rectangular tracts. Applications to less precise work will be considered in a subsequent section.

Horizontal point-sampling is by far the most useful precise angle-gauge technique. A procedure for precise estimates is recommended below; it takes into account all possible biases previously discussed and therefore assumes: locating sample points in unbiased manner within tract to be sampled; calipering and recording maximum (D) and minimum (d) for each sample tree;measuring slope dihedral through sampling points, usually perpendicular to contour; using a constant horizontal angle-gauge with optical magnification, unadjusted for slope but properly calibrated on the level by the user; defining peripheral zones and using partial sweep techniques and appropriate tree weights where slopover bias might be important; measuring auxiliary tree variables such as form class, total or merchantable height so that volume is directly calculable as a function of $\sqrt{Dd}$ and auxiliary variables (volume tables, if used,should be expressed as regression surfaces depending on the measured variables); checking distance to doubtfully gauged trees against the unadjusted Horizontal Distance Factor times $\sqrt{\frac{D^2+d^2}{2}}$; selecting gauge-angle and using diligence so as to detect all qualified sample trees; tilting horizontal angle-gauge by the exact cross-level component of leaning trees; employing electronic data-processing of individual tree cards (all computations will be done automatically and practically instantaneously, with figures blown up to a per-acre basis and punched on the individual tree cards).

On these assumptions,' the mensurational field record for a sample tree would appear as follows (omitting all but the most pertinent figures):

D	d	H	F	HPF	Slope	Slopover	Shape divisor
17.0	15.1	66	.80	1833	1.02	2	258.5

↑ not entered in field.

This field record indicates use of a 104. l&minute horizontal angle-gauge (HPF = 1833) with Horizontal Distance Factor 2.75 for check of doubtful trees (substituting 1833 into the appropriate formulae in fig. 6). It also indicates that secant of slope perpendicular to contour was 1.02 (so slope must have been between 17.4 and 22. 5 percent). The center of the sample tree lay in a "side zone" near a boundary where slopover might have occurred, hence tree was allowed to generate only a half circle away from the tract boundary, and slopover factor was 2 , The shape area divisor, calculated electronically by a data-processing machine, will be $\frac{(17.0)^2+(15.1)^2}{2}$ = 258. 5. The effective d. b. h. is machine-calculated as $\sqrt{Dd}$ = $\sqrt{(17.0)(15.1)} = \sqrt{256.7}$ = 16.0 inches. Elliptical basal area in square feet is machine-calculated as . 0054546d = 1.400 sq. ft. Tree volume will be machine-calculated as some function of Dd, H, and F, where H is merchantable height in feet to a specified top and F is Girard form class (or $\frac{\text{d.i.b. at 17.3 ft.}}{\text{d.b.h. o.b.}}$).

Other variables could of course be used, and heights for several different products could be recorded (or lengths of stem could be classified as to product suitability, grade, etc.). A regression surface can be readily fitted to any volume table now by machine techniques. Growth is also readily handled as an individual-tree variable. The multiplier that will blow up sample frequency (1), sample basal area (1.400 sq. ft.), sample volume (say 288 board feet scaled by International log rule with 1/4-inch kerf), sample. growth, etc. to a per-acre basis is:

$$\frac{\text{(HPF) (Slope Factor) (Slopover Factor)}}{\text{(Shape Divisor)}} = \frac{(1833)(1.02)(2)}{(258.5)} = 14.46$$

The machine will compute this automatically and multiply it by every desired individual-tree variable to put each quantity on a per-acre basis. The per-acre quantities that might be punched out on the tree card are:

(14.46)(1)	= 14.46	trees per acre
(14.46)(16. 0)	= 231.3	inches of diameter per acre
(14.46)(66)	= 954	lineal feet of merchantable height per acre
(14.46)(1.400)	= 20.24	sq. ft. of basal area per acre
(14.46)(288)	= 4164	board feet of sawtimber per acre

The above recommendations involve an extremely simple and foolproof field procedure, with a constant horizontal gauge-angle unaffected by slope and with a constant HDF. This latter can be incorporated into a 100-foot tape graduated so that each mark reads the smallest diameter--to tenths of inches--qualifying at that distance. This coupled with a double-entry table of $\sqrt{\frac{D^2+d^2}{2}}$ for D and d in tenths of inches, will make field checks of doubtful trees quite simple. A single slope factor at a point applies to all trees tallied therefrom. Adjustments for slope, slopover, and elliptical tree cross sections are automatically made by machine in the blowup factor. Volumes are regression-computed by machine. Machines adapted to such inventory are IBM 650, 704, 705, 709, or similar electronic devices.

Had the same tree been line-sampled with the same angle-gauge, HLF = 120 would replace HPF = 1833, the Slope Factor would have been measured perpendicular to line-of-sight to each tree (instead of perpendicular to contour), slopover factor would have been 2 (the same as before, in this case), and shape divisor would have been $\sqrt{258.5}$ = 16. 1 instead of 258. 5. The same HDF = 2.75 would be used to check doubtful trees in line-sampling as in plot-sampling.

As a last step, whenever tree variables (blown up as above in either point-sampling or line-sampling) have been sorted and tabulated for any physical stratum or population, the totals and subtotals must be divided by the pertinent number of points or length of line involved (in chains) to reduce figures to an acre basis. They must then be multiplied by stratum or tract acreage to get population totals.

The sampling error of estimates based on points or line-segments (exclusive of whatever volume-table errors may be involved) can be easily calculated as the variance of frequency, basal area, volume, or growth estimates per acre (or quantities proportional to them) among randomly located points or line-segments. Variance of the calibration factor used to blow up sample-tree estimates at a point is reflected in the variation among point-sampling estimates if the calibration was bias-free. This point seems to have been misunderstood by many. Calibration-factor variance is also reflected by variance among lines in line-sampling.

Precise estimates of various components of tree growth per acre can be obtained either from periodic remeasurement of each tree initially selected by a permanent sample point or line, or from a single measurement and an increment core from each tree selected by a temporary sample point or line. HDF should, of course, be employed to resolve doubts as to sample-tree qualification.

Use of permanent sample points or lines involves selection of individually identifiable sample trees whose subsequent harvest, death, or growth can be observed periodically. With such an installation, periodic measurements of net change in initial stand per acre can be analyzed into 3 major components: harvest, unsalvaged mortality, and growth of surviving trees initially at least 4-1/2 feet tall, A fourth component--ingrowth of trees initially less than 4-1/2 feet tall or initially nonexistent--can be estimated, if desired, by coring small trees newly qualifying at time of remeasurement and then measuring and tallying only those having a breast-height age equal to or less than the growth period, Alternatively, a small permanent plot (milacre or less) could be used to estimate ingrowth originating subsequent to initial tally. Trees originating subsequent to initial tally will not contribute much volume to ingrowth into merchantable size classes in a 5- to 10-year period, however, and could be ignored if only merchantable volume change were of interest.

Below is an illustration of point-sampling growth analysis of a sample-tree tally at a single permanent point where a 104. 18-minute gauge with HPF = 1833 (i. e., Basal Area Factor is 10) has been used. The permanent-point record shows all trees initially tallied and measured; their subsequent fate (mortality, harvest, survival) and remeasurement if surviving after 10 years; subsequent measurement (at same time as survivor remeasurement) of newly qualifying trees younger than 10 years of age at breast height (determined from core).

D.b.h.		Basal area			Volume		
Initial	After 10 years	Initial	After 10 years	Difference	Initial	After 10 years	Difference
- - Inches - -		- - Square feet - -			- - Cubic feet - -		
4.0	6.0	.0873	.1963	+ .1090	0.0	3.0	+ 3.0
10.0	12.0	.5455	.7854	+ .2399	12.0	19.0	+ 7.0
10.0	.0 (dead)	.5455	.0	- .5455	12.0	.0	-12.0
20.0	.0 (cut)	2.1817	.0	-2.1817	75.0	.0	-75.0
.0	1.0	.0	.0055	+ .0055	.0	.0	.0
.0	6.0	.0	.1963	+ .1963	.0	3.0	+ 3.0

N.B.: Newly qualifying trees more than 10 years of age at the end of the 10-year growth period would be recorded separately elsewhere.

The use of electronic data-processing machines readily allows differences for each tree with initial d. b. h. greater than zero to be multiplied by $\frac{1833}{(\text{initial d.b.h.})^2}$, and differences for each tree with initial d. b. h. equal to zero to be multiplied by $\frac{1833}{(\text{terminal d. b. h.})^2}$. In case elliptical biases were to be eliminated, the appropriate shape divisors would replace (d. b. h.)^2 as denominators of these fractions. If slope and slopover were involved, the fractions would, of course, be multiplied by the appropriate factors. Each fraction, corrected for shape, slope, or slopover,is actually the number of trees represented by each sample tree.

Any sample tree with zero initial diameter represents ingrowth originating subsequent to initial tally. Any sample tree with zero terminal diameter represents mortality or harvest cut, with a code to distinguish them. Any sample tree with neither initial nor terminal diameter equal to zero represents survivor growth. Of course, segregation of so-called ingrowth into or out of various size classes (pulpwood, sawlogs, etc.) could easily be achieved. A good rule of thumb to use in coring newly qualifying trees at remeasurement time is to core only those smaller than 10 inches in d. b. h. to ascertain age.

From the above periodic remeasurement of sample trees selected by a permanent point, the 4 major components of net change in basal area and volume per acre over a lo-year period are readily available:

	SUBTRACT				ADD				
Initial d.b.h. (inches)	Subsequent mortality Basal area	Volume	Subsequent harvest Basal area	Volume	Terminal d.b.h. (inches)	Survivor growth Basal area	Volume	Ingrowth Basal area	Volume
	Sq. ft.	cu. ft.	sq. ft.	cu. ft.		sq. ft.	cu. ft.	sq. ft.	cu. ft.
4.0	..	...	..	...	6.0	12.49	343.7	..	...
10.0	..	...	..	...	12.0	4.40	li8.3	..	...
10.0	10.00	220.0	..	...	...	...	...	..	...
20.0	..	...	10.00	343.5	...	...	...	..	...
.0					1.0	...	...	10.00	...
.0					6.0	...	...	10.00	152.8
Totals	10.00	220.0	10.00	343.5	Totals	16.89	472.0	20.00	152.8

If a 1 percent per year simple annual growth rate (or some locally more valid estimate) is assumed for the slower growing mortality and harvest components (assumed to cease growth in midperiod), the harvest and mortality totals should be increased by multiplication by 1 t (5) (. 01) = 1. 05. Hence, the basal-area estimates for a IO-year period would be 10. 5 sq. ft. per acre mortality, 10.5 sq. ft. per acre harvest, 16. 89 sq. ft. per acre survivor growth, and 20 sq. ft. per acre ingrowth--the gross growth being 37. 89 sq. ft. per acre, from which 1.0. 5 sq. ft. of mortality and 10. 5 sq. ft. of harvest must be deducted to get the net gain in growing stock per acre over the lo-year period. A similar calculation can be carried out for cubic feet or board feet, and frequency, of course, is directly available as

$$\frac{\text{HPF}}{(\text{initial or terminal d. b. h.})^2}$$

It will frequently be desirable to express these change components as simple annual percents of initial or terminal stand. The simple annual percents can then be applied to a much larger sample derived from supplementary temporary points (by species, tree class, and diameter or height class if desired).

If permanent points are used for a second growth period, the newly qualifying trees older than 10 years at breast height at the start,of the second period should be included in the new sample, as well as the newly qualifying ingrowth trees 10 years of age or younger (which in the second growth period are no longer classed as ingrowth from the zero d. b. h. class, but will become survivors, harvest, or mortality). The second growth-period calculations are then based on tree diameters measured at the start of the second growth period, just as though there had never been a first growth-period.

Use of temporary sample points or lines, coupled with coring all tallied trees, will furnish estimates of growth of surviving trees and ingrowth (to any size class from trees originating at any time). Harvest and unsalvaged mortality must be obtained by auxiliary techniques familiar to all users of growth techniques involving stand projection. Obviously, if temporary points and coring are used for estimating survivor growth and ingrowth, the factor by which basal area and volume differences should be multiplied is $\frac{1833}{(\text{current d. b. h.})^2}$ throughout (properly corrected for shape, slope, slopover, etc.).

Although the preceding discussions bring together in usable form all theory necessary to obtain precise and unbiased estimates of tree frequency, basal area, volume, or growth on a per-acre basis by means of point-sampling or line-sampling, it might be helpful to compare plot-sampling, line-sampling, and point-sampling in a specific situation. For simplicity, each tree'will be assumed to have 1 of 3 exact diameters (6. 0 inches, 12. 0 inches,, 18. 0 inches). Actually, the distribution of trees in diameter would be spread over a continuum, but this is immaterial to the purpose of the example.

The 3 diameters will have the following basal areas in square feet: 6. 0 inches, $\frac{\pi}{16}$ sq. ft.; 12. 0 inches, $\frac{4\pi}{16}$ sq. ft.; 18. 0 inches, $\frac{9\pi}{16}$ sq. ft. Assume that the following stand of trees of the above 3 sizes exists on a specified acre:

Tree D (inches)	*Frequency per acre*	*Diameter per acre*	*Basal area per acre*
	Number of trees	*Inches*	*sq. ft.*
6. 0	96	576 (=48x 12)	18. 8 (= 6π)
12.0	64	768 (=64x 12)	50.3 (=16π)
18. 0	16	288 (=24 x 12)	28.3 (= 9π)
	176	1632(=136x12)	97.4 (=31π)

The expectation (or average) for a single plot-sample, line-sample, and point-sample is compared below for a one-tenth-acre plot, a one-chain line-sample (with a 104. 18-minute angle-gauge being used on both sides of the line), and a single point-sample (with a 104. l&minute angle-gauge being used on a full 360° sweep):

Tree D (inches)	*Expected tally for-- One- tenth-acre plot sample*	*104.18-minute fine-sample*	*104.18-minute point-sample*
	-Number of trees		
6.0	9.6	4. 8	1. 88
12. 0	6.4	6.4	5. 03
18. 0	1.6	2.4	2. 83
Total	17.6	13. 6	9. 74

Note that number of trees tallied by one-tenth-acre plot, multiplied by 10, gives frequencies per acre; that number of trees tallied by one-chain line-sample, multiplied by 10, gives aggregate diameter in feet per acre, or multiplied by 120 (= HLF), gives aggregate diameter in inches per acre; and that number of trees tallied by point- sample, multiplied by 10 (=BAF), gives aggregate basal area in square feet per acre, or multiplied by 1833 (= HPF), gives aggregate squared diameter per acre. Further note that l&e-sampling with a 104. 18-minute angle-gauge tallied fewer trees smaller than 12 inches than did one-tenth-acre plot-sampling, and that point- sampling with a 104. l&minute angle-gauge tallied fewer trees smaller than 15. 28 inches than did line-sampling, and fewer trees smaller than 13. 54 inches than did one-tenth-acre plot-sampling.

It is now apparent that frequency per acre can be estimated by multiplying each tree in the plot-sample by 10, or by multiplying each tree in the line-sample by $\frac{120}{D}$, or by multiplying each tree in the point- sample by $\frac{1833}{D^2}$, where D is measured in inches. If D is measured in feet and if basal area replaces D^2, both numerators will become 10. Thus, HLF (= 120) and HPF (= 1.833) are analogs of plot blowup factor, except that an appropriate function of D must also be employed. Slope, slopover, and shape would, of course, modify the factors actually used.

Line-samplers will find additional convenient HLF's to be 165 and 330 (with HDF's of 2 and 1, respectively). These will involve prisms of about 4 and 8 diopter strength, respectively.

LESS PRECISE ESTIMATES

Some less precise adaptations of point- and line- sampling technique are worthy of note. Bell and Alexander (1) have gauged tree diameter at the top of a 16-foot log instead of at breast height. This approach has advantages when used with volume tables entered with first-log diameter, and when half-sweeps gauge only trees on the downhill side of the point. However, it does not appear to be a convenient dimension for ordinary use. The top of the first 16-foot log is expensive to locate, a gauge adjustment is always needed before it can be gauged, and a check on doubtful trees is exorbitantly expensive.

Hirata (7) has point-sampled total height above breast height with a vertical angle-gauge counting all qualifying trees, then counting all trees on small circular plots concentric about each sampling point. Without any tree measurements, an unbiased estimate of quadratic mean tree height in feet $\left(\sqrt{\frac{\sum H_i^2}{M}}\right)$ is calculated as 4-1/2 feet plus the product of plot radius (in feet) times tan vertical gauge-angle times square root of the ratio of sum of point-sample counts to sum of plot- sample counts. Since quadratic mean tree height has few uses, it seems unlikely that the technique is of more than theoretical interest, especially in view of the difficulty of detecting all qualified tree tops, and the expense of checking doubtful trees.

Besides Hirata's quadratic mean height technique, several other combinations of point- or line-sampling with plot-sampling are feasible and do not require any tree measurements.

Quadratic mean tree diameter in inches ($\sqrt{\frac{\sum^{M} D_i^2}{M}}$) can be estimated from a horizontally point-sampled tree count and the concentric plot-sample tree count. It will be plot diameter in inches times sin 1/2 horizontal gauge-angle times square root of ratio of point-sample count to plot-sample count.

Arithmetic mean tree diameter in inches ($\frac{\sum^{M} D_i}{M}$) can be estimated from a horizontally line-sampled tree count (samples on both sides) and the tree count on the strip-sample bisected by the line. It will be strip width in inches times sin 1/2 horizontal gauge-angle times the ratio of line-sample count to strip-sample count.

Arithmetic mean tree height in feet can similarly be estimated from a vertically line-sampled tree count (with samples taken on both sides of the line) and a tree count on the strip-sample bisected by the line. It will be 4-1/2 feet plus the product of one-half strip width in feet times tan vertical gauge-angle times the ratio of vertical line-sample count to strip-sample count.

Although convenient gauge-angles, plot radii, and strip widths can readily be calculated, it is not believed that the above substitutes will be as efficient as point-sampling or plot-sampling with direct measurement of the desired variables, except under unusual circumstances.

Another combination technique, requiring direct measurement of diameter only, is that of Strand (8). He employed a short segment of line to select two sets of sample trees--one set selected by a horizontal line-sample with probability proportional to diameter, and another by a vertical line-sample with probability proportional to total height above breast height, If samples are taken on both sides of the line, mean basal area per unit land area is then estimated by the sum of diameters of the horizontally line-sampled trees times .785398 times sin 1/2 horizontal gauge-angle, all divided by line-segment length (using common units of measure--multiply by 43,560 to convert to per-acre basis). Total basal area times height (above breast height) per unit of land area is estimated by the sum of squared diameters of the vertically line-sampled trees times .392699 times tan vertical gauge-angle, all divided by line-segment length in common units of measure; multiply by 43,560 to convert to per-acre basis. Obviously, convenient angles and line- segment lengths can be chosen to simplify calculations, and basal area times tree height below breast height can be readily added to the second estimate. Dividing the second estimate by the first will give an estimate of mean height (weighted by basal area). If sample trees are selected on only one side of line, the above formulae must be multiplied by 2. This combination technique has the same disadvantage of all vertical sampling--without expensive check of doubtful vertical gauging, it is subject to bias. Since each line-sampled tree must be visited for diameter measurement, it would seem that horizontal point-sampling with measurement of height from a convenient point where it is easily visible (and with the usual d. b. h. measurement) would be either more efficient or less subject to bias.

Nevertheless vertical line-sampling may have merit in estimating lineal feet (above' breast height) per unit of land area for pole estimates by pole class, or for correlating ground-samples with aerial photo-interpretation in terms of lineal feet of tree height per unit area of land. From page 6, it may be seen that a vertical angle-gauge tree count on both sides of a sample line (without any diameter or height measurement) will allow an estimate of lineal feet (above breast height) per unit of land area--it is merely the sum of counted trees in a given class times 1/2 times tangent vertical gauge-angle, all divided by line-segment length in common units of measure; multiply by 43,560 to convert to per-acre basis. Even here, the expense of checking doubtful trees and the crude approximation needed to allow for the omitted lengths between stump and breast height probably make point-sampling preferable if accuracy is desired.

The most useful crude application of horizontal line-sampling will probably be to get the sum of cull tree diameters per unit of land area in need of girdling or poisoning,for costs of such work are closely correlated with circumference or sum of diameters per unit of land area. As has been explained on page 6, this can be obtained by simply counting horizontally gauged cull trees on both sides of the sample line. The sum of diameters per unit of land area will be the tree count times sin [redacted] -1/2 gauge-angle, all divided by line length (all measurements in common units). If tree diameters are desired in inches (1/12 foot), if desired unit of land area is an acre (43,560 square feet), and if line length is measured in chains (66 feet each), the above formula should be multiplied by 7920.

Probably the most useful crude applications of horizontal point-sampling are included in the diagnostic tally devised by Grosenbaugh (4). These include the original tree count for basal area earlier devised by Bitterlich (2). Convenient tables of gauge-angles, calibration distance factors, basal *area* factors, horizontal distance factors (called plot radius factors), and slope correction factors may be found in (4).

LITERATURE CITED

(1) Bell, J.F., and Alexander, L.B. 1957. *Application of the variable plot method of sampling forest stands.* Ore. State Board Forestry Res. Note 30, 22pp., illus.

(2) Bitterlich, W. 1948. *Die Winkelzählprobe*. Allgemeine Forst- und Holzwirtschaftliche Zeitung 59(1/2): 4-5.

(3) Crosenbaugh, L.R. 1952. *Plotless timber estimates--new, fast, easy.* Jour. Forestry 50: 32-37, illus.

(4) __________. 1955. *Better diagnosis and prescription in southern forest management.* U.S. Forest Serv. South. Forest Expt. Sta. Occas. Paper 145, 27 pp.

(5) __________ and Stover, W.S. 1957. *Point-sampling compared with plot-sampling in southeast Texas.* Forest Sci. 3: 2-14.

(6) Hansen, M.H., Hurwitz, W.N., and Madow, W.G. 1953. *Sample survey methods and theory.* 2 vols., illus. N.Y. and London. (Esp. vol. 1, pp. 341-342.)

(7) Hirata, T. 1955. *Height, estimation through Bitterlich's method--vertical angle-count sampling [in Japanese].* Japanese Jour. Forestry 37: 479-480.

(8) Strand, L. 1957. "*Relaskopisk*" *hoyde- og kubikkmassebestemmelse.* Norsk Skogbruk 3 : 535-538, illus.