A chance constraint estimation approach to optimizing resource management under uncertainty

Michael Bevers

Abstract: Chance-constrained optimization is an important method for managing risk arising from random variations in natural resource systems, but the probabilistic formulations often pose mathematical programming problems that cannot be solved with exact methods. A heuristic estimation method for these problems is presented that combines a formulation for order statistic observations with the sample average approximation method as a substitute for chance constraints. The estimation method was tested on two problems, a small fire organization budgeting problem for which exact solutions are known and a much larger and more difficult habitat restoration problem for which exact solutions are unknown. The method performed well on both problems, quickly finding the correct solutions to the fire budgeting problem and repeatedly finding identical solutions to the habitat restoration problem.

Résumé : L'optimisation à contrainte aléatoire est une importante méthode pour gérer le risque associé aux variations aléatoires dans les systèmes de ressources naturelles mais les formulations probabilistes posent souvent des problèmes de programmation mathématique qui ne peuvent être résolus avec des méthodes exactes. Une méthode d'estimation heuristique pour résoudre ces problèmes est présentée dans cet article. Cette méthode combine une formulation pour les observations de statistiques d'ordre avec une méthode d'approximation de la moyenne de l'échantillon comme substitut pour les contraintes aléatoires. La méthode d'estimation a été testée avec deux problèmes : un problème budgétaire d'une petite organisation de lutte contre les incendies pour lequel les solutions exactes sont connues et un problème beaucoup plus gros et difficile de restauration d'habitat pour lequel les solutions exactes sont inconnues. La méthode a eu une bonne performance dans les deux cas en trouvant rapidement les bonnes solutions au problème budgétaire et des solutions identiques au problème de restauration d'habitat de façon répétitive.

[Traduit par la Rédaction]

Introduction

Random disturbances ranging from small perturbations to large shocks produce variations from expected resource conditions that can be important to forest managers. Recent occurrences of catastrophic forest fires, insect and disease outbreaks, and hurricane damage offer notable examples (Prestemon and Holmes 2004). Smaller perturbations in forest growth, wildlife population demographics, market prices, and other influences on resource management can also have substantial effects (Goodman 1987; Gove and Fairweather 1992).

Exact methods for optimizing resource management with random outcomes have been used to develop management guidelines and investigate policy implications. For example, optimal control theory has been used by Reed (1987) to explore optimal forest fire protection patterns and harvest policies, by Lenhart et al. (1999) to control undesirable effects at habitat boundaries, and by Bhat et al. (1999) to control river contamination by surface runoff. Stochastic dynamic programming has been used by Fina et al. (2001) to account for the effects of random price arrivals on optimal rotations, by Gong and Yin (2004) to account for the effects of correlated product prices on optimal harvest strategies, by Spring and Kennedy (2005) to account for random fire effects on optimal timber and wildlife management, and by Spring et al. (2005) to explore climate change effects on optimal catchment management for water and timber production and carbon sequestration. Markov decision models have been used by Boychuck and Martell (1988) to roughly determine the number of firefighters needed for a season and to evaluate the benefits of centralized control and by Rollin et al. (2005) to account for the effects of random growth and prices on optimal uneven-aged management of a mixed-species forest. Multistage stochastic programming has been used by Gassmann (1989) to investigate the effects of forest fires on optimal harvesting and by Boychuck and Martell (1996) to investigate the effects of forest fires on sustainable timber supplies. Chance-constrained programming has been used by Hunter et al. (1976) to optimally allocate forage and by Hof et al. (1996) to account for the effects of spatial autocorrelation in timber yields on optimal timber harvesting.

Solving these types of models can be quite challenging, even for moderately small problems. Heuristic methods, often developed and tested against small problems with known solutions (e.g., Strange et al. 2006), are commonly

Received 24 July 2006. Accepted 13 April 2007. Published on the NRC Research Press Web site at cjfr.nrc.ca on 17 November 2007.

M. Bevers.[1] USDA Forest Service, Rocky Mountain Research Station, 2150A Centre Avenue, Suite 361, Fort Collins, CO 80526, USA (mbevers@fs.fed.us).

 doi:10.1139/X07-076

used when increasing realism creates stochastic problems too large to solve exactly. In this paper, we develop and test a heuristic method for reformulating and solving stochastic programming and, especially, chance-constrained programming problems (see Birge and Louveaux (1997) for more background on these general classes of problems). The contribution to the resource management literature is in three parts. First, the sample average approximation method (Kleywegt et al. 2001), which has been used informally by the resource modeling community to address large scenario optimization problems (e.g., Snyder et al. 2004), is described in detail. Second, we show how the order statistic approach used by Fuessle et al. (1987) to exogenously model chance constraints can be combined with sample average approximation to create an endogenous linear formulation for these difficult problems. Third, a search algorithm is presented that exploits these formulations. This algorithm combines the indifference-zone approach from the ranking and selection literature (see Goldsman and Nelson 1998) with a probability-based method for search termination. Details of these contributions are presented in the Methods section below using two artificial test cases. The first test case poses a small firefighting budget allocation problem that has been greatly simplified and contrived to provide substantial experimental control. The second test case poses a more realistic wildlife habitat restoration problem.

Methods

We begin by defining a test model (Problem EVP, the simplified firefighting budget allocation problem) small enough to be solved exactly by enumeration given the necessary probability distributions. We then reformulate the exact random variable problem as a sequence of separate, instantiated integer programming estimation problems. Each estimation problem (Problem SEVP-*k*) uses sample average approximation to provide an estimate of the optimal expected objective function value for the full problem. The sample average approximation approach is recommended for problems where a large number of random variables makes scenario optimization impractical (Shapiro 2008).

Order statistics are then introduced, forming Problem SCCP-*k*, to address chance constraints. We describe a search algorithm (Algorithm SCCP-*k*) that combines simulation with optimization of separate estimation problems (indexed by *k*) to select solutions for the full problem. A larger and more complicated test model that includes joint chance constraints (for a habitat restoration problem) is then defined exactly in Problem JCCP and reformulated in Problem SJCCP-*k*. Solutions and search times for the two test cases are reported in the Results section.

Expected value stochastic programming problems

Consider a fire planning unit with nine subunits where the problem is to allocate one of three possible fire organization budgets to each subunit so as to minimize either the expected cost of fires for the fire season or the seasonal cost associated with a prespecified, more risk-averse probability level (i.e., a "probable" or "chance-constrained" cost). The expected value stochastic programming problem (Problem EVP) is formulated and revised below; formulations for the chance-constrained programming problem (beginning with Problem CCP) follow.

Problem EVP

[1] $$\text{Minimize } \mathrm{E}[Y]$$

subject to

[2] $$Y - \sum_i \sum_j F_{ij} X_{ij} = 0$$

[3] $$\sum_j X_{ij} = 1 \quad \forall i$$

[4] $$X_{ij} \in \{0, 1\} \quad \forall i, j$$

[5] $$\sum_i \sum_j c_{ij} X_{ij} \le \zeta$$

where i indexes the I = 9 subunits of the fire planning unit, j indexes the J = 3 fire organization budgets being considered for each subunit, F_{ij} is a random variable representing unbudgeted fire suppression cost for the season in subunit i given fire organization budget j, c_{ij} is the fixed seasonal presuppression cost of using fire organization j in subunit i, ζ is the total budget for fixed seasonal presuppression costs available to the planning unit, X_{ij} is a binary decision variable set to 1 when fire organization budget j is selected for subunit i (and set to zero otherwise), and Y is a simple recourse variable (where no subsequent choices are available except "to pay the bills") representing the total random fire suppression cost across the planning unit for the season.

Equation 1 minimizes the expected seasonal total of variable fire suppression costs (referred to hereafter as "fire costs") for the planning unit as summed across subunits and fire budget choices in eq. 2. Equations 3 and 4 require that each subunit be allocated exactly one of the three fire organization budgets being considered for that subunit. Equation 5 requires that the total fixed presuppression costs of employing the nine selected fire organizations ("budget costs") not exceed the available fixed-cost budget. The presuppression budget costs constrained in eq. 5 would often be minimized in eq. 1 along with expected fire suppression costs in stochastic programming problems; eq. 5 is used here instead so that we can examine results for two different budget levels. For ease of illustration, Problem EVP uses spatial autocorrelations of fire costs (see Results, Test case 1) as a surrogate for travel costs (e.g., MacLellan and Martell 1996) or travel times (e.g., Donovan and Rideout 2003) more typically used in firefighting resource allocation models. Problem EVP also has the desirable property for testing purposes that it can be solved exactly, providing a basis for comparison with estimation results.

Problem EVP is straightforward if the mean subunit fire suppression costs for the season (ϕ_{ij}) can be estimated exogenously and entered directly into the problem, replacing the random variables (F_{ij}) in eq. 2. The optimal fire organization decision X^*_{EVP} resulting from Problem EVP is then based on knowledge of the expected fire costs for the season (although actual fire costs in any one season might differ considerably). Where a complex system of constraints could affect expected values and might preclude estimating expected values a priori, one alternative is to estimate the expected values endogenously using sample average approximation as in Problem SEVP below.

Problem SEVP

$$[6] \quad \text{Minimize} \sum_{n=1}^{N} \frac{1}{N} Y_n$$

subject to

$$[7] \quad Y_n - \sum_i \sum_j f_{ijn} X_{ij} = 0 \quad \forall n$$

$$[8] \quad \sum_j X_{ij} = 1 \quad \forall i$$

$$[9] \quad X_{ij} \in \{0, 1\} \quad \forall i, j$$

$$[10] \quad \sum_i \sum_j c_{ij} X_{ij} \leq \zeta$$

where n indexes sample values f_{ijn} of the random variables (F_{ij}) representing fire costs for the season in each subunit under each budget. The optimal budget allocation decision (X^*_{SEVP}) is now based on minimizing sample average fire cost using knowledge of the N sample outcomes (hence, imperfect knowledge of the expected values), i.e., the decision is only optimal with respect to the samples drawn. Accuracy in finding the true optimal solution of interest (X^*_{EVP}) is sacrificed to an unknown extent, and an additional computing cost is incurred due to the expansion from eq. 2 with dimension 1 to eq. 7 with dimension N.

When total expected cost is a function of numerous constraints that create a large mathematical programming problem, sample size N might have to be kept small. In practice, optimal solutions to the sample average approximation problem could easily miss the mark of identifying an optimal solution to the true expected value problem (see Linderoth et al. 2006). A further step then is to solve random replications of Problem SEVP, referred to below as Problem SEVP-*k*.

***Problem SEVP*-k**

$$[11] \quad \text{Minimize} \sum_{n=1}^{N} \frac{1}{N} Y_{nk}$$

subject to

$$[12] \quad Y_{nk} - \sum_i \sum_j f_{ijnk} X_{ijk} = 0 \quad \forall n$$

$$[13] \quad \sum_j X_{ijk} = 1 \quad \forall i$$

$$[14] \quad X_{ijk} \in \{0, 1\} \quad \forall i, j$$

$$[15] \quad \sum_i \sum_j c_{ij} X_{ijk} \leq \zeta$$

Problem SEVP-*k* is identical to Problem SEVP except that index k has been added to specify random replications of the problem. This highlights that each mathematical programming model replication k produces one estimated solution. Simulations, statistical procedures, and (or) heuristic methods might then be used to select an estimated global optimum from the K replications examined. Problem SEVP-*k* is the building block of the expected value optimization approach tested in this study.

Chance constraint problems

We noted above that after selecting the budgets and implementing the fire organizations that minimize expected fire costs, actual fire costs for a given season (such as the upcoming fire season) might differ considerably. For example, in the tests reported here based on lognormal distributions of fire costs in each subunit, observed seasonal fire costs exceeded average seasonal fire costs in about 40% of the cases simulated. Had fire costs been normally instead of lognormally distributed, average fire costs would have been exceeded in about 50% of the simulations. Risk-averse fire managers might prefer to minimize and plan for cost levels that have a smaller probability of exceedance. Chance-constrained programming is one method for optimizing such "probable" rather than "expected" outcomes (Bevers and Kent 2007). Problem CCP, formulated below, redefines the fire budget problem as a chance-constrained programming problem.

Problem CCP

$$[16] \quad \text{Minimize}\, B$$

subject to

$$[17] \quad \Pr(Y > B) < p$$

$$[18] \quad Y - \sum_i \sum_j F_{ij} X_{ij} = 0$$

$$[19] \quad \sum_j X_{ij} = 1 \quad \forall i$$

$$[20] \quad X_{ij} \in \{0, 1\} \quad \forall i, j$$

$$[21] \quad \sum_i \sum_j c_{ij} X_{ij} \leq \zeta$$

The chance constraint (eq. 17) sets an upper level B for total fire cost Y and requires that the probability (Pr) of exceeding that cost be less than parameter p, an accepted level of risk (e.g., 0.05). The resulting probable fire cost B is minimized in eq. 16. Because eqs. 17 and 18 can make solving Problem CCP quite difficult, we are again interested in using endogenous estimation to simplify the problem. To that end, nonparametric order statistics (Larsen and Marx

1986) are used as a substitute for chance constraints in the problem that follows. Previously, Fuessle et al. (1987) used order statistics in an exogenous simulation procedure to solve a chance-constrained air quality management problem. We build on that idea here by formulating order statistic observations directly in the mathematical programming model and using sample average approximation to estimate expected values of those order statistics.

We observe that in each replicate of Problem SEVP-*k*, the objective function (eq. 11) is not just an estimate of the minimum expected fire cost that can be achieved, it is also (trivially) an estimate of the minimum expected value of the first-order statistic of fire cost from an order statistic sample of size 1. In Problem SCCP-*k* (described below), this idea is extended to provide estimation of the minimum expected value of the *M*th-order statistic from an order statistic sample of size *M*. Our motivation is that as order statistic sample size *M* increases, the expected value of the extreme fire cost observation and the associated cumulative probability both increase; we will be minimizing a fire cost in the right-hand tail of the distribution associated with the resulting decision vector, where the probability p_M of exceeding that fire cost is relatively small. The solution to Problem SCCP-*k* provides an estimated solution to Problem CCP for the case where $p = p_M$ in eq. 17. Even if during optimization we do not know the probability p_M associated with the *M*th-order statistic (because we might not know beforehand the probability density functions of overall fire cost for all possible decision vectors), that probability can be estimated easily with postoptimization simulations of the resulting decision vector. Algorithm SCCP-*k* (described below) and Experiment 2 in the Results section will help demonstrate this concept. Problem SCCP-*k* is formulated as

***Problem SCCP*-k**

$$[22] \qquad \text{Minimize} \sum_{n=1}^{N} \frac{1}{N} Y_{nk}$$

subject to

$$[23] \qquad Y_{nk} - \sum_{i} \sum_{j} f_{ijmnk} X_{ijk} \geq 0 \quad \forall m, n$$

$$[24] \qquad \sum_{j} X_{ijk} = 1 \quad \forall i$$

$$[25] \qquad X_{ijk} \in \{0, 1\} \quad \forall i, j$$

$$[26] \qquad \sum_{i} \sum_{j} c_{ij} X_{ijk} \leq \zeta$$

where *m* now indexes the set of observations used for each of the *N* independent observations of the *M*th-order statistic. The dimension of eq. 12 from Problem SEVP-*k* is multiplied by *M* in eq. 23, which is now an inequality constraint providing *N* order statistic observations. The expected value estimated as a sample average in eq. 22 is now the expected value of the *M*th-order statistic of fire cost for the season.

Note that Problem SCCP-*k* adds a third level to our hierarchical sampling effort: (*i*) we take independent, identically distributed samples indexed by *m* up to some number *M* to obtain each observation of the *M*th-order statistic of fire cost, (*ii*) we make multiple observations of the *M*th-order statistic to endogenously estimate the expected value of that order statistic using the *N* overarching samples indexed by *n*, and (*iii*) we make one independent observation of the minimized average value of the *M*th-order statistic of fire cost for the season with the solution of each replicate mathematical program, indexed by *k*, of Problem SCCP-*k*. The instantiated estimation Problems SEVP-*k* and SCCP-*k* are easy to build and to solve, as would be many more realistic problems. How the number of problems to be solved (*K*) is determined and how a single solution is chosen from the set of *K* resulting solutions are addressed in the section that follows.

Before proceeding, we also note that the expected value estimation Problem SEVP-*k* is simply the special case of the expected order statistic estimation Problem SCCP-*k* where $M = 1$. Hereafter, we refer only to Problem SCCP-*k*, relying on the setting for parameter *M* to indicate whether we are searching for solutions to a stochastic integer (expected value) programming problem ($M = 1$) or for solutions to a chance-constrained integer programming problem ($M > 1$).

A heuristic solution algorithm

The heuristic solution procedure used in this study is one of many possible procedures that might be used (e.g., see Reeves 1993). As a heuristic method, this algorithm makes no attempt to run until it converges to a true optimal solution (i.e., a solution to either Problem EVP or to Problem CCP, whichever is intended by the magnitude of parameter *M* in our estimation Problem SCCP-*k*). Instead, the algorithm continues searching by constructing and solving Problem SCCP-*k* replicates until the estimated probability (τ) of finding a better solution with the next replicate decreases to a prespecified stopping point (τ_0). The steps of the algorithm are as follows.

***Algorithm SCCP*-k**

Step 0: Initialize the best objective function value found (μ^) to an arbitrarily large number. Initialize the stack of best solution vectors found and the stack of discarded solution vectors to zeroes; initialize to zeroes counters for each decision variable, a counter *k*^ for the number of replications tested since finding the best solution examined, and the counter *k* for the number of the replication currently being examined. Set k_{MIN} = the minimum number of Problem SCCP-*k* replications to examine, k_{MAX} = the maximum number of replications to examine, τ_0 = the probability point for terminating the search based on τ, the probability of finding a better solution with the next replication, α = an acceptable probability of rejecting hypothesis H_0: $\tau = \tau_0$ when H_0 is true (the probability of committing a Type I error), β = an acceptable probability of accepting H_0 when it is false (the probability of committing a Type II error), $\tau_0 + \varepsilon$ = the probability of finding a better solution assumed true for computing $1 - \beta$, the probability of accepting H_1: $\tau > \tau_0$ when $\tau = \tau_0 + \varepsilon$ (with $\varepsilon > 0$), δ = the width of an interval of indifference, or tolerance, used as a confidence interval to distinguish estimated objective function values, Θ = the confidence level imposed for interval estimation of μ_k such that $\Pr(\mu_k^* - \delta/2 \le \mu_k \le \mu_k^* + \delta/2) \ge \Theta$ where μ_k is the true objective function value for replicate k decision vector $\boldsymbol{X}^*_{\mathrm{SCCP}-k}$ (abbreviated $\boldsymbol{X}_k^*$) and μ_k^* is the observed mean objective function value, M and N.

Step 1: Set $k = k + 1$. Build and solve replicate Problem SCCP-k. Increment counters by 1 for the selected decision variables.

Step 2: Check stacks for decision vector $\boldsymbol{X}_k^*$. If found, go to Step 5.

Step 3: Simulate the selected decision vector $\boldsymbol{X}_k^*$ until confidence level Θ is achieved for the δ-interval specified by observed value μ_k^*.

Step 4: Test μ_k^*. If $(\mu_k^* + \delta/2) < (\mu^\wedge - \delta/2)$, then μ_k^* is declared a better solution than the previous best solution $\mu^\wedge$. Set $k^\wedge = 1$ and $\mu^\wedge = \mu_k^*$. Move decision vectors from the best solution stack to the discarded solution stack. Place $\boldsymbol{X}_k^*$ on the best solution stack. Else if $(\mu_k^* - \delta/2) \le (\mu^\wedge + \delta/2)$, then μ_k^* is declared an alternate best solution. Set $k^\wedge = k^\wedge + 1$. Place $\boldsymbol{X}_k^*$ on the best solution stack. Else μ_k^* is declared a suboptimal solution. Set $k^\wedge = k^\wedge + 1$. Place $\boldsymbol{X}_k^*$ on the discarded solution stack.

Step 5: Construct a new decision vector $\boldsymbol{X}_k$, prioritizing allocations based on the counts recorded for each decision variable from the preceding k solutions. Repeat Steps 2–4 for $\boldsymbol{X}_k$ in place of $\boldsymbol{X}_k^*$, but without returning to Step 5 and without incrementing $k^\wedge$ if $\boldsymbol{X}_k$ is not a better solution.

Step 6: If $k^\wedge \ge 100$, test hypothesis H_0 as described below. If ($k < k_{\mathrm{MIN}}$) or if ($k < k_{\mathrm{MAX}}$ and H_0 tests false or was untested), go to Step 1.

Step 7: Accept $\mu^\wedge$ as a qualified estimate of the true global optimal solution (μ^*), as described below. Simulate the decision vectors from the best solution stack to estimate p, the probability that $\mu^\wedge$ will be exceeded in any given fire season. Report those decision vectors and probability estimates along with $\mu^\wedge$. Stop.

The decision variable counters incremented in Step 1 and used in Step 5 to construct a new decision vector are intended to account for the possibility in large problems that none of the replicates may actually produce a globally optimal decision vector. For some problems, the greater number of times some decision variables are selected relative to others might be indicative that those variables belong in the optimal decision (Bertsimas et al. 2007).

Maintaining a stack of the discarded decision vectors, as well as the set of best decision vectors found, allows the current decision vector to be checked against all previously examined decision vectors in Step 2. This prevents unnecessarily repeating simulations to estimate the objective function, which might be quite time-consuming depending on the indifference interval and the confidence level specified for Step 3.

The test in Step 4 for whether a better, worse, or equivalent solution has been found is relatively strict. Ordinarily, a two-sample t test might be used, but that test can be problematic if the variances differ for the two solutions. Here, each of the two solutions is simulated in batches of 100. The sample mean from each batch is used as a single observation, and the distribution of sample means is assumed to be approximately normal. Batches of simulations continue until the probability that the true mean value is within a ±δ/2 interval of the overall sample mean is at least confidence level Θ. At least 200 batches are simulated and z values of the standard normal distribution are used to approximate Student t statistics for determining the confidence interval achieved with each new batch. The test in Step 4 then establishes that new objective function values declared to be either superior or inferior to the previous best value are smaller or larger, respectively, by more than δ with confidence Θ^2. Solutions that do not differ by more than δ (with confidence Θ^2) are treated as being equally good, i.e., if optimal, they are alternate optima.

The decision to terminate the search for better solutions is based on the likelihood of finding a better solution given another replication effort. Each time a solution better than the previous best solution is found, the probability of finding an even better solution decreases and the magnitude reached by counter $k^\wedge$ before finding the next "even better solution" tends to increase. Following each replication, $k^\wedge$ is treated as the size of a sample from the binomial distribution for which the observations are one success (i.e., a better solution was found) and $k^\wedge - 1$ failures (i.e., a better solution was not found). At $k^\wedge \ge 100$, we use the approximation

$$W = \frac{1 - (k^\wedge)(\tau_0)}{\sqrt{(k^\wedge)(\tau_0)(1-\tau_0)}} \sim N(0,1)$$

to test H_0.

Type II errors pose the greater concern for this test because incorrectly accepting H_0 leads to stopping the search prematurely. Consequently, power of the test is calculated as

$$1 - \beta^\wedge = \int_{t_a, k^\wedge - 1}^{\infty} f_{T_\Delta}(t)\mathrm{d}t$$

where T_Δ is a noncentral T variable with $k^\wedge - 1$ degrees of freedom and noncentrality parameter

$$\gamma = (k^\wedge)(\varepsilon)\sqrt{k^\wedge}$$

When $k^\wedge$ is sufficiently large that $\beta^\wedge \le \beta$, hypothesis H_0 is accepted as true and the search terminates in Step 6 unless the number of replications has not yet reached k_{MIN}, a prespecified lower limit. Alternatively, the search also terminates in Step 6 if the number of replications has reached k_{MAX}, a prespecified upper limit.The idea here is that the best solutions reported in Step 7 are accepted, whether or not they are truly optimal, on the basis that more searches are too unlikely to find better solutions to be worth the effort. Whether the reported solutions are truly equivalent or are merely quite similar in performance is treated as a matter of indifference to the decision maker; results are similar enough to be considered equivalent. While this approach borrows ideas from the literature on ranking and selection methods (Goldsman and Nelson 1998), it is important to note that Algorithm SCCP-k is designed for problems with many candidate solutions and stops short of rigorously identifying optimal solutions. Optionally, the algorithm could be revised to produce a manageable number of alternatives for more rigorous testing using ranking and selection to discern which solutions are defensibly the best of those examined (see Boesel et al. 2003).

Joint chance constraints

While the chance-constrained fire Problem CCP could be difficult to solve exactly if it was not small enough to solve by complete enumeration, the simple formulation does not highlight the kind of complexity that can be addressed with the estimation approach described here. To that end, we define a second test problem that introduces greater spatial complexity, hundreds of random variables, multiple time periods, and joint chance constraints. Problem JCCP, an iterative habitat restoration problem, uses information on initial locations and demographics of a wildlife population to optimize a schedule of habitat restorations, similar to the black-footed ferret (*Mustela nigripes*) population reintroduction and recovery problem addressed by Bevers et al. (1997). However, unlike the ferret problem, which was modeled deterministically and with continuous-variable representations of abundance, Problem JCCP stochastically models a hypothetical species using binary variables to represent the presence or absence of individual animals in habitat territories. The formulation is described below.

Problem JCCP at iteration 1 of 5

[27] $\quad$ Maximize B_1

subject to

[28] $$\Pr(\Lambda_1,\ \Lambda_2,\ \Lambda_3,\ \Lambda_4,\ \Lambda_5 \geq\ B_1)\ \geq\ q$$

[29] $$\Lambda_t\ = \sum_i Y_{it}; t = 1, ..., 5$$

[30] $$Y_{i0} = y_i \quad \forall i$$

[31] $$Y_{it} \leq L_{it} Y_{i(t-1)} + \sum_j L_{jt} Y_{j(t-1)} R_{jit} \quad \forall i; t = 1, ..., 5$$

[32] $$Y_{it} \leq 1; i = 61, ..., 103; t = 1, ..., 5$$

[33] $$Y_{it} \leq \sum_{h=1}^{5} v_{iht} X_{ih}; i = 1, ..., 60; t = 1, ..., 5$$

[34] $$\sum_{h=1}^{5} X_{ih} = 1; i = 1, ..., 60$$

[35] $$X_{ih} \in \{0, 1\} \quad \forall h; i = 1, ..., 60$$

[36] $$\sum_{i=1}^{60} X_{ih} = 12; h = 1, ..., 5$$

where t indexes population model years beginning with initial conditions at t = 0, h indexes the five alternative years for habitat restoration, i and j each index the set of 60 restorable territories (numbered 1–60) and 43 already suitable territories (numbered 61–103), X_{ih} is a 0–1 binary decision variable indicating restoration (when X_{ih} = 1) of territory i in year h, Y_{it} is a naturally 0–1 response variable indicating occupancy (when Y_{it} = 1) of territory i in year t by an adult female, Λ_t is the total abundance of adult females on the landscape in year t, B_ω is the smallest probable annual abundance of adult females in a given iteration of the solution procedure (e.g., ω = 1 above), as described below, q is a reliability parameter (e.g., q = 0.95) equivalent to 1 – p in Problem CCP, y_i is the initial occupancy state (0 = unoccupied, 1 = occupied) for each territory i, L_{it} is a random variable indicating survival (when L_{it} = 1, else 0) from year t – 1 to year t of the adult female (if any) in territory i, R_{jit} is a random variable indicating the integer number of female offspring born (given an adult female occupant) in territory j in year t – 1 that disperse and become adults in territory i in year t, and v_{iht} is a deterministic carrying capacity parameter for territory i set to zero in years prior to habitat restoration (i.e., $t < h$) and set to 1 otherwise.

Here, the problem is to maximize in eq. 27 a lower level B_ω representing the most limiting probable future annual abundance. The joint chance constraint in eq. 28 has been formulated in terms of reliability q rather than risk p. After each iteration, eq. 28 is revised to reflect the optimal probable abundance found in the preceding iteration. For example, suppose that at reliability q, probable abundance declines each year despite optimal restoration efforts. Then at iteration ω = 1, Λ_5 would provide the limiting value in eq. 28 for the optimal level B_1^*. Along with changing the variable in eq. 27 from B_1 to B_2 for iteration ω = 2, eq. 28 would be updated to $\Pr(\Lambda_1, \Lambda_2, \Lambda_3, \Lambda_4 \geq B_2; \Lambda_5 \geq B_1^*) \geq q$ reflecting the first iteration result. Following all five iterations, an optimal decision vector $\boldsymbol{X}_{\mathrm{JCCP}}^*$ is found that meets the joint chance constraint

$$\Pr(\Lambda_1 \geq B_5^*; \Lambda_2 \geq B_4^*; \Lambda_3\ \geq B_3^*; \Lambda_4 \geq B_2^*; \Lambda_5 \geq B_1^*) \geq q$$

(still under the supposition that abundance declines over time) along with meeting the other constraints reflected in eqs. 29–36.

Equation 29 totals adult female abundance across the 103 territories each year. Equation 30 sets the initial occupancy in each territory to either 1 or 0. Equation 31 models growth and dispersal of the adult female population as individual-based limiting random processes from initial conditions at year 0 through year 5. Territory occupancy is further constrained by habitat availability in eqs. 32 and 33; some habitat already exists (eq. 32), while habitat availability in eq. 33 results from the restoration decisions made in eqs. 34 and 35. Equation 36 is analogous to a budget constraint, requiring restoration of exactly 12 of the 60 restorable territories each year.

Like fire Problem CCP, this habitat restoration model is substantially simplified. Besides leaving out age and sex structure as well as other biological details that could be important, we are treating a multistage stochastic problem as a single-stage decision problem which assumes that we collect no new occupancy or demographic data during the restoration process. Unfortunately, adding full recourse decision stages would present too large a problem and, in practice, new data likely would be dealt with by updating and reanalyzing the single-stage formulation. Nonetheless, Problem JCCP presents considerable size and complexity. With approximately 3.3×10^{38} alternative restoration schedules in the feasible decision space, exact solutions to this problem are unknown and joint probability distributions of total abundance vectors cannot be estimated exogenously for all possible solutions.

Problem SJCCP-k at iteration 1 of 5

Much like Problem CCP, reformulating Problem JCCP to use sample average approximation of expected order statistic values requires replacing the chance constraint in eq. 28 and abundance summation in eq. 29 with eqs. 38–40 below, replacing the random variables in eq. 31 with sample observations in eq. 42, and adding sample indices throughout the model. More subtlely, the discrete annual abundance variables are replaced with continuous variables representing the average of N discrete first-order statistic observations of annual abundance:

[37] $\text{Maximize } B_{1k}$

subject to

[38] $B_{1k} \leq A_{tk}; t = 1, ..., 5$

[39] $A_{tk} \leq \sum_{n} \frac{1}{N} O_{tnk}; t = 1, ..., 5$

[40] $O_{tnk} \leq \sum_{i} Y_{itmnk} \, \forall m, n; t = 1, ..., 5$

[41] $Y_{i0mnk} = y_i \; \forall i, m, n$

[42] $Y_{itmnk} \leq l_{itmnk} Y_{i(t-1)mnk} + \sum_{j} l_{jtmnk} Y_{j(t-1)mnk} r_{jitmnk} \, \forall i, m, n; t = 1, ..., 5$

[43] $Y_{itmnk} \leq 1 \, \forall m, n; i = 61, ..., 103; t = 1, ..., 5$

[44] $Y_{itmnk} \leq \sum_{h=1}^{5} v_{iht} X_{ihk} \, \forall m, n; i = 1, ..., 60; t = 1, ..., 5$

[45] $\sum_{h=1}^{5} X_{ihk} = 1; i = 1, ..., 60$

[46] $X_{ihk} \in \{0, 1\} \, \forall h; i = 1, ..., 60$

[47] $\sum_{i=1}^{60} X_{ihk} = 12; h = 1, ..., 5$

where O_{tnk} is the nth observation of the first-order statistic from M observations of adult female abundance for each time period t and each mathematical programming replicate k.

Results

Fire Problem SCCP-k and habitat restoration Problem JSCCP-k provide two useful test cases. The fire problem allows us to look at the effects of sample average sample size N and the effects of order statistic sample size M for a case where solutions both to the expected value (M = 1) problem and to chance-constrained (M > 1) problems can be found by enumeration. The habitat restoration problem allows us to test the consistency of repeated heuristic solutions for a larger and more complicated case where exact solutions are unknown.

The search times reported below should be treated as relative values useful primarily for comparison with each other. The study was exploratory and mathematical programming steps were constructed with substantial amounts of processing to and from files rather than attempting to handle as many procedures in core memory as possible. Likewise, little effort was made to design the simulations to run more efficiently.

Test case 1: fire problem SCCP-*k*

In our first test problem (Problem SCCP-k), we introduced symmetry to intentionally create cases that had alternate optimal solutions, demonstrating the use of indifference interval parameter δ. Thus, we treated the subunits of the fire planning unit as square areas arranged in a 3 × 3 grid. Subunits were numbered 1 through 9 from upper left to lower right so that subunit 3 was in the upper right-hand corner and subunit 7 was in the lower left-hand corner of the grid. Three alternative decisions were possible for each subunit, and decisions for the entire planning unit were described by vectors ranging from $\boldsymbol{X}$ = (1, 1, 1, 1, 1, 1, 1, 1, 1), where all subunits were assigned the least expensive alternative, to $\boldsymbol{X}$ = (3, 3, 3, 3, 3, 3, 3, 3, 3), where all subunits were assigned the most expensive alternative. We used this notation for convenience; the actual decision vectors had 27 binary (0–1) elements.

The budget cost on each subunit for alternative 1 was set to \$1 million, the cost for alternative 2 was \$1.5 million, and the cost for alternative 3 was \$2 million. The distribution of fire cost for the season on each subunit was a lognormally distributed function of the selected alternative (as suggested but not tested by Strauss et al. 1989). These distributions are described by the mean and variance of the underlying normal distributions. Under alternative 2, the underlying mean parameter for fire cost was set to 12.6 and the standard deviation to 0.8 for each subunit. Random fire costs by subunit were then generated for that alternative as

$$f_{i,(j=2),mnk} = \mathrm{e}^{(12.6+0.8z)}$$

where z is a random deviate from the standard normal distribution. Under alternative 1, the means and standard deviations of the underlying normal distributions were increased by 10%; under alternative 3, they were decreased by 10%. Seasonal fire costs were spatially correlated in such a way that the correlations of z values between subunits that shared a common edge were 0.40, the correlations between subunits that shared only a common vertex were 0.15, and no correlations existed between subunits that did not adjoin.

Johnson and Kotz (1970) provide a formula that gives ex-

Table 1. Estimated objective function values, probability of exceedance values, and search times for various magnitudes of M in Experiment 2 on the Mth-order statistic estimation Problem SCCP-k.

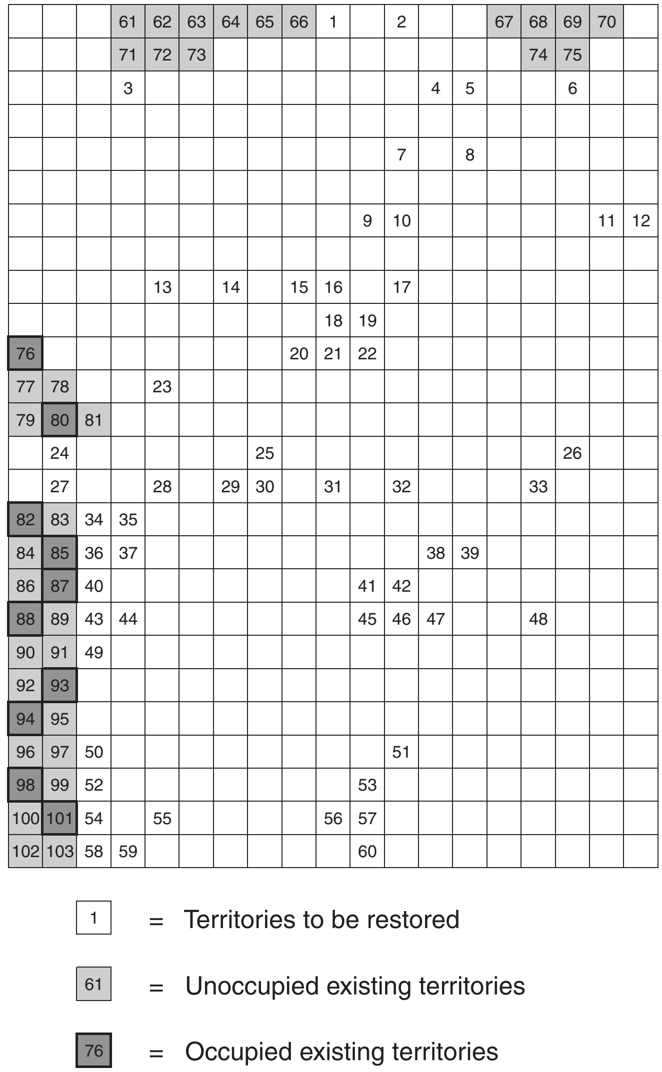
M	Objective function value (million \$)	Exceedance probability	Search time (min)
1	4.8	0.39	1.0
2	6.0	0.22	1.5
3	6.8	0.15	2.2
4	7.4	0.12	2.8
5	7.9	0.10	3.4
6	8.3	0.08	4.0
7	8.6	0.07	4.8
8	8.9	0.06	5.5
9	9.2	0.055	6.4
10	9.4	0.048	7.2
15	10.4	0.032	11.6
20	11.1	0.024	16.6
40	12.9	0.012	41.0

pected values of lognormal random variables based on the parameters of the underlying normal distribution. Using that formula, the expected value of fire cost for the entire planning unit was directly computed for each of the $3^9 = 19\,683$ possible decision vectors as the sum of the expected values computed for each subunit. The optimal budget was found to be \$13.5 million, after which the cost of additional budget increments exceeded incremental reductions in expected fire cost. The sole optimal decision vector based on expected value was $\boldsymbol{X} = (2, 2, 2, 2, 2, 2, 2, 2, 2)$ and the expected fire cost for the planning unit was about \$3.7 million.

The expected fire cost calculations showed \$13 million to be another interesting budget level to examine. The nine decision vectors ranging from $\boldsymbol{X} = (1, 2, 2, 2, 2, 2, 2, 2, 2)$ to $\boldsymbol{X} = (2, 2, 2, 2, 2, 2, 2, 2, 1)$ were all optimal based on expected value, as we might anticipate because spatial correlations had no impact on expected values. The expected fire cost under each of these alternatives was about \$4.8 million.

No simple formula is available for the variance of the sum of lognormal random variables, which affects optimal chance-constrained solutions. Instead, 10 million simulations were used to estimate the overall standard deviation of fire cost for all alternatives with budget levels of \$13 million or \$13.5 million. The simulations for a budget level of \$13.5 million revealed that the decision vector $\boldsymbol{X} = (2, 2, 2, 2, 2, 2, 2, 2, 2)$ had the smallest variance as well as the smallest expected value among those planning alternatives, indicating that vector would also be optimal for any chance-constrained problem we defined. The simulations for a budget level of \$13 million revealed that four of the nine decision vectors that were optimal based on expected value also had the smallest variance among those planning alternatives. The four vectors $\boldsymbol{X} = (1, 2, 2, 2, 2, 2, 2, 2, 2)$, $\boldsymbol{X} = (2, 2, 1, 2, 2, 2, 2, 2, 2)$, $\boldsymbol{X} = (2, 2, 2, 2, 2, 2, 1, 2, 2)$, and $\boldsymbol{X} = (2, 2, 2, 2, 2, 2, 2, 2, 1)$ would be optimal for any chance-constrained problem we defined. Because spatial correlations were largest for adjacent subunits in this problem, reduced staffing in the corner subunits was preferable to reduced staffing elsewhere when covariance was considered. The effect is similar to the central placement of fire-fighting resources we would expect to be optimal (all else being equal) in models that include travel costs or travel times.

Fig. 1. Initial habitat arrangement and territory occupancy status for a hypothetical population starting with 10 adult females, 43 territories with suitable habitat, and 60 territories requiring habitat restoration.



Experiment 1

We began by testing the effect of sample size N when $M = 1$ for a case with fairly stringent settings for the heuristic algorithm parameters $k_{\mathrm{MIN}} = 100$, $k_{\mathrm{MAX}} = 5000$, $\alpha = 0.05$, $\beta = 0.01$, $\tau_0 = 0.001$, $\varepsilon = 0.0001$, $\delta = (0.001)(\mu_k^*)$, and $\Theta = 0.99$. With the budget parameter $\zeta = \$13.5$ million and $N = 1$, the search concluded after 7 h and 12 min at $K = 1170$. The optimal decision was constructed from decision variable counts and correctly identified (i.e., it was the last "better" solution found) at $k = 6$; the rest of the search was required to satisfy the requirements $\beta = 0.01$ and $\tau_0 + \varepsilon = 0.0011$. The search time was quite long for this small problem because 575 different decision vectors were found and had to be simulated at length to estimate the resulting objective function values with high precision. The optimal solution was found 50 times

Table 2. Estimated optimal expected values of first-order statistics of annual abundance ($\hat{A}_t^*$) and estimated joint reliability ($\hat{q}$) values for various magnitudes of M, N, Θ, and K in Experiment 3 on first-order statistic estimation Problem JSCCP-k.

Run No.	M	N	Θ	K	$(\hat{A}_1^*, \hat{A}_2^*, \hat{A}_3^*, \hat{A}_4^*, \hat{A}_5^*)$	$\hat{q}$
1	1	999	0.995	251	(10.2, 10.7, 11.1, 11.4, 11.7)	0.21–0.24
2	2	750	0.995	50	(8.8, 8.6, 8.6, 8.6, 8.7)	0.47–0.50
3	3	500	0.995	50	(8.0, 7.6, 7.4, 7.3, 7.2)	0.55–0.57
4	4	400	0.995	50	(7.6, 7.0, 6.7, 6.4, 6.2)	0.65–0.66
5	5	300	0.995	50	(7.2, 6.6, 6.1, 5.8, 5.6)	0.71–0.72
6	10	150	0.995	50	(6.3, 5.3, 4.7, 4.2, 3.8)	0.83–0.84
7	50	35	0.995	268	(4.6, 3.2, 2.3, 1.7, 1.2)	0.95–0.96
8	50	35	0.995	255	(4.6, 3.2, 2.3, 1.7, 1.2)	0.95–0.96
9	50	35	0.995	287	(4.6, 3.2, 2.3, 1.7, 1.2)	0.95–0.96
10	50	35	0.995	500	(4.6, 3.2, 2.3, 1.7, 1.2)	0.95–0.96
11	50	35	0.995	50	(4.6, 3.2, 2.3, 1.7, 1.2)	0.95–0.96
12	50	25	0.995	50	(4.6, 3.2, 2.3, 1.7, 1.2)	0.95–0.96
13	50	35	0.98	50	(4.6, 3.2, 2.3, 1.7, 1.2)	0.95–0.96
14	50	25	0.98	50	(4.6, 3.2, 2.3, 1.7, 1.2)	0.95–0.96
15	50	25	0.98	50	(4.6, 3.2, 2.3, 1.7, 1.2)	0.95–0.96

(although not initially) in Step 1, so most of the processing was used to identify and test suboptimal decision vectors. Nevertheless, the decision vectors constructed from decision variable counts in Step 5 stabilized on the optimal solution without further variation at k = 18. The objective function value and p (the probability of exceedance) were estimated at \$3.7 million and 0.41, respectively.

With N = 10, the search concluded after 26 min at K = 1165. The optimal decision vector was found and correctly identified at k = 1. The search time was reduced considerably because only 38 different decision vectors had to simulated; the optimal solution was found 1104 times in Step 1. Estimates of the objective function value and p were the same as reported above.

With N = 100, the search concluded after 2 min at K = 1165 with other results much the same as before. The optimal decision vector was found without exception.

Experiment 2

For Experiment 2, varying M at a budget level of ζ = \$13 million, the heuristic parameter settings were relaxed somewhat: k_{MIN} = 100, k_{MAX} = 5000, α = 0.05, β = 0.01, τ_0 = 0.001, ε = 0.0005, $\delta = (0.005)(\mu_k^*)$, and Θ = 0.99. With N set to 100, M was tested across the values (1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 15, 20, 40). All searches were completed at either K = 399 or K = 400 because one of the optimal solutions was found at either k = 1 or k = 2. With the fairly small indifference zone defined above by δ, the correct alternate optima (nine for M = 1, four for M > 1) were reported in all cases; if δ was increased, additional decision vectors eventually would be accepted as "optimal." The estimated objective function values, p values, and search times are reported in Table 1. For these runs, search times were lengthened during Step 1 at large M by mathematical programming problem size. A few additional tests indicated that successful searches could be completed for M = 40 in about 10 min, a fourfold reduction in search time, with N set to 25 (instead of 100), which was about ideal for this case. The study by Linderoth et al. (2006) suggests, however, that ideal settings for sample size N will vary from problem to problem. More work is needed on this important issue.

Test case 2: habitat restoration problem JSCCP-*k*

For this test case, the habitat layout and initial conditions shown in Fig. 1 were used. Random sample values for eq. 42 were selected in accordance with the probabilities $\Pr[l_{itmnk} = 0, 1] = (0.3, 0.7)$ and $\Pr[r_{jtmnk} = 0, 1, 2, 3, 4] = (0.2, 0.15, 0.25, 0.25, 0.15)$ where r_{jtmnk} is the total number of female offspring born in territory j in year t – 1 that would survive to become adults in year t if they dispersed to suitable, vacant territories in random instance mnk. Sample disperser numbers r_{jitmnk} then were determined by drawing dispersal destinations for each offspring. Dispersal distances were selected from an exponential distribution (see Johnson and Kotz 1970) with a minimum distance of 0.707 territory edges (measured from the center of the natal cell) and a mean distance of four territory edges; dispersal directions were selected from a uniform distribution.

Experiment 3

Table 2 shows the estimated optimal expected values of first-order statistics of annual abundance $(\hat{A}_1^*, \ldots, \hat{A}_5^*)$ and estimated joint reliability ($\hat{q}$) for various settings of M, N, Θ, and K. The total number of trials K in each test run was controlled by the settings α = 0.05, β = 0.01, τ_0 = 0.05, and ε = 0.01 for runs 1, 7, 8, and 9 and by settings of k_{MIN} and k_{MAX} for the others. Because annual results varied a full order of magnitude, depending on the size of parameter M, indifference interval settings were varied from $\delta = (0.002)(A_{tk}^*)$ at M = 1 to $\delta = (0.01)(A_{tk}^*)$ at M = 50 to estimate annual results at about the hundredths decimal place (with confidence Θ) to report results with reasonable accuracy to the nearest tenth. Confidence level Θ was set high for estimating annual values so that overall confidence in the 5-year vectors of results was at least $0.995^5 \cong 0.975$ for runs 1–12 and $0.98^5 \cong 0.90$ for runs 13–15. Although an optimal solution was found within the first 37 trials in all runs (and usually much sooner), numerous alternate optima and unique suboptimal solutions were also found in all cases. The large number of solutions to be simulated in each test run combined with stringent parameter settings resulted in long run times, e.g., ranging from almost 20 h for run 1 to more than 102 h for run 7. After observing that initial optimal solutions were found quickly in those two tests, most runs were limited to 50 trials to save time.

Runs 1–7 in Table 2 show the effects of order statistic sample size M on overall reliability and resulting chance-constrained solutions. At $M = 1$, $\hat{A}_1^*$ limits the optimal first iteration level; at $M = 2$, $\hat{A}_3^*$ is initially limiting and at $M \geq 3$, $\hat{A}_5^*$ is initially limiting. This qualitative change from a recovering population with reliability around 0.23 to a declining population when reliability 0.55 or higher is required would be noteworthy to conservation biologists and managers. Reliabilities are reported as ranges because estimates came from simulations of numerous equally acceptable solutions.

Runs 8–15 were used to test repeatability of run 7 results for $M = 50$ using these methods on a difficult chance-constrained problem where true optimal solutions are unknown. As Table 2 shows, results for all of these runs were consistent.

Conclusion

The results reported in the preceding section indicate that endogenous estimation formulations combined with heuristic solution algorithms can help solve difficult stochastic programming and chance-constrained programming resource management problems. Many other applications will be required, however, to more fully characterize the usefulness of this approach. The formulations and algorithm tested here offer a promising start while mathematical programming research continues to address convergence issues for these complex problems (e.g., Bastin et al. 2007; Bayraksan and Morton 2007; Blomvall and Shapiro 2007; Haneveld et al. 2007).

Acknowledgements

The author is grateful to Laurie Travis and to Eldon Gunn for many helpful discussions and to three anonymous referees and an associate editor for their helpful review comments.

References

Bastin, F., Cirillo, C., and Toint, P. 2007. Convergence theory for nonconvex stochastic programming with an application to mixed logit. Math. Program. Ser. B, **108**: 207–234. (Published online 25 April 2006)

Bayraksan, G., and Morton, D.P. 2007. Assessing solution quality in stochastic programs. Math. Program. Ser. B, **108**: 495–514. (Published online 25 April 2006)

Bertsimas, D., Natarajan, K., and Teo, C.-P. 2007. Persistence in discrete optimization under data uncertainty. Math. Program. Ser. B, **108**: 251–274. (Published online 25 April 2006)

Bevers, M., and Kent, B. 2007. Managing risk with chance-constrained programming. Chap. 12. *In* Wildfire risk: human perceptions and management implications. *Edited by* W.E. Martin, C. Raish, and B. Kent. Resources for the Future, Washington, D.C. In press.

Bevers, M., Hof, J., Uresk, D.W., and Schenbeck, G.L. 1997. Spatial optimization of prairie dog colonies for black-footed ferret recovery. Oper. Res. **45**: 495–507.

Bhat, M.G., Fister, K.R., and Lenhart, S. 1999. An optimal control model for surface runoff contamination of a large river basin. Nat. Resour. Model. **12**: 175–195.

Birge, J.R., and Louveaux, F. 1997. Introduction to stochastic programming. Springer-Verlag, New York.

Blomvall, J., and Shapiro, A. 2007. Solving multistage asset investment problems by the sample average approximation method. Math. Program. Ser. B, **108**: 571–595. (Published online 2 June 2006)

Boesel, J., Nelson, B.L., and Kim, S. 2003. Using ranking and selection to "clean up" after simulation optimization. Oper. Res. **51**: 814–825.

Boychuck, D., and Martell, D.L. 1988. A Markov chain model for evaluating seasonal forest fire fighter requirements. For. Sci. **34**: 647–661.

Boychuck, D., and Martell, D.L. 1996. A multistage stochastic programming model for sustainable forest-level timber supply under risk of fire. For. Sci. **42**: 10–26.

Donovan, G.H., and Rideout, D.B. 2003. An integer programming model to optimize resource allocation for wildfire containment. For. Sci. **49**: 331–335.

Fina, M., Amacher, G.S., and Sullivan, J. 2001. Uncertainty, debt, and forest harvesting: Faustmann revisited. For. Sci. **47**: 188–196.

Fuessle, R.W., Brill, E.D., Jr., and Liebman, J.C. 1987. Air quality planning: a general chance-constraint model. J. Environ. Eng. **113**: 106–123.

Gassmann, H.I. 1989. Optimal harvest of a forest in the presence of uncertainty. Can. J. For. Res. **19**: 1267–1274.

Goldsman, D., and Nelson, B.L. 1998. Comparing systems via simulation. Chap. 8. *In* Handbook of simulation: principles, methodology, advances, applications, and practice. *Edited by* J. Banks. John Wiley & Sons, New York. pp. 273–306.

Gong, P., and Yin, R. 2004. Optimal harvest strategy for slash pine plantations: the impact of autocorrelated prices for multiple products. For. Sci. **50**: 10–19.

Goodman, D. 1987. Consideration of stochastic demography in the design and management of biological reserves. Nat. Resour. Model. **1**: 205–234.

Gove, J.H., and Fairweather, S.E. 1992. Optimizing the management of uneven-aged forest stands: a stochastic approach. For. Sci. **38**: 623–640.

Haneveld, W.K.K., Stougie, L., and van der Vlerk, M.H. 2007. Simple integer recourse models: convexity and convex approximations. Math. Program. Ser. B, **108**: 435–473. (Published online 25 April 2006)

Hof, J., Bevers, M., and Pickens, J. 1996. Chance-constrained optimization with spatially autocorrelated forest yields. For. Sci. **42**: 118–123.

Hunter, D.H., Bartlett, E.T., and Jameson, D.A. 1976. Optimum forage allocation through chance-constrained programming. Ecol. Model. **2**: 91–99. doi:10.1016/0304-3800(76)90027-2.

Johnson, N.L., and Kotz, S. 1970. Distributions in statistics: continuous univariate distributions — 1. John Wiley & Sons, New York.

Kleywegt, A.J., Shapiro, A., and Homem-De-Mello, T. 2001. The sample average approximation method for stochastic discrete optimization. SIAM J. Optim. **12**: 479–502. doi:10.1137/S1052623499363220.

Larsen, R.J., and Marx, M.L. 1986. An introduction to mathematical statistics and its applications. 2nd ed. Prentice-Hall, Englewood Cliffs, N.J.

Lenhart, S., Liang, M., and Protopopescu, V. 1999. Optimal control of the effects of boundary habitat hostility. Math. Methods Appl. Sci. **22**: 1061–1077. doi:10.1002/(SICI)1099-1476(19990910)22:13<1061::AID-MMA70>3.0.CO;2-I.

Linderoth, J., Shapiro, A., and Wright, S. 2006. The empirical behavior of sampling methods for stochastic programming. Ann. Oper. Res. **142**: 215–241. doi:10.1007/s10479-006-6169-8.

MacLellan, J.I., and Martell, D.L. 1996. Basing airtankers for forest fire control in Ontario. Oper. Res. **44**: 677–686.

Prestemon, J.P., and Holmes, T.P. 2004. Market dynamics and optimal timber salvage after a natural catastrophe. For. Sci. **50**: 495–511.

Reed, W.J. 1987. Protecting a forest against fire: optimal protection patterns and harvest policies. Nat. Resour. Model. **2**: 23–53.

Reeves, C.R. (*Editor*). 1993. Modern heuristic techniques for combinatorial problems. Halstead Press, New York.

Rollin, F., Buongiorno, J., Zhou, M., and Peyron, J.-L. 2005. Management of mixed-species, uneven-aged forests in the French Jura: from stochastic growth and price models to decision tables. For. Sci. **51**: 64–75.

Shapiro, A. 2008. Stochastic programming approach to optimization under uncertainty. Math. Program. Ser. B, **112**: 183–220. doi:10.1007/s10107-006-0090-4.

Snyder, S.A., Haight, R.G., and ReVelle, C.S. 2004. A scenario optimization model for dynamic reserve site allocation. Environ. Model. Assess. **9**: 179–187. doi:10.1023/B:ENMO.0000049388.71603.7f.

Spring, D.A., and Kennedy, J.O.S. 2005. Existence value and optimal timber-wildlife management in a flammable multistand forest. Ecol. Econ. **55**: 365–379. doi:10.1016/j.ecolecon.2004.11.012.

Spring, D.A., Kennedy, J.O.S., and Mac Nally, R. 2005. Optimal management of a forested catchment providing timber and carbon sequestration benefits: climate change effects. Glob. Environ. Change, **15**: 281–292.

Strange, N., Thorsen, B.J., and Bladt, J. 2006. Optimal reserve selection in a dynamic world. Biol. Conserv. **131**: 33–41. doi:10.1016/j.biocon.2006.02.002.

Strauss, D., Bednar, L., and Mees, R. 1989. Do one percent of forest fires cause ninety-nine percent of the damage? For. Sci. **35**: 319–328.