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Ranking Forestry Investments With Parametric Linear Programming

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Parametric linear programming is introduced as a technique for ranking forestry investments under multiple constraints; it combines the advantages of simple ranking and linear programming as capital budgeting tools.

Additional keywords. Capital budgeting, financial analysis, investment selection.

Forest land managers and public agencies are often faced with the task of allocating limited funds to a number of competing investments such as precommercial thinning, type conversion, or timber stand Improvement. If the only constraint is the amount of money, project selection Is straightforward. The ratio of the present value of money received to money spent is calculated for each project, the ratios are ranked, and projects are chosen by moving down the rankings until the funds are exhausted (Lorie and Savage 1955).

When multiple constraints exist, linear programming is needed. Weingartner (1963) applied mathematical programming to capital budgeting including allocation of a fixed sum. Two other studies (Teeguarden and Von Sperber 1968, Buongiorno and Teeguarden 1973) showed that Douglas-fir reforestation projects could best be selected by linear programming.

However, linear programming by itself lacks a useful feature of simple ranking. Ranking provides a solution for all levels of funding; linear programming gives a solution for only'one. Other constraints such as manpower, equipment, and nursery capacity are usually known in advance, but funding is different. Annual budgets may be allocated at a different organization level. Consequently, the exact amount is not known far in advance and may change. A ranking of projects would circumscribe this problem, if it could be done under multiple constraints.

This paper describes a procedure for ranking forestry investments that are bound by multiple constraints. This procedure-which uses parametric linear programming-combines the advantages of simple ranking and linear programming.

Model Formulation

Parametric linear programming is applied to a problem only after an initial solution has been found by the simplex method. The dual simplex method-the algorithm used for parameterization-is then applied to the primal problem. Both methods use the same notation. For these reasons, the simplex method will be outlined briefly, and then parametric linear programming and its application to ranking investments will be discussed. The notation and development used here are from Hadley (1962).

The typical linear programming problem is

$$\max z = c'x,$$

subject to the restrictions,

$$Ax = b, \tag{1}$$

$$x \geq o, \tag{2}$$

where A is an m x n matrix, x and b are n x 1 column vectors, c' is a 1 x n row vector, and o is an n x 1 vector of zeroes.

If the matrix B is composed of m linearly independent columns of A, then any column a_j of A can be written as a linear combination of the columns of B,

$$a_j = By_j. \tag{3}$$

This matrix B also provides a basic solution to the simultaneous linear equations represented by equation (1). This solution x_B is determined as follows:

$$x_B = B^{-1}b.$$

Corresponding to a basic solution x_B, there is an associated price vector c_B, which is composed of m elements from vector c.

A basic solution which satisfies restriction (2) is called a basic feasible solution. Assuming there is a basic feasible solution, the simplex method provides a means of improving it-that is, increasing the value of

$$z = c_B' x_B. \tag{4}$$

Equation (3) may be rewritten as

$$a_j = \sum_{i=1}^{m} y_{ij} b_i.$$

Any vector b_r in the basis may be written in terms of the remaining basis vectors and a vector a_j not in the basis.

$$b_r = a_j/y_{rj} - \sum_{\substack{i=1\\ i\neq r}}^{m} y_{ij} b_i/y_{rj}.$$

By substituting a_j for b_r, a new basic solution is obtained,

$$b = \sum_{\substack{i=1\\ i\neq r}}^{m} x_{Bi} b_i + x_{Br}\left[a_j/y_{rj} - \sum_{i=1}^{m} y_{ij} b_i/y_{rj}\right]$$

For the new solution to be feasible,

$$x_{Bi} - x_{Br} y_{ij}/y_{rj} \geq 0, \ i \neq r$$
$$\frac{x_{Br}}{y_{rj}} \geq 0, \ i = r.$$

To maintain feasibility, the vector b_r to be replaced is determined by

$$x_B r/y_{rj} = \min_j \{x_{Bi}/y_{ij}, y_{ij} > 0\}$$

The vector to enter should improve the basic feasible solution. Equation (4) may be rewritten with a_j substituted for b_r,

$$z = \sum_{\substack{i=1\\ i\neq r}}^{m} [x_{Bi} - x_{Br} y_{ij}]\, c_{Bi} + x_{Br} c_j/y_{rj}. \tag{5}$$

Since

$$c_{Br}\,(x_{Br} - x_{Br} y_{ij}/y_{ij}) = 0,$$

the $i=r$ term can be included in the summation, and equation (5) becomes

$$\hat{z} = \sum_{i=1}^{m} x_{Bi} c_{Bi} - x_{Br}/y_{rj} \sum_{i=1}^{m} y_{ij} c_{Bi} + x_{Br} c_j/y_{rj},$$

which is reduced to

$$\hat{z} = z + \left[c_j - \sum_{i=1}^{m} y_{ij} c_{Bi}\right] x_{Br}/y_{rj}. \tag{6}$$

For there to be an increase in the objective function.

$$c_j - \sum_{i=1}^{m} y_{ij} c_{Bi} > 0.$$

The usual criterion for selecting the vector to enter is to pick the one with the maximum $(c_j - z_j)$, where

$$z_j = \sum_{i=1}^{m} y_{ij} x_{Bi}. \tag{7}$$

Given a basic feasible solution, a new basic feasible solution with an improved objective function can be found by the simplex method. The process can terminate in two ways:

(1) one or more $z_j - c_j < 0$, and for each $z_j - c_j < 0$, $y_{ij} \leq 0$ for all $i = 1, \ldots, m$

(2) all $z_j - c_j \geq 0$ for the columns of A not in the basis.

If situation (2) occurs, there is an optimal basic feasible solution.

Suppose there is an optimal basic feasible solution. We want to increase or decrease the constraints by changing the requirements vector b,

$$b^* = b + \theta r.$$

The vector r is specified, and θ is a non-negative scalar (Hndley 1962, p. 382). Changing b changes the solution, which becomes

$$x_B^* = B^{-1}(b + \theta r),$$
$$= x_B + \theta v.$$

Even though the solution x_B^* changes, optimality is maintained as long as the solution remains feasible because the values of the $z_j - c_j$'s are affected only by the basis vectors and not by the solution vector. If any of the v_i's in vector v are negative, the point where the first x_{Bi} becomes less than or equal to zero is

$$\theta = \min_{r}\ \min_{i} \{-x_{Bi}/v_i, v_i < 0\}. \tag{8}$$

The dual simplex algorithm can be applied in this situation. Given a solution that is infeasible but where all $z_j - c_j$'s ≥ 0 for vectors not in the basis, the dual simplex can be used to obtain a basic feasible solution while preserving $z_j - c_j \geq 0$ for all nonbasis vectors. Once a basic feasible solution is reached, it is also optimal.

The vector to leave the basis is already known by equation (8). Only the vector to enter the basis remains to be determined.

Given the primal problem,

$$Ax = b,\ x \geq 0,\ \max z = c'x,$$

the dual formulation is

$$A'w \geq c,\ \min Z = b'w,$$
$$A'w \geq c, \tag{9}$$

where the w_i's are unrestricted in sign.

If the primal solution has all $z_j - c_j \geq 0$ for all vectors not in the basis and one of the x_{Bi}'s is negative, the corresponding solution to the dual w is not optimal.

Consider a new vector $\hat{w}$, which is given by

$$\hat{w} = w - \theta\beta^r,$$

where β^r is a vector of B^{-1} (Hadley 1962, p. 245). For all a_j not in the primal basis,

$$\begin{aligned}\hat{w}'a_j &= w'a_j - \theta\beta_r{}'a_j \\ &= w'a_j - \theta y_{rj}\end{aligned}$$

Substituting $\hat{w}'a_j$ into the inequality from (9), we have

$$w'a_j - \theta y_{rj} \geq c_j.$$

Let us now consider $c_B{}'B^{-1}$ as a solution for w', or

$$w' = c_B{}'B^{-1}$$

For it to be a solution, only the inequality $A'w \geq c$ need be satisfied.

$$\begin{aligned}c_B{}'B^{-1}A &\geq c'.\\ c_B{}'B^{-1}a_j &\geq c_j.\end{aligned}$$

Since $y_j = B^{-1}a_j$,

$$c_B{}'y_j \geq c_j.$$

From relation (7), it may be written as

$$z_j \geq c_j, \text{ or } z_j - c_j \geq 0,$$

which is a characteristic of the original problem. Hence,

$$\begin{aligned}&c_B{}'B^{-1}a_j - \theta y_{rj} \geq c_j,\\ &z_j - \theta y_{rj} \geq c_j\\ &\theta y_{rj} \leq z_j - c_j. \qquad (10)\end{aligned}$$

Before we derive the criterion for the vector to enter from expression (10), let us look again at equation (6). For there to be an increase in the objective function

$$x_{Br}(c_j - z_j)/y_{rj} \geq 0.$$

If x_{Br} were just slightly negative, y_{rj} would have to be less than zero for $z \geq z$, since $c_j - z_j \leq 0$. Therefore $y_{rj} < 0$ in expression (IO). and it follows that $\theta < 0$.

For a maximum increase in the objective function, the vector a_k to enter the basis is determined by

$$\theta = (z_k - c_k)/y_{rk} = \max\,(z_j - c_j)/y_{rj},\quad y_{rj} < 0. \qquad (11)$$

As elements of the requirements vector are changed by right-hand side parameterization, a series of optimal bases develops. The vector to leave and the vector to enter the basis at each step are determined by expressions (8) and (1 1), respectively.

Let us return to our original problem. We are faced with deriving a set of priorities from a set of investment projects under multiple constraints. Given a certain budget, linear programming can select the set of projects that maximizes net present value. With parametric linear programming, sets of investment projects can be selected for different ranges of funding. A list of priorities that is valid for any level of funding can be derived in the following manner.

The funding level in the original requirements vector is set equal to or less than the amount of money required for the smallest project. This procedure insures that no more than one project enters into the initial solution. Then the simplex method or other procedure is used to arrive at an optimal basic feasible solution. Next the budget level is varied-that is, increased-by means of parametric linear programming. As the budget level is increased, projects enter the solution one by one. The order in which the investments enter the solution determines their priority for two reasons. A given solution remains optimal over the region in which it remains feasible, and once a solution is no longer feasible, the dual simplex procedure selects a vector to enter the basis such that the new solution is both optimal and feasible.

Examples

Most linear programming systems now available can also do either right-hand side or objective row parameterization. The UNIVAC 1108 Linear Programming System was used for the following examples. Ranking was first done with the funding level as the only constraint to illustrate that ranking with parametric linear programming is equivalent to simple ranking when only one restriction exists. The second example illustrates the effectiveness of ranking with parametric linear programming when multiple constraints exist.

As the result of succession often aided by timber cutting practices, extensive acreages of pine sites in the South are now dominated by hardwood forest types'. Converting these hardwood stands to pine offers one of the best ways of increasing the region's pine timber supply. In southern Alabama alone, over 3 million acres of private, nonindustrial forest land could be converted to pine (table 1). To set priorities these lands were classified according to the size of the hardwood stand now occupying the site, the pine species to be regenerated, the site index of the

Table 1.—*Type conversion opportunities on pine sites in southern Alabama, 6 percent interest rate, miscellaneous private owners, 1972*

Number	Present stand size	Species	Type of regeneration	Site index	Area	Net present value	Conversion cost
				Feet	*Thousand acres*	- - *Dollars/acre* - -	
1	sawtimber	shortleaf[1]	natural	92	22.7	104.36	39.70
2	"	"	"	80	34.7	68.38	39.70
3	all other	"	"	89	18.5	96.07	38.60
4	"	"	"	79	34.8	66.78	33.50
5	"	"	"	65	21.6	30.40	38.80
6	"	slash[2]	"	76	25.6	11.53	38.90
7	sawtimber	"	"	72	23.6	1.17	38.50
8	all other	"	planting	81	135.6	55.60	56.62
9	"	"	"	74	170.2	12.92	56.42
10	"	longleaf[3]	natural	78	87.2	-14.32	38.60
11	"	"	"	61	58.7	-26.00	38.40
12	sawtimber	"	"	79	35.2	-13.25	38.40
13	all other	loblolly[4]	"	100	44.3	218.08	38.80
14	"	"	"	86	304.8	140.90	38.60
15	"	"	"	72	232.5	91.04	38.70
16	sawtimber	"	"	101	102.2	226.66	40.60
17	"	"	"	87	164.9	146.80	40.00
18	"	"	"	75	111.3	101.09	39.90
19	all other	"	planting	100	57.0	341.12	57.92
20	"	"	"	84	618.4	132.01	57.02
21	"	"	"	73	529.0	42.56	57.12
22	"	"	"	56	34.9	-28.41	56.32
23	sawtimber	"	"	106	28.5	445.57	59.82
24	"	"	"	86	187.1	150.57	59.22
25	"	"	"	75	67.1	53.32	59.72
					3,150.4		

[1] *Pinus echinata* Mill.
[2] *Pinus elliottii* Engelm.
[3] *Pinus palustris* Mill.
[4] *Pinus taeda* L.

pine species, and the type of regeneration. Information for formulating the linear programming problem included the acreage, net present value, and conversion cost for each class.

Yet present value was calculated assuming a perpetual series of rotations to insure comparability among the classes. If only one rotation were considered, the net present value of one conversion class would not be directly comparable to that of another class with a different rotation length. To further assure comparability, classes with artificial regeneration were assumed to be followed by a sequence of natural stands rather than plantations.

The problem is

$$\max \sum_{i=1}^{n} p_i x_i,$$

subject to the constraints

$$0 \leq x_i \leq a_i, i = 1,n$$

$$\sum_{i=1}^{n} c_i x_i \leq d,$$

where x_i is the acreage in the ith opportunity class to be converted, a_i is the amount of land in the ith class available for type conversion, p_i is the net present value of converting 1 acre of the ith class to pine, c_i is the cost of converting 1 acre of the ith class to pine, and d is the amount of funds available for type conversion. Initially, d is set at a level such that.

$$d \leq \min\{c_i a_i\}$$

For this problem, the third type conversion opportunity in table 1 has the minimum $c_i a_i$, which is $714,100. Hence. the initial value of d should be less than this amount; I set it at $1,000.

An optimal basic feasible solution to the problem was found. Right-hand side parameterization was then applied to the problem. A column vector r was specified so that the value of d was increased through right-hand side parameterization.

The priorities that resulted (table 2) are the same as those obtained by simple ranking of the ratios of net present value to conversion cost.

A more realistic approach would be to include nursery capacity constraints in the problem.

The total acreage that could be planted to slash (*Pinus elliottii* Engelm.) or loblolly (*P. taeda* L.) pine is 1.8 million acres. If planting density is 8 x 8 feet or 680 trees per acre, the total number of seedlings required would be 1.2 billion.

Let s and f be the maximum acreages that can be planted to slash and loblolly pines, respectively. The following constraints are then added to the problem,

$$\Sigma x_j \leq s, \; j = 8,9$$
$$\Sigma x_k \leq f, \; k = 19,20, \ldots ,25$$

The maximum acreage that can be planted to slash pine will be 33,000 acres; the maximum for loblolly pine will be 67,000 acres. The remaining variables are unchanged.

When the nursery constraints are added, only three classes with artificial regeneration are in the rankings (table 3) ; there were eight in table 2. The maximum aggregate cost is reduced by two-thirds as the result of the imposition of these additional restrictions.

Of course, the type conversion classes could have been ranked under nursery capacity restrictions without parametric linear programming. By totaling the acreages of each class that utilized planting, one could determine the point in the rankings where nursery capacity would be exhausted. All classes below this point that used planting would be deleted from the list. However, the problems presented in tables 2 and 3 are simple. More complex problems would be exceedingly difficult, if not impossible, to solve in this manner.

This technique should prove useful in two ways. It lessens the necessity to resolve problems because of unanticipated changes in constraint levels, and it increases the capability of linear

Table 2.—*Ranking of type conversion opportunities for southern Alabama*

Rank	Number	Net present value / Conversion cost	Cumulative cost	Cumulative area planted: Slash	Cumulative area planted: Loblolly
			Thousand dollars	— — *Thousand acres* — —	
1	23	7.44	1,705		28.6
2	19	5.89	5,006		85.6
3	13	5.62	6,725		85.5
4	16	5.58	10,874		85.6
5	17	3.67	17,470		85.5
6	14	3.65	29,236		85.6
7	1	2.63	30,137		85.5
8	24	2.54	41,217		272.6
9	18	2.53	45,657		272.6
10	3	2.49	46,372		272.6
11	15	2.35	55,370		272.6
12	20	2.32	90,631		891.0
13	4	1.73	91,971		891.0
14	2	1.72	93,348	0.0	891.0
15	8	.98	101,026	135.6	891.0
16	25	.89	105,033	135.6	958.1
17	5	.78	105,871	135.6	1,487.1
18	21	.75	136,087	135.6	1,487.1
19	6	.30	137,083	135.6	1,487.1
20	9	.23	146,686	305.8	1,487.1
21	7	.03	147,595	305.8	1,487.1

Table 3.—*Ranking of type conversion opportunities for southern Alabama with nursery capacity as a constraint*

Rank	No.	Cumulative cost	Cumulative area planted		Solution value
			Slash	Loblolly	
		Thousand dollars	— — *Thousand acres* -		-
1	23	1,705		28.5	28.5
2	19	3,335		28.5	38.5
3	13	5,654		67.0	44.3
4	16	9,803		67.0	102.2
5	17	16,399		67.0	164.9
6	14	28,164		67.0	304.8
7	1	29,065		67.0	22.7
8	18	33,506		67.0	ill.3
9	3	34,220		67.0	3 8.5
10	15	43,217		67.0	232.5
11	4	44,558		F7.0	34.8
12	2	45,936	0.0	67.0	34.7
13	8	47,804	33.0	G7.0	33.0
14	5	48,642	33.0	67.0	21.6
15	6	49,638	33.0	67.0	25.6
16	7	50,547	33.0	67.0	23.6

programming as an investigative and planning tool. In addition, using parametric linear programming to rank projects under multiple constraints has other uses. For example, one can investigate the sensitivity of the rankings to different interest rates or stumpage prices.

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