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A Nonlinear Theory for Elastic Plates With Application to Characterizing Paper Properties

A theory of thin plates which is physically as well as kinematically nonlinear is developed and used to characterize elastic material behavior for arbitrary stretching and bending deformations. It is developed from a few clearly defined assumptions and uses a unique treatment of strain energy. An effective strain concept is introduced to simplify the theory to a special theory having fewer parameters and being capable of characterizing paper. A procedure is given for matching the special theory to edgewise compressive data of paperboard.

1 Introduction

Our primary objective is to develop a theory of thin elastic plates which allows physical as well as kinematic nonlinearity. In the following section this theory is developed using an asymptotic analysis of the three-dimensional equations of finite elasticity. Plate stress-strain relations and equilibrium equations are derived. These equations are adequate for the analysis of engineering problems involving finite deformation and anisotropic nonlinear material behavior.

Our current interest is to use our plate stress-strain relations to characterize the elastic properties of paper. In Section 3 we develop a simple special set of stress-strain relations through the introduction of an effective strain. The material parameters in this special theory can be determined from experimental data currently available. In Section 4 comparison of our theory with data on paper is presented. The special theory seems capable of modeling the available experimental data sufficiently well for engineering purposes.

There is only a small literature on elastic plate theory which allows physical as well as geometrical nonlinearity. Completely general theories are presented by Naghdi [1] and Wempner [2]. The theory given here is developed by the scaling-asymptotic technique [3]. The analysis has the desirability of using a few assumptions that are clearly defined. Further contributions of our approach are its unique treatment of the strain energy and the use of an effective strain measure which allows the material properties to be characterized by a small number of coefficients. We note that the effective strain concept has been found useful in plasticity theory [4].

In a previous work [5] on the strength of corrugated fiberboard it was found necessary to model paper as a nonlinear elastic material. The need for a nonlinear model has

also been shown by Moody [6]. While the model used in [5] proved successful for the problem considered there, we stress that it is ad hoc to that problem and does not provide a general theory that can be applied to a wide range of engineering problems. Our purpose here is to provide such a theory.

2 Derivation of Plate Equations

We start with the equations of nonlinear three-dimensional elasticity and simplify them by a scaling argument appropriate for a thin plate. The resulting equations are then partially integrated to yield two-dimensional plate equations. The resulting plate equations represent a generalization of the von-Karman equations [7] to account for nonlinear anisotropic stress-strain relations as well as nonlinear kinematics.

The basic equations of nonlinear three-dimensional elasticity [7] are the equation of equilibrium,

$$\frac{\partial P_{ij}}{\partial x_j} = 0 \quad (2.1)$$

the stress-strain relations

$$P_{ij} = T_{ik} \left(\delta_{jk} + \frac{\partial u_j}{\partial x_k} \right), \quad T_{ij} = \frac{\partial H(E)}{\partial E_{ij}} \quad (2.2)$$

and the strain-displacement relations

$$2E_{ij} = \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} + \frac{\partial u_k}{\partial x_i} \frac{\partial u_k}{\partial x_j} \quad (2.3)$$

In the foregoing, P is the Piola-Kirchhoff stress tensor of the first kind, E is the material strain tensor, H is the strain energy density, u is the displacement, and x_i are Cartesian material coordinates. The equations are written in terms of Cartesian components and the summation convention is used.

Let the middle surface on the undeformed plate be $x_3 = 0$, the thickness of the plate be h , and the lateral length scale be L . For a thin plate $\epsilon = h/L$ is a small parameter. We assume that coordinates, displacement components, and strain components are scaled the same as in the von-Karman theory,

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$$x_1, x_2 = 0(L), \quad x_3 = 0(h) \quad (2.4)$$

$$u_1, u_2 = 0(\epsilon h), \quad u_3 = 0(h) \quad (2.5)$$

$$\left. \begin{aligned} E_{11}, E_{22}, E_{12}, E_{33} &= 0(\epsilon^2) \\ E_{13}, E_{23} &= 0(\epsilon^3) \end{aligned} \right\} \quad (2.6)$$

With this scaling, the lowest order terms in the strain displacement relations (2.3) are

$$\left. \begin{aligned} E_{11} &= \frac{\partial u_1}{\partial x_1} + \frac{1}{2} \left(\frac{\partial u_3}{\partial x_1} \right)^2 \\ E_{22} &= \frac{\partial u_2}{\partial x_2} + \frac{1}{2} \left(\frac{\partial u_3}{\partial x_2} \right)^2 \\ E_{12} &= \frac{1}{2} \left(\frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1} + \frac{\partial u_3}{\partial x_1} \frac{\partial u_3}{\partial x_2} \right) \end{aligned} \right\} \quad (2.7)$$

$$\left. \begin{aligned} 0 &= \frac{\partial u_1}{\partial x_3} + \frac{\partial u_3}{\partial x_1} + \frac{\partial u_3}{\partial x_1} \frac{\partial u_3}{\partial x_3} \\ 0 &= \frac{\partial u_2}{\partial x_3} + \frac{\partial u_3}{\partial x_2} + \frac{\partial u_3}{\partial x_2} \frac{\partial u_3}{\partial x_3} \\ 0 &= \frac{\partial u_3}{\partial x_3} \left(1 + \frac{1}{2} \frac{\partial u_3}{\partial x_3} \right) \end{aligned} \right\} \quad (2.8)$$

Equations (2.8) can be partially integrated to yield:

$$\left. \begin{aligned} u_3 &= w(x_1, x_2) \\ u_1 &= U_1(x_1, x_2) - x_3 \frac{\partial w}{\partial x_1} \\ u_2 &= U_2(x_1, x_2) - x_3 \frac{\partial w}{\partial x_2} \end{aligned} \right\} \quad (2.9)$$

where U_1 , U_2 , and w are components of displacement of the middle surface. With (2.9) equations (2.7) take the form

$$\left. \begin{aligned} E_{11} &= \epsilon_1 - \kappa_1 x_3 \\ E_{22} &= \epsilon_2 - \kappa_2 x_3 \\ E_{12} &= \epsilon_{12} - \kappa_{12} x_3 \end{aligned} \right\} \quad (2.10)$$

where ϵ_1 , ϵ_2 , and ϵ_{12} are middle surface strain components and κ_1 , κ_2 , and κ_{12} are middle surface curvature components given by:

$$\left. \begin{aligned} \epsilon_1 &= \frac{\partial U_1}{\partial x_1} + \frac{1}{2} \left(\frac{\partial w}{\partial x_1} \right)^2 \\ \epsilon_2 &= \frac{\partial U_2}{\partial x_2} + \frac{1}{2} \left(\frac{\partial w}{\partial x_2} \right)^2 \\ \epsilon_{12} &= \frac{1}{2} \left(\frac{\partial U_1}{\partial x_2} + \frac{\partial U_2}{\partial x_1} + \frac{\partial w}{\partial x_1} \frac{\partial w}{\partial x_2} \right) \end{aligned} \right\} \quad (2.11)$$

$$\kappa_1 = \frac{\partial^2 w}{\partial x_1^2}, \quad \kappa_2 = \frac{\partial^2 w}{\partial x_2^2}, \quad \kappa_{12} = \frac{\partial^2 w}{\partial x_1 \partial x_2} \quad (2.12)$$

Next, let p have the dimension of stress. The following scaling of the stress components is consistent with the equilibrium equations (2.1).

$$\left. \begin{aligned} P_{11}, P_{22}, P_{21}, P_{12} &= 0(p) \\ P_{31}, P_{13}, P_{23}, P_{32} &= 0(\epsilon p) \\ P_{33} &= 0(\epsilon^2 p) \end{aligned} \right\} \quad (2.13)$$

The terms in each of the equilibrium equations are the same

order so that all terms are to be retained. Define force and moment resultants by

$$N_{ij} = \int_{-h/2}^{h/2} P_{ij} dx_3, \quad M_{ij} = \int_{-h/2}^{h/2} P_{ij} x_3 dx_3 \quad (2.14)$$

where the integrations are taken over the plate thickness in the undeformed configuration. We assume the surface conditions,

$$\left. \begin{aligned} P_{31}(h/2) &= P_{31}(-h/2) = 0 \\ P_{32}(h/2) &= P_{32}(-h/2) = 0 \\ P_{33}(h/2) - P_{33}(-h/2) &= q(x_1, x_2) \end{aligned} \right\} \quad (2.15)$$

That is, there is an applied surface load in the three-direction. Integration of the three-dimensional equilibrium equations and use of conditions (2.15) yield the plate force equilibrium equations:

$$\left. \begin{aligned} \frac{\partial N_{11}}{\partial x_1} + \frac{\partial N_{21}}{\partial x_2} &= 0 \\ \frac{\partial N_{12}}{\partial x_1} + \frac{\partial N_{22}}{\partial x_2} &= 0 \\ \frac{\partial N_{13}}{\partial x_1} + \frac{\partial N_{23}}{\partial x_2} + q &= 0 \end{aligned} \right\} \quad (2.16)$$

Next, multiply the first two equations of (2.1) by x_3 and integrate to obtain the equations of plate moment equilibrium

$$\left. \begin{aligned} \frac{\partial M_{11}}{\partial x_1} + \frac{\partial M_{21}}{\partial x_2} &= N_{31} \\ \frac{\partial M_{12}}{\partial x_1} + \frac{\partial M_{22}}{\partial x_2} &= N_{32} \end{aligned} \right\} \quad (2.17)$$

The stress-strain relations (2.2) are equivalent to the work principle

$$\dot{H} = P_{ki} \dot{F}_{ik} \quad (2.18)$$

where

$$F_{ij} = \delta_{ij} + \partial u_i / \partial x_j \quad (2.19)$$

is the deformation gradient tensor and the dot indicates the material time derivative.

Equations (2.2) cannot be approximated directly to obtain plate stress-strain relations because E_{33} is the same order of magnitude as E_{11} , E_{22} , and E_{12} . That is, if E_{33} is retained in the stress-strain relations, the result is an inconsistent theory. The way to resolve this problem is to use the reciprocal stress-strain relation written in terms of the complementary energy. Assume that the stress-strain relation (2.2) is uniquely invertible for strain E in terms of tensor T . Define complementary energy $H_c(T)$ by the Legendre transformation:

$$H_c(T) = T_{ij} E_{ij} - H(E) \quad (2.20)$$

Note that

$$E_{ij} = \frac{\partial H_c(T)}{\partial T_{ij}} \quad (2.21)$$

We now assume H_c is correctly approximated by retaining only the lowest order components of T . This is appropriate if H_c is a polynomial, for example. Noting that the components of T have the same scaling in ϵ as the components of P , we obtain

$$H_c(T) \sim \bar{H}_c(T_{11}, T_{22}, T_{12}) \quad (2.22)$$

$$E_{11} = \frac{\partial \bar{H}_c}{\partial T_{11}}, \quad E_{22} = \frac{\partial \bar{H}_c}{\partial T_{22}}, \quad E_{12} = \frac{\partial \bar{H}_c}{\partial T_{12}} \quad (2.23)$$

We note that approximation (2.22) cannot be adequate if E_{13} ,

E_{23} , or E_{33} is to be obtained from (2.21). We further assume that equations (2.23) can be uniquely inverted to give T_{11} , T_{22} , and T_{12} in terms of E_{11} , E_{22} , and E_{12} . Using (2.22) and noting that

$$T_{ij}E_{ij} \sim T_{11}E_{11} + T_{22}E_{22} + 2T_{12}E_{12} \quad (2.24)$$

we use (2.20) to approximate the strain energy function as follows

$$H(E) \sim \bar{H}(E_{11}, E_{22}, E_{12}) = T_{\alpha\beta}E_{\alpha\beta} - \bar{H}_C(T_{11}, T_{22}, T_{12}) \quad (2.25)$$

where Greek indices take on the values 1, 2. It is important that $\bar{H}(E_{11}, E_{22}, E_{12}) \neq H(E_{11}, E_{22}, E_{12}, 0, 0, 0)$.

We next make an appropriate thin plate approximation by expanding H in a Taylor series in x_3 about $x_3 = 0$ and keeping the leading terms.

$$H(E) \sim \bar{H}(\epsilon_1, \epsilon_2, \epsilon_{12}) + \frac{\partial \bar{H}}{\partial x_3} \bigg|_0 x_3 + \frac{1}{2} \frac{\partial^2 \bar{H}}{\partial x_3^2} \bigg|_0 x_3^2 \quad (2.26)$$

Use of a chain rule and equation (2.10) yields:

$$\frac{\partial^2 \bar{H}}{\partial x_3^2} \bigg|_0 = \bar{H}_{11} \kappa_1^2 + \bar{H}_{22} \kappa_2^2 + \bar{H}_{33} \kappa_{12}^2 + 2\bar{H}_{12} \kappa_1 \kappa_2 + 2\bar{H}_{13} \kappa_1 \kappa_{12} + 2\bar{H}_{23} \kappa_2 \kappa_{12} \quad (2.27)$$

where $\bar{H}_{11}(\epsilon_1, \epsilon_2, \epsilon_{12})$, $\bar{H}_{12}(\epsilon_1, \epsilon_2, \epsilon_{12})$, etc., are given by

$$\bar{H}_{11} = \frac{\partial^2 \bar{H}}{\partial \epsilon_1^2}, \quad \bar{H}_{12} = \frac{\partial^2 \bar{H}}{\partial \epsilon_1 \partial \epsilon_2}, \text{ etc.} \quad (2.28)$$

The integrated form of the work principle (2.18) is

$$\dot{\Sigma} = \int_{-h/2}^{h/2} P_{ji} \partial \dot{u}_i / \partial x_j dx_3 \quad (2.29)$$

where

$$\Sigma = \int_{-h/2}^{h/2} H dx_3 \quad (2.30)$$

is the plate strain energy density. Using (2.26), we obtain

$$\Sigma(\epsilon_1, \epsilon_2, \epsilon_{12}, \kappa_1, \kappa_2, \kappa_{12}) \sim h \bar{H}(\epsilon_1, \epsilon_2, \epsilon_{12}) + \frac{1}{24} h^3 \frac{\partial^2 \bar{H}}{\partial x_3^2} \bigg|_0 \quad (2.31)$$

Introducing forms (2.9) for the displacement components in (2.29) and using definitions (2.14) for force and moment resultants yields:

$$\begin{aligned} & \left(\frac{\partial \Sigma}{\partial \epsilon_1} - N_{11} \right) \frac{\partial \dot{U}_1}{\partial x_1} + \left(\frac{\partial \Sigma}{\partial \epsilon_2} - N_{22} \right) \frac{\partial \dot{U}_2}{\partial x_2} \\ & + \left(\frac{1}{2} \frac{\partial \Sigma}{\partial \epsilon_{12}} - N_{21} \right) \frac{\partial \dot{U}_1}{\partial x_2} + \left(\frac{1}{2} \frac{\partial \Sigma}{\partial \epsilon_{12}} - N_{12} \right) \frac{\partial \dot{U}_2}{\partial x_1} \\ & + \left(\frac{\partial \Sigma}{\partial \epsilon_1} \frac{\partial w}{\partial x_1} + \frac{1}{2} \frac{\partial \Sigma}{\partial \epsilon_{12}} \frac{\partial w}{\partial x_2} + N_{31} - N_{13} \right) \frac{\partial \dot{w}}{\partial x_1} \\ & + \left(\frac{\partial \Sigma}{\partial \epsilon_2} \frac{\partial w}{\partial x_2} + \frac{1}{2} \frac{\partial \Sigma}{\partial \epsilon_{12}} \frac{\partial w}{\partial x_1} - N_{23} + N_{32} \right) \frac{\partial \dot{w}}{\partial x_2} \\ & + \left(\frac{\partial \Sigma}{\partial \kappa_1} + M_{11} \right) \frac{\partial^2 \dot{w}}{\partial x_1^2} + \left(\frac{\partial \Sigma}{\partial \kappa_2} + M_{22} \right) \frac{\partial^2 \dot{w}}{\partial x_2^2} \\ & + \left(\frac{\partial \Sigma}{\partial \kappa_{12}} + M_{12} + M_{21} \right) \frac{\partial^2 \dot{w}}{\partial x_1 \partial x_2} = 0 \end{aligned} \quad (2.32)$$

Assuming that the rates in (2.32) are independent and arbitrary, leads us to set each of their coefficients to zero. These are the desired plate stress-strain relations. If Σ is approximated by (2.31) with the second derivative given by (2.27), these plate stress-strain relations take the form

$$\left. \begin{aligned} N_{11} &= h \frac{\partial \bar{H}}{\partial \epsilon_1}, \quad N_{22} = h \frac{\partial \bar{H}}{\partial \epsilon_2} \\ N_{12} &= N_{21} = \frac{1}{2} h \frac{\partial \bar{H}}{\partial \epsilon_{12}} \end{aligned} \right\} \quad (2.33)$$

$$\left. \begin{aligned} M_{11} &= -\frac{1}{12} h^3 [\bar{H}_{11} \kappa_1 + \bar{H}_{12} \kappa_2 + \bar{H}_{13} \kappa_{12}] \\ M_{22} &= -\frac{1}{12} h^3 [\bar{H}_{12} \kappa_1 + \bar{H}_{22} \kappa_2 + \bar{H}_{23} \kappa_{12}] \\ M_{12} + M_{21} &= -\frac{1}{12} h^3 [\bar{H}_{13} \kappa_1 + \bar{H}_{23} \kappa_2 + \bar{H}_{33} \kappa_{12}] \end{aligned} \right\} \quad (2.34)$$

$$\left. \begin{aligned} N_{13} &= N_{31} + h \frac{\partial \bar{H}}{\partial \epsilon_1} \frac{\partial w}{\partial x_1} + \frac{1}{2} h \frac{\partial \bar{H}}{\partial \epsilon_{12}} \frac{\partial w}{\partial x_2} \\ N_{23} &= N_{32} + h \frac{\partial \bar{H}}{\partial \epsilon_2} \frac{\partial w}{\partial x_2} + \frac{1}{2} h \frac{\partial \bar{H}}{\partial \epsilon_{12}} \frac{\partial w}{\partial x_1} \end{aligned} \right\} \quad (2.35)$$

where we have dropped higher order terms. Combining equations (2.33) and (2.35) yields

$$\left. \begin{aligned} N_{13} &= N_{31} + N_{11} \frac{\partial w}{\partial x_1} + N_{21} \frac{\partial w}{\partial x_2} \\ N_{23} &= N_{32} + N_{12} \frac{\partial w}{\partial x_1} + N_{22} \frac{\partial w}{\partial x_2} \end{aligned} \right\} \quad (2.36)$$

It is interesting that $N_{12} = N_{21}$ but $N_{13} \neq N_{31}$ and $N_{23} \neq N_{32}$.

The basic plate equations are the strain-displacement equations (2.11) and (2.12), the equilibrium equations (2.16) and (2.17), and the stress-strain relations (2.33), (2.34), and (2.36). These are 19 equations in 20 variables. To resolve this apparent discrepancy, we derive a combined equation of transverse equilibrium by substituting expressions (2.17) for N_{31} and N_{32} into equations (2.36) and then substituting the resulting expressions for N_{13} and N_{23} into equation (2.16). The result is

$$\begin{aligned} & \frac{\partial^2 M_{11}}{\partial x_1^2} + \frac{\partial^2}{\partial x_1 \partial x_2} (M_{12} + M_{21}) + \frac{\partial^2 M_{22}}{\partial x_2^2} \\ & + \frac{\partial}{\partial x_1} \left(N_{11} \frac{\partial w}{\partial x_1} + N_{12} \frac{\partial w}{\partial x_2} \right) \\ & + \frac{\partial}{\partial x_2} \left(N_{22} \frac{\partial w}{\partial x_2} + N_{21} \frac{\partial w}{\partial x_1} \right) + q = 0 \end{aligned} \quad (2.37)$$

The equations are now reduced to 15 in number: equations (2.11), (2.12), (2.16)₁, (2.16)₂, (2.33), (2.34), and (2.37). The variables are also 15 in number: ϵ_1 , ϵ_2 , ϵ_{12} , κ_1 , κ_2 , κ_{12} , U_1 , U_2 , w , N_{11} , N_{22} , N_{12} , M_{11} , M_{22} , $M_{12} + M_{21}$. Note that equation (2.37) is also obtained in the von-Karman theory.

Our particular interest in what follows is with the stress-strain relations (2.33) and (2.34) which express the nonlinear elastic mechanical properties of the plate. These properties are determined by the strain energy density H which is a function only of the middle surface strains ϵ_1 , ϵ_2 , and ϵ_{12} . Linear isotropic plate theory corresponds to the strain energy

$$\bar{H} = \frac{E}{2(1-\nu^2)} (\epsilon_1 + \epsilon_2)^2 - \frac{E}{1+\nu} (\epsilon_1 \epsilon_2 - \epsilon_{12}^2) \quad (2.38)$$

where E is Young's modulus and ν Poisson's ratio. Linear orthotropic plate theory corresponds to

$$\bar{H} = \frac{E_1}{2(1-\nu_1 \nu_2)} (\epsilon_1^2 + \nu_2 \epsilon_1 \epsilon_2)$$

$$+ \frac{E_2}{2(1-\nu_1\nu_2)} (\epsilon_2^2 + \nu_1\epsilon_1\epsilon_2) + 2G\epsilon_{12}^2 \quad (2.39)$$

where E_1 and E_2 are Young's moduli, ν_1 , ν_2 Poisson's ratios, and G the shear modulus. Note that the moduli satisfy $\nu_2 E_1 = \nu_1 E_2$. Equations (2.38) and (2.39) are obtained, of course, by starting with the expressions for the three-dimensional strain energy density $H(E)$ and following the steps indicated from equations (2.20) to (2.25).

For orthotropic (rhombic) material symmetry it can be shown that $\bar{H}$ depends on variable ϵ_{12} as the square of that variable; that is

$$\bar{H}(\epsilon_1, \epsilon_2, \epsilon_{12}^2) \quad (2.40)$$

The proof of this is given in [8]. This symmetry is appropriate to paper or wood.

We note that the constitutive equations (2.34) are linear in curvature components with stiffness moduli that depend only on middle-surface strain components. This form is consistent with the original scaling assumptions (2.4), (2.5), and (2.6) on which this entire plate theory is based. In each equation only terms of lowest order in ϵ are retained. One could produce terms in (2.34) that are nonlinear in curvature components by retaining additional terms in the expansion (2.26) but that would be inconsistent with the approximations made in the kinematic and equilibrium equations.

It is important that middle-surface strains are small, being of order ϵ^2 , but general nonlinear, anisotropic constitutive behavior is retained. In fact, this is how the present theory represents an extension of the von-Karman theory.

We remark that equations (2.36) can also be derived from the fact that tensor T_{ij} , defined in equation (2.2), is symmetric. This symmetry has been built into the preceding analysis because T_{ij} is defined as $\partial H / \partial E_{ij}$ and E_{ij} is symmetric by definition.

From equation (2.2),

$$P_{ij} = T_{ik} F_{jk}$$

which inverts to give

$$T_{ij} = P_{ik} F_{jk}^{-1}$$

The symmetry condition $T_{ij} = T_{ji}$ leads to the following equation to be satisfied by tensor P_{ij} .

$$P_{ki} F_{jk} = F_{ik} P_{kj}$$

or

$$P_{ji} + P_{ki} \frac{\partial u_j}{\partial x_k} = P_{ij} + \frac{\partial u_i}{\partial x_k} P_{kj} \quad (2.41)$$

Equation (2.41) is an identity when $i = j$. For $i \neq j$, if we use scalings (2.5) and (2.13), drop higher order terms and write $u_3 = w(x_1, x_2)$. Equation (2.41) yields

$$\left. \begin{aligned} P_{12} &= P_{21} \\ P_{13} &= P_{31} + P_{11} \frac{\partial w}{\partial x_1} + P_{21} \frac{\partial w}{\partial x_2} \\ P_{23} &= P_{32} + P_{12} \frac{\partial w}{\partial x_1} + P_{22} \frac{\partial w}{\partial x_2} \end{aligned} \right\} \quad (2.42)$$

The integration of these equations over the thickness of the plate yields $N_{12} = N_{21}$ and equations (2.36).

We also note that tensor T_{ij} is related to the Cauchy stress tensor C_{ij} by

$$C_{ij} = \frac{\rho}{\rho_0} F_{ik} T_{km} F_{jm}$$

where ρ and ρ_0 are densities of the deformed and undeformed body. The symmetry of C_{ij} , and hence of T_{ij} , follows from the law of moment equilibrium in continuum mechanics. In

this sense, equations (2.36) represent plate equilibrium equations. In linear plate theory, five equilibrium equations of the form (2.16) and (2.17) are used. Equations (2.36) represent additional equilibrium equations' that arise in the present theory because of the use of the nonsymmetric stress tensor P_{ij} .

3 A Special Plate Theory

As pointed out in the foregoing the nonlinear elastic properties of a thin plate are described by equations (2.33) and (2.34) in terms of a single function, the strain energy density H . H is a function of three variables, the middle-surface strains ϵ_1 , ϵ_2 , ϵ_{12} . To completely determine $\bar{H}$, it would be necessary to have a complete set of data on biaxial stretching and shear of the plate. Since such data is not yet available for paper, we formulate a simpler set of stress-strain relations in this section based on an assumed structure for the strain energy density function. All the parameters in these equations can be determined from experimental data that is now available on paper. Beyond this, the modified theory is interesting in its own right.

Note that, for linear orthotropic elasticity the plate strain energy function (2.39) can be written in the form

$$\bar{H} = \frac{\nu_2 E_1}{2(1-\nu_1\nu_2)} e \quad (3.1)$$

where e is given by

$$e = \nu_2^{-1} \epsilon_1^2 + \nu_1^{-1} \epsilon_2^2 + 2\epsilon_1 \epsilon_2 + c \epsilon_{12}^2 \quad (3.2)$$

Constant c is related to the shear modulus G by

$$c = \frac{4(1-\nu_1\nu_2)}{\nu_1 E_2} G \quad (3.3)$$

The quantity $\epsilon_e = \sqrt{e}$ is an "effective strain" measure, a measure of the magnitude of strain at a point in the material.

For nonlinear orthotropic material behavior we construct an approximate theory by assuming that the plate strain energy function is a function of the single variable e as it is in the linear theory, $\bar{H}(e)$. Constants ν_1 , ν_2 , and c as well as the functional form of $\bar{H}(e)$ are to be found by comparison with the data. The material functions in (2.33) and (2.34) now take the form

$$\left. \begin{aligned} N_{11} &= h \bar{H}'(e) 2(\nu_2^{-1} \epsilon_1 + \epsilon_2) \\ N_{22} &= h \bar{H}'(e) 2(\nu_1^{-1} \epsilon_2 + \epsilon_1) \\ N_{12} &= N_{21} = h \bar{H}'(e) c \epsilon_{12} \end{aligned} \right\} \quad (3.4)$$

$$\left. \begin{aligned} \bar{H}_{11} &= \bar{H}''(e) 4(\nu_2^{-1} \epsilon_1 + \epsilon_2)^2 + 2\nu_2^{-1} \bar{H}'(e) \\ \bar{H}_{22} &= \bar{H}''(e) 4(\nu_1^{-1} \epsilon_2 + \epsilon_1)^2 + 2\nu_1^{-1} \bar{H}'(e) \\ \bar{H}_{12} &= \bar{H}''(e) 4(\nu_2^{-1} \epsilon_1 + \epsilon_2)(\nu_1^{-1} \epsilon_2 + \epsilon_1) + 2\bar{H}'(e) \\ \bar{H}_{33} &= \bar{H}''(e) 4c^2 \epsilon_{12}^2 + 2c \bar{H}'(e) \\ \bar{H}_{13} &= \bar{H}''(e) 4(\nu_2^{-1} \epsilon_1 + \epsilon_2) c \epsilon_{12} \\ \bar{H}_{23} &= \bar{H}''(e) 4(\nu_1^{-1} \epsilon_2 + \epsilon_1) c \epsilon_{12} \end{aligned} \right\} \quad (3.5)$$

For uniaxial extension in the two direction, we have $N_{11} = 0$ which leads to $\epsilon_1 = -\nu_2 \epsilon_2$. That is, constant ν_2 is the Poisson ratio associated with two-direction extension. Similarly, ν_1 is the Poisson ratio associated with one-direction extension. It is interesting that, while this theory is nonlinear, it predicts constant Poisson ratios. With $\epsilon_{12} = 0$, equations (3.4) and (3.5) take the following form for two-direction uniaxial extension:

¹ We thank Professor Gerald Wempner for pointing this out.

$$e = (\nu_1^{-1} - \nu_2)\epsilon_2^2 \quad (3.6)$$

$$N_{22} = 2h(\nu_1^{-1} - \nu_2)\bar{H}'(e)\epsilon_2 \quad (3.7)$$

$$\left. \begin{aligned} \bar{H}_{11} &= 2\nu_2^{-1}\bar{H}'(e) \\ \bar{H}_{22} &= 4(\nu_1^{-1} - \nu_2)^2\bar{H}''(e)\epsilon_2^2 + 2\nu_1^{-1}\bar{H}'(e) \\ \bar{H}_{12} &= 2\bar{H}'(e), \quad \bar{H}_{33} = 2c\bar{H}'(e) \\ \bar{H}_{13} &= \bar{H}_{23} = 0 \end{aligned} \right\} \quad (3.8)$$

Similar relations hold for uniaxial extension in the one direction. The axial stress-strain relation is, in this case:

$$N_{11} = 2h(\nu_2^{-1} - \nu_1)\bar{H}'(e)\epsilon_1 \quad (3.9)$$

where

$$e = (\nu_2^{-1} - \nu_1)\epsilon_1^2.$$

Either extension in the two direction or in the one direction can be used to determine the function $\bar{H}(e)$. To use (3.7), we combine it with (3.6) to obtain

$$\frac{d\bar{H}}{d\epsilon_2} = \sigma_2(\epsilon_2) \quad (3.10)$$

where σ_2 is the axial stress in uniaxial extension. With $\sigma_2(\epsilon_2)$ given experimentally, we can integrate (3.10) to find function $\bar{H}$. Substituting this form for H back into (3.4) and (3.5) determines the stress-strain relations once constant c has been found. If extension in the one direction is used to determine H the relevant equation is:

$$\frac{d\bar{H}}{d\epsilon_1} = \sigma_1(\epsilon_1) \quad (3.11)$$

One interesting relationship predicted by this special theory says that the stress σ_1 in uniaxial extension in the one direction is related to the stress σ_2 in uniaxial extension in the two direction by

$$\epsilon_1\sigma_1(\epsilon_1) = \epsilon_2\sigma_2(\epsilon_2) \quad (3.12)$$

when the corresponding strains are related by

$$\sqrt{\nu_1}\epsilon_1 = \sqrt{\nu_2}\epsilon_2 \quad (3.13)$$

These relations are compared with experimental data for paper in Section 4. They provide a critical test of the special plate theory. A complete test of this theory will be possible if and when biaxial stress-strain data are available.

Any strain energy function of form (2.40) describes an orthotropic material. Hence, the special strain energy form of this section given by $\bar{H}(e)$ where e is given by (3.2) describes a particular orthotropic material.

4 Application to Paper Properties

This section compares the Special Plate Theory to data on paper. We identify the one direction with the machine direction (M.D.) of paper and the two directions with the cross machine direction (C.D.). Our current application of the theory is as part of a theory to predict the behavior of paperboard in a corrugated fiberboard structure. We therefore limit strains to those normally encountered in paperboard during the compressive loading of fiberboard boxes. Up to these strains we can adequately assume that compression and tension stress-strain curves are identical.

Average values for Poisson ratios ν_1 and ν_2 are given by Kellicutt for Fourdrinier kraft paperboards [11]. These values are

$$\nu_1 = 0.328, \nu_2 = 0.219 \quad (4.1)$$

An effective Poisson ratio, $\nu = 0.268$, defined as the geometric mean $\nu = \sqrt{\nu_1\nu_2}$, was used in the engineering calculations of [5] and [10].

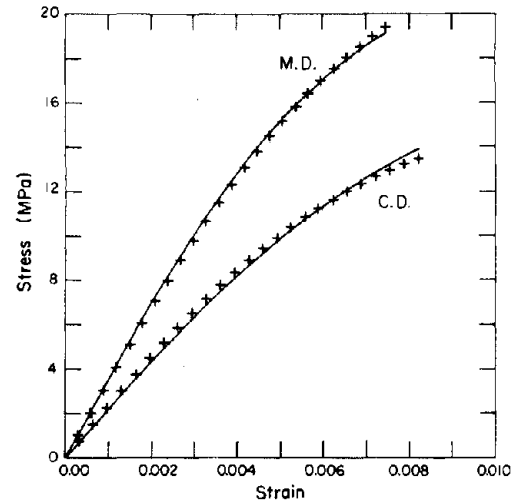


Fig. 1 Stress-strain curves for linerboard material. The crosses represent the average fit of equation (4.3) to tests of 18 C.D. specimens and the average fit to tests of 18 M.D. specimens. The curves represent a fit of equation (4.7) to the data.

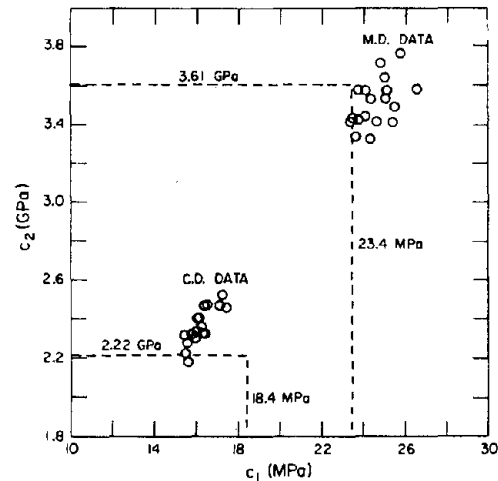


Fig. 2 Values of C_1 and C_2 . The points are obtained by fitting equation (4.3) to tests of 18 C.D. specimens and 18 M.D. specimens. The dashes represent the values obtained from the regression analysis of equation (4.7).

Poisson ratios for several papers tested in tension have been measured by Götsching and Baumgarten [12] and by Baum and Bornhoeft [13]. The range of values found is large, ν_1 ranging from 0.15–0.50, and ν_2 from 0.04–0.20. There is a large scatter in the results given in [12], which illustrates the difficulty in the measurement of Poisson ratios.

Experimental values of the initial shear modulus G for several papers are given by Setterholm, Benson, and Kuenzi [14]. These values range from 1.23 to 1.47 GPa. For situations where the shear modulus is not known, some writers [15] recommend the formula

$$G = \frac{\sqrt{E_1 E_2}}{2(1 + \sqrt{\nu_1 \nu_2})} = \frac{\sqrt{\nu_1 / \nu_2} E_2}{2(1 + \sqrt{\nu_1 \nu_2})} \quad (4.2)$$

where E_1 and E_2 are initial Young's moduli. For the stress-strain data on several papers given in [5] and with ν_1 and ν_2 given by equation (4.1), equation (4.2) predicts G values from 1.01–1.59 GPa, which are not unreasonable when compared to the results in [14].

The prediction of constant Poisson ratios is confirmed up to the strains of interest by tensile data given by Götsching and Baumgarten [12]. For several papers (Figs. 3, 4, 6, 7, 10 of reference [12]), the Poisson ratios are almost constant. For

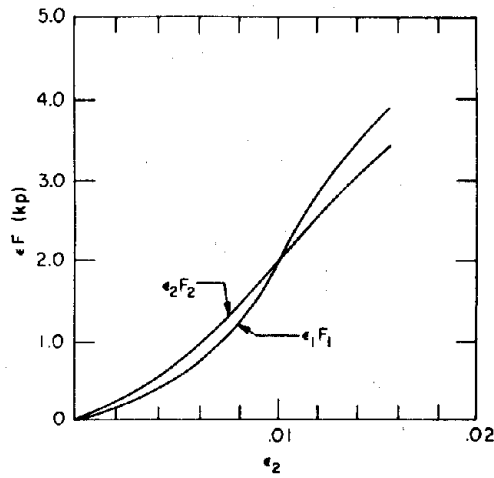


Fig. 3 Curves of $\epsilon_1 F_1$ and $\epsilon_2 F_2$ determined from data given in [12]. $\nu_1 = 0.42$, $\nu_2 = 0.11$. The theory of equations (3.12) and (3.13) predicts that the two curves should be identical.

the other papers tested, there is only a small deviation, so that assuming constant Poisson ratios would seem an adequate engineering approximation. In fact, throughout reference [12], Poisson's ratio is implicitly assumed to be constant, that is, independent of strain.

An empirical relation for the compressive stress-strain relation $\sigma(\epsilon)$ is given in [5]. This relation makes use of three parameters. A more accurate formula, which uses only two parameters, is given in [9] along with a procedure for matching the formula to data. That formula is

$$\sigma(\epsilon) = c_1 \tanh(c_2 \epsilon / c_1) \quad (4.3)$$

where c_1 and c_2 are parameters to be determined from the data. While (4.3) can be identified with either the C.D. or M.D. stress-strain curves, we here match it with the C.D. two direction.

$$\sigma_2(\epsilon_2) = c_1 \tanh(c_2 \epsilon_2 / c_1) \quad (4.4)$$

Equation (3.10) then reads

$$\frac{d\bar{H}}{d\epsilon_2} = c_1 \tanh(c_2 \epsilon_2 / c_1)$$

Integrating this equation yields the strain energy density function

$$\bar{H} = (c_1^2 / c_2) \log \cosh(c_2 \epsilon_2 / c_1)$$

Replacing ϵ_2 by e according to equation (3.6) gives the strain energy density in terms of e .

$$\bar{H}(e) = (c_1^2 / c_2) \log \cosh \left(\frac{c_2}{c_1} \sqrt{\frac{\nu_1 e}{1 - \nu_1 \nu_2}} \right) \quad (4.5)$$

For uniaxial deformation in the one direction, the stress-strain relation is given by equation (3.11). Using equations (3.9) and (4.5), we obtain

$$\sigma_1(\epsilon_1) = N_{11} / h = c_1 \sqrt{\frac{\nu_1}{\nu_2}} \tanh \left(\frac{c_2}{c_1} \sqrt{\frac{\nu_1}{\nu_2}} \epsilon_1 \right) \quad (4.6)$$

Stress-strain data have been taken at the Forest Products Laboratory for linerboard material tested in edgewise compression. Eighteen C.D. specimens and 18 M.D. specimens have been tested. The results of an analysis of these data are shown on Figs. 1 and 2. While it makes some sense to determine c_1 and c_2 by fitting (4.3) to C.D. data only, we have found that a good fit to both C.D. and M.D. data can be made with the single equation

$$\sigma = q_1 c_1 \tanh(c_2 \epsilon / c_1) + q_2 c_1 \sqrt{A} \tanh \left(\frac{c_2}{c_1} \sqrt{A} \epsilon \right) \quad (4.7)$$

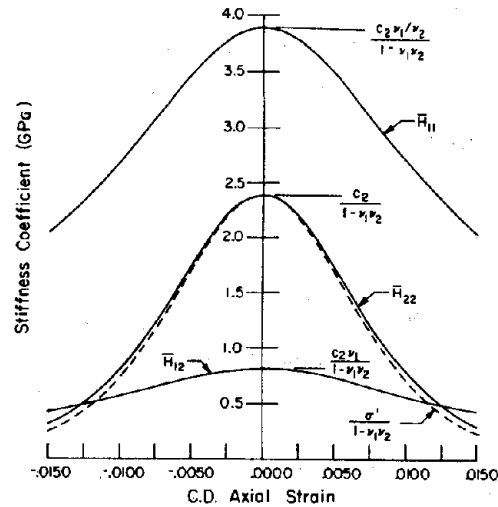


Fig. 4 Form of the plate stiffness coefficients in uniaxial strain. $c_1 = 18.4$ MPa, $c_2 = 2.22$ GPa, $A = 1.63$, $\nu = 0.268$ ($\nu_1 = 0.342$, $\nu_2 = 0.210$). Also shown is the tangent modulus coefficient $\sigma' / (1 - \nu_1 \nu_2)$ used as an approximation to H_{22} in [5].

where A stands for the ratio ν_1 / ν_2 . Equation (4.7) reduces to (4.4) when $q_1 = 1$, $q_2 = 0$, and to (4.6) when $q_1 = 0$, $q_2 = 1$.

Values of c_1 , c_2 , and A are determined from a nonlinear regression analysis of (4.7) using σ as the dependent variable and ϵ , q_1 , and q_2 as the independent variables. For C.D. data we use $q_1 = 1$, $q_2 = 0$, and for M.D. data $q_1 = 0$, $q_2 = 1$. The values of c_1 and c_2 obtained, which are C.D. values, are c_1 (C.D.) = 18.4 MPa and c_2 (C.D.) = 2.22 GPa. The value of A obtained is $A = 1.628$.

From equations (4.3), (4.4), and (4.6) we see that

$$\left. \begin{aligned} c_1(\text{M.D.}) &= \sqrt{A} c_1(\text{C.D.}) \\ c_2(\text{M.D.}) &= A c_2(\text{C.D.}) \end{aligned} \right\} \quad (4.8)$$

These equations are equivalent to equations (3.12) and (3.13) of the special theory. We use them and the preceding results for c_1 (C.D.) and c_2 (C.D.) to obtain c_1 (M.D.) = 23.4 MPa and c_2 (M.D.) = 3.61 GPa. These parameter values are used in equation (4.3) to plot the curves in Fig. 1 which show good agreement with the data.

On Fig. 2 are shown values of c_1 and c_2 obtained by matching equation (4.3) to the 18 C.D. and 18 M.D. stress-strain curves. These are compared with values of c_1 and c_2 obtained from equation (4.7). Figures 1 and 2 are a good test of the special theory and especially of relations (3.12) and (3.13) predicted by the special theory.

In Fig. 8 of reference [12] are given load and lateral contraction data for tensile tests in the machine and cross-machine directions of offset printing paper. These results can also be used to check prediction (3.12) and (3.13). The Poisson ratios determined from Fig. 8 of [12] are $\nu_1 = 0.42$ and $\nu_2 = 0.11$. The check of equation (3.12) is shown in Fig. 3. There is especially good agreement for strains less than 0.01. Note that the data are not given in terms of stress, but rather in tensile force. Thus, forces rather than stresses have been used in equation (3.12) and on Fig. 3. This is justified because the widths of all test specimens are the same and it was assumed that the thicknesses of M.D. and C.D. specimens were the same.

The plate stiffness coefficients appear in constitutive equations (2.34) and are given by equations (3.5) for the special theory. With the strain energy given by equation (4.5), equations (3.5) take the following form for uniaxial extension in the two direction.

$$\left. \begin{aligned}
 \bar{H}_{11} &= \frac{c_1 \nu_1 / \nu_2}{1 - \nu_1 \nu_2} \frac{\tanh(c_2 \epsilon_2 / c_1)}{\epsilon_2} \\
 \bar{H}_{22} &= \frac{c_2}{\cosh^2(c_2 \epsilon_2 / c_1)} + \frac{\nu_1 \nu_2 c_1}{1 - \nu_1 \nu_2} \frac{\tanh(c_2 \epsilon_2 / c_1)}{\epsilon_2} \\
 \bar{H}_{12} &= \frac{c_1 \nu_1}{1 - \nu_1 \nu_2} \frac{\tanh(c_2 \epsilon_2 / c_1)}{\epsilon_2} \\
 \bar{H}_{33} &= c \bar{H}_{12}, \quad \bar{H}_{13} = \bar{H}_{23} = 0
 \end{aligned} \right\} \quad (4.9)$$

The variation of stiffness components (4.9) with axial strain is illustrated in Fig. 4. There are no experimental data available to compare with these predictions of the theory.

Acknowledgment

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