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Inference

Confidence Bounds and Hypothesis Tests for Normal Distribution Coefficients of Variation

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For normally distributed populations, we obtain confidence bounds on a ratio of two coefficients of variation, provide a test for the equality of k coefficients of variation, and provide confidence bounds on a coefficient of variation shared by k populations.

Keywords Asymptotics; Coefficient of variation; $\sqrt{n}$-consistent estimators; Cramér conditions; Efficient likelihood estimators; Likelihood ratio test; Newton's method; One-step Newton estimators; Quadratic mean differentiability; Rao test; Risk to return ratio; Signal to noise ratio; Wald test.

Mathematics Subject Classification Primary 62F03; Secondary 62F25.

1. Introduction

The coefficent of variation (COV) of a distribution with mean μ and variance σ^2 is defined as the noise to signal ratio or σ/μ. (Sometimes this ratio is multiplied by 100 and reported as a percentage.) Building materials are often evaluated not only on the basis of mean strength but also on relative variability. Laboratory techniques are often compared on the basis of their COVs. Financial managers treat coefficients of variation as measures of risk to return. Thus, scientists, engineers, and portfolio managers are interested in obtaining confidence intervals on population COVs and in testing for the equality of COVs.

For normally distributed populations, Vangel (1996) and Verrill (2003) focused on techniques for obtaining confidence intervals on single coefficients of variation. In this article, we use likelihood based methods to obtain large sample solutions to

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three additional problems:

1. Obtain a confidence interval on the ratio of two coefficients of variation.
2. Perform a test of the hypothesis that the coefficients of variation associated with k populations are all equal.
3. Obtain a confidence interval on a coefficient of variation that is shared by k populations.

A likelihood ratio test of the hypothesis that k coefficients of variation are equal has previously appeared in the literature (Doornbos and Dijkstra, 1983; Nairy and Rao, 2003). However, no rigorous demonstration of its asymptotic distribution has previously appeared. In addition, several other tests of the equality of k coefficients of variation have been proposed (Bennett, 1976; Feltz and Miller, 1996; Miller, 1991; Nairy and Rao, 2003; Shafer and Sullivan, 1986). Simulation studies (Doornbos and Dijkstra, 1983; Fung and Tsang, 1998; Nairy and Rao, 2003; Shafer and Sullivan, 1986) have suggested that the tests of Bennett (1976), Miller (1991) (and their modifications—Feltz and Miller, 1996; Shafer and Sullivan, 1986), and the ICV Wald test of Nairy and Rao (2003) have statistical sizes that are near nominal for small sample sizes while the statistical size of the likelihood ratio test is overly liberal. This has led some authors to recommend the tests of Bennett (1976) and Miller (1991) over the likelihood ratio test. However, we provide Web-based simulation tools that protect a user from an overly liberal test. We illustrate the use of one of these tools in the next section.

In a subsequent article we will compare the power properties of the tests when actual statistical sizes for all tests are near nominal.

A confidence interval on a ratio of two coefficients of variation has not previously appeared in the literature. Tian (2005) has taken a Weerahandi (1993) approach to obtain a generalized confidence interval on a coefficient of variation that is shared by k populations.

We make use of likelihood ratio tests in Theorems A.2, A.4, and A.6. The asymptotic distributions of our test statistics are given by Theorems H.1 and H.4 of Verrill and Johnson (2007). We establish the conditions that permit us to invoke these theorems in Appendices B, E, and G of Verrill and Johnson (2007). These conditions would also have permitted us to invoke Theorems H.2, H.3, H.5, and H.6 of Verrill and Johnson (2007) to obtain the asymptotic distributions of the corresponding Wald and Rao test statistics.

In Appendix H of Verrill and Johnson (2007) we also demonstrate that a solution of the likelihood equations can be replaced in likelihood ratio, Wald, and Rao tests by a Newton step refinement of a $\sqrt{n}$-consistent estimator. ($\hat{\mathbf{a}}$ is a $\sqrt{n}$-consistent estimator of $\mathbf{a}$ if $\sqrt{n}(\hat{\mathbf{a}} - \mathbf{a}) = \mathbf{O}_p(1)$.)

An outline of an alternative proof of results closely related to our Theorems H.1–H.3 is provided in Sec. 6e.3 of Rao (1973). Taking a quadratic mean differentiability approach rather than the Cramér conditions approach that we take, Lehmann and Romano (2005) outline the proof of results (see their Sect. 12.4.2–12.4.4) closely related to our Theorems H.1–H.6. However, their Newton step estimator results actually rely on unstated assumptions about Cramér conditions. In Verrill and Johnson (2007) we make our Cramér condition assumptions and their use explicit. The original work on asymptotic tests of composite hypotheses was due to Wilks (1938) and Wald (1943).

2. Application of our Results to Laboratory Quality Control Data

In this section we illustrate the use of one of our programs to test the hypothesis that five coefficients of variation are equal. The data set consists of quality control measurements of the Xylan percentage (Xylan is a polysaccharide) in a standard measured in the USDA Forest Products Laboratory's Analytical Chemistry Laboratory over the course of two years. (We thank Dr. Mark Davis for providing this data.) The data are presented in Table 1. The measurements were time sequential. For the purposes of illustration we have grouped observations 1–20 into Group 1, observations 21–40 into Group 2, and so on. Ideally the quality control measurements represent a steady state process so the coefficient of variation should not be changing. We plot the data in Fig. 1. The line is a loess smooth.

From the top plot in Fig. 1, it is clear that for practical purposes, the Xylan measurement is quite stable. However, the bottom plot in Fig. 1 suggests that at a micro level, there might be statistically significant local trends in the data. (There were 102 observations in the full data set. The dashed lines in the bottom plot of Fig. 1 demarcate the 5 groups of 20.)

We plot box plots of the data in Fig. 2. The sample means and standard deviations are presented in Table 2. Normal probability plots and formal tests of normality indicated that Groups 1, 2, and 5 do not violate the normality assumption. Group 3 appears to be bimodal and nonnormal. Group 4 appears to have two outliers that cause the normality assumption to be violated. We have performed three analyses. For the first we accepted all of the data. For the second we removed two "outliers." For the third we removed all "outliers." The Web program that we used to analyze the three cases is available at http://www1.fpl.fs.fed.us/covtestk.html; See Fig. 3. As indicated in Fig. 3 a user need only provide the program with the number of groups, the sample sizes, means, and standard deviations of the groups, and an integer starting value for the random number generator that is used in the small sample simulation. The program returns the p-value calculated from the asymptotic test, and a simulation based estimated p-value. For the data presented in Table 1, the asymptotic p-value is 0.144, and the small sample simulation test is not significant at a 0.10 level. However, if the two "outliers" of Group 4 are removed, the asymptotic p-value is 0.016 and the estimated p-value from the small sample simulation is 0.025. If all "outliers" in all groups are removed, the asymptotic p-value is 0.006 and the estimated p-value from the small sample simulation is 0.012. A user should give greater credence to the results of the small sample tests (see Sec. 3).

We emphasize that we have presented this example solely for the purpose of illustrating the use of the program. A fully defensible analysis in this case would have to consider issues of serial correlation, non normal data, and the legitimacy of discarding outliers.

We have also developed two additional Web-based programs. A program that calculates a confidence interval on a ratio of two coefficients of variation can be found at http://www1.fpl.fs.fed.us/covratio.html. A program that calculates a confidence interval on a coefficient of variation that is shared by k normally distributed populations can be found at http://www1.fpl.fs.fed.us/covconfk.html.

In Appendix A we present the statistical theory that underlies these programs.

Table 1
Measurement order and Xylan percent

Order	Xylan percent	Order	Xylan percent	Order	Xylan percent
1	6.34	46	6.22	91	6.42
2	6.35	47	6.37	92	6.43
3	6.37	48	6.24	93	6.28
4	6.55	49	6.19	94	6.38
5	6.28	50	6.39	95	6.32
6	6.30	51	6.38	96	6.36
7	6.39	52	6.39	97	6.25
8	6.29	53	6.37	98	6.35
9	6.26	54	6.40	99	6.34
10	6.11	55	6.37	100	6.41
11	6.25	56	6.43	101	6.24
12	6.22	57	6.23	102	6.33
13	6.25	58	6.45		
14	6.36	59	6.36		
15	6.30	60	6.28		
16	6.33	61	6.33		
17	6.41	62	6.35		
18	6.48	63	6.25		
19	6.38	64	6.30		
20	6.31	65	6.31		
21	6.28	66	6.35		
22	6.31	67	6.35		
23	6.37	68	6.28		
24	6.15	69	6.34		
25	6.45	70	6.48		
26	6.25	71	6.59		
27	6.18	72	6.22		
28	6.36	73	6.37		
29	6.26	74	6.26		
30	6.26	75	6.20		
31	6.30	76	6.28		
32	6.27	77	6.34		
33	6.19	78	6.37		
34	6.31	79	6.25		
35	6.28	80	6.33		
36	6.23	81	6.34		
37	6.28	82	6.34		
38	6.34	83	6.34		
39	6.31	84	6.33		
40	6.22	85	6.43		
41	6.22	86	6.19		
42	6.27	87	6.40		
43	6.14	88	6.28		
44	6.13	89	6.34		
45	6.17	90	6.36		

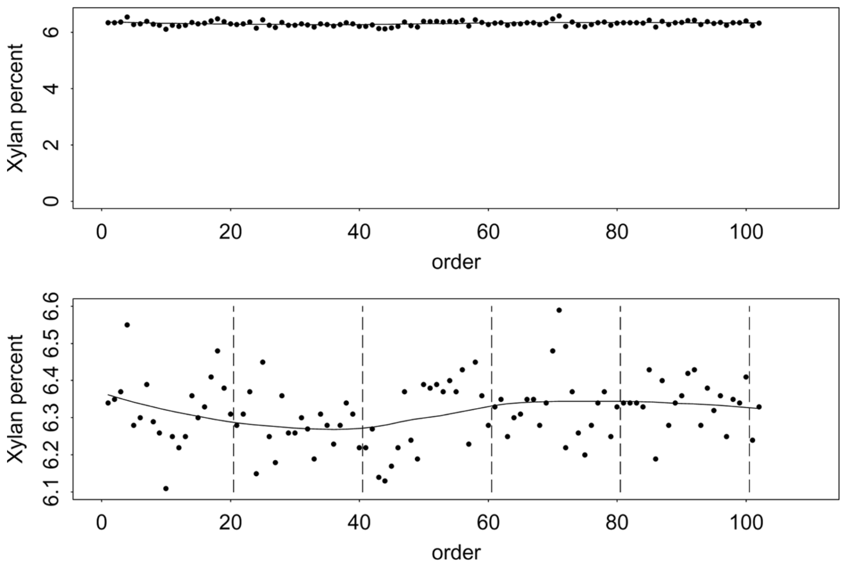

Figure 1. Plots of Xylan percentage vs. time order.

3. Small Sample Tools and Further Research

The theory that underlies the theorems presented in Appendix A is asymptotic theory. That is, it yields good approximations for large data sets but poorer approximations for small data sets. In particular, for small data sets, it leads to confidence intervals that are too narrow and to tests of hypotheses that reject true null hypotheses too frequently. We are currently engaged in research that

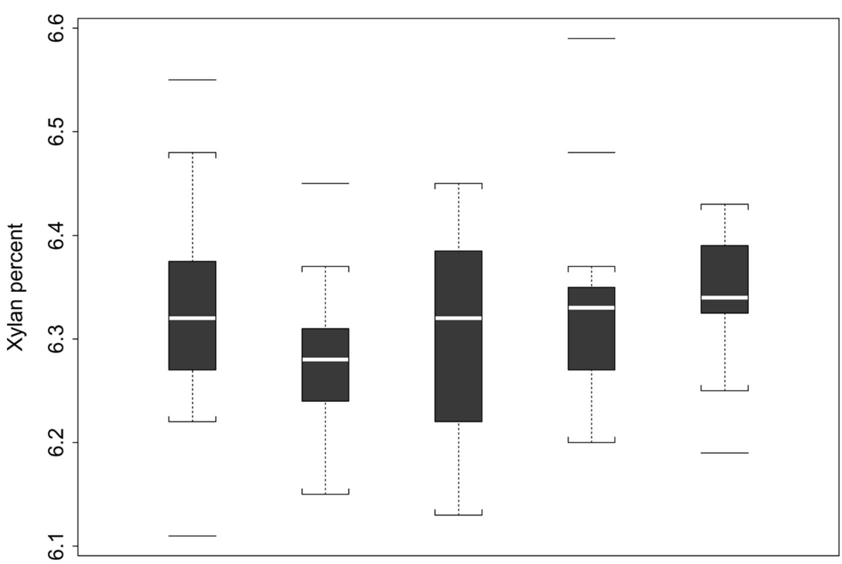

Figure 2. Boxplots of the data from the five groups.

Table 2
Sample means and standard deviations of the Xylan data

	"Outliers" in		"Outliers" out	
Group	Mean	SD	Mean	SD
1	6.3265	.09477	6.3261	.06590
2	6.2800	.06996	6.2711	.05896
3	6.3000	.10167	6.3000	.10167
4	6.3275	.08813	6.3044	.05193
5	6.3445	.06134	6.3526	.05075

should lead to improved small sample approximations. In the interim, however, we have provided a simulation-based fix to the problem. Our Web-based programs (see http://www1.fpl.fs.fed.us/covconfk.html) perform the theoretical calculations needed to obtain confidence intervals or to test hypotheses. However, they also perform tests and calculate confidence intervals that are based on simulations. In

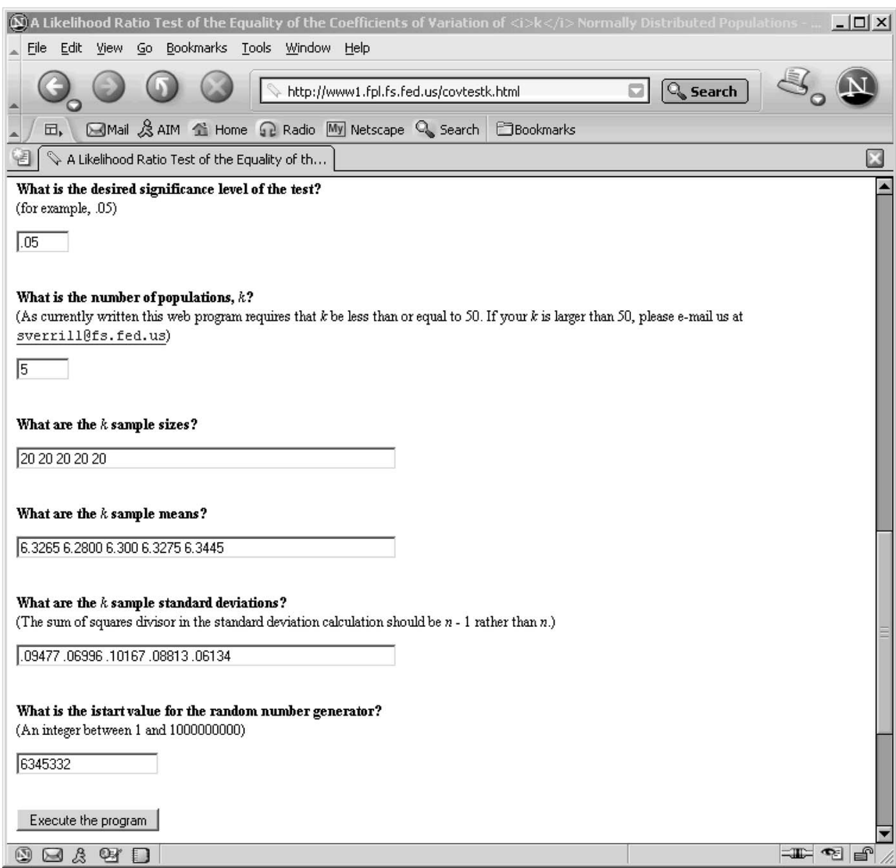

Figure 3. Web page for a test of the equality of k coefficients of variation.

particular, small sample critical values are obtained via simulations in which 10,000 samples are generated from the normal distributions estimated from the original data. The value of the appropriate likelihood ratio statistic is calculated for each of these samples. Empirical estimates of the 90th, 95th, and 99th percentiles of the distribution of the likelihood ratio statistic are then obtained from these 10,000 values, and used to perform small sample tests and to calculate small sample confidence intervals.

These simulations might be suspect because they make use of estimated population parameters rather than the true (and unknown) population parameters.

Table 3
Confidence interval coverage of the ratio[a] of two coefficients of variation (based on Theorem A.2), $n = n_1 = n_2$

			Asymptotic procedure		Simulation procedure	
COV	n	Nominal coverage	Estimate[b] of coverage	95% CI[b] on coverage	Estimate[b] of coverage	95% CI[b] on coverage down
0.05	5	0.90	0.8372	[0.8299, 0.8444]	0.8985	[0.8925, 0.9043]
		0.95	0.9007	[0.8948, 0.9065]	0.9490	[0.9446, 0.9532]
		0.99	0.9723	[0.9690, 0.9754]	0.9885	[0.9863, 0.9905]
	10	0.90	0.8732	[0.8666, 0.8796]	0.9033	[0.8974, 0.9090]
		0.95	0.9314	[0.9264, 0.9363]	0.9518	[0.9475, 0.9559]
		0.99	0.9830	[0.9804, 0.9854]	0.9908	[0.9888, 0.9926]
	20	0.90	0.8863	[0.8800, 0.8924]	0.8990	[0.8930, 0.9048]
		0.95	0.9403	[0.9356, 0.9449]	0.9484	[0.9440, 0.9526]
		0.99	0.9853	[0.9828, 0.9876]	0.9892	[0.9871, 0.9911]
0.15	5	0.90	0.8288	[0.8214, 0.8361]	0.8990	[0.8930, 0.9048]
		0.95	0.9009	[0.8950, 0.9067]	0.9504	[0.9461, 0.9546]
		0.99	0.9718	[0.9685, 0.9750]	0.9913	[0.9894, 0.9930]
	10	0.90	0.8715	[0.8649, 0.8780]	0.9009	[0.8950, 0.9067]
		0.95	0.9294	[0.9243, 0.9343]	0.9490	[0.9446, 0.9532]
		0.99	0.9825	[0.9798, 0.9850]	0.9911	[0.9892, 0.9928]
	20	0.90	0.8869	[0.8806, 0.8930]	0.9001	[0.8941, 0.9059]
		0.95	0.9427	[0.9381, 0.9472]	0.9507	[0.9464, 0.9549]
		0.99	0.9868	[0.9845, 0.9889]	0.9902	[0.9882, 0.9920]
0.25	5	0.90	0.8349	[0.8276, 0.8421]	0.8956	[0.8895, 0.9015]
		0.95	0.8974	[0.8914, 0.9033]	0.9496	[0.9452, 0.9538]
		0.99	0.9700	[0.9666, 0.9733]	0.9900	[0.9880, 0.9919]
	10	0.90	0.8670	[0.8603, 0.8736]	0.8966	[0.8906, 0.9025]
		0.95	0.9257	[0.9205, 0.9308]	0.9491	[0.9447, 0.9533]
		0.99	0.9828	[0.9802, 0.9853]	0.9900	[0.9880, 0.9919]
	20	0.90	0.8896	[0.8834, 0.8957]	0.9039	[0.8980, 0.9096]
		0.95	0.9439	[0.9393, 0.9483]	0.9538	[0.9496, 0.9578]
		0.99	0.9855	[0.9831, 0.9877]	0.9886	[0.9864, 0.9906]

[a]Ratio = 1 in these trials.
[b]Based on 10,000 trials (and 10,000 trials within each of these to determine the simulation-based critical values used to construct small-sample confidence intervals).

However, we have performed "simulations of simulations" that indicate that this small sample approach works quite well. For a variety of cases, we have performed 10,000 trial simulations in which we generated a sample data set from known normal distributions, estimated population parameters, drew 10,000 samples from the estimated populations, calculated estimates of the percentiles of the likelihood ratio statistic, and then used these to perform tests and obtain confidence intervals. Test sizes and confidence interval coverages were very near nominal for the small sample simulation approach. A subset of results from this simulation of simulations appears in Tables 3–5. For these three tables, $k = 2$ and $n = n_1 = n_2$. An expansion of these tables to include COVs equal to .01 and .40 appears in Verrill and Johnson (2007). The tables suggest that the small sample approach works well.

Table 4
Size of a test that k coefficients of variation are equal (based on Theorem A.4), $k = 2$, $n = n_1 = n_2$

			Asymptotic procedure		Simulation procedure	
COV	n	Nominal size	Estimate[b] of size	95% CI[b] on size	Estimate[b] of size	95% CI[b] on size
0.05	5	0.10	0.1639	[0.1567, 0.1712]	0.0981	[0.0923, 0.1040]
		0.05	0.0961	[0.0904, 0.1020]	0.0477	[0.0436, 0.0520]
		0.01	0.0291	[0.0259, 0.0325]	0.0112	[0.0092, 0.0134]
	10	0.10	0.1334	[0.1268, 0.1401]	0.1054	[0.0995, 0.1115]
		0.05	0.0742	[0.0691, 0.0794]	0.0520	[0.0477, 0.0564]
		0.01	0.0184	[0.0159, 0.0211]	0.0113	[0.0093, 0.0135]
	20	0.10	0.1163	[0.1101, 0.1227]	0.1009	[0.0951, 0.1069]
		0.05	0.0589	[0.0544, 0.0636]	0.0503	[0.0461, 0.0547]
		0.01	0.0148	[0.0125, 0.0173]	0.0119	[0.0099, 0.0141]
0.15	5	0.10	0.1678	[0.1605, 0.1752]	0.1026	[0.0967, 0.1086]
		0.05	0.1009	[0.0951, 0.1069]	0.0514	[0.0472, 0.0558]
		0.01	0.0290	[0.0258, 0.0324]	0.0094	[0.0076, 0.0114]
	10	0.10	0.1287	[0.1222, 0.1353]	0.0964	[0.0907, 0.1023]
		0.05	0.0675	[0.0627, 0.0725]	0.0498	[0.0456, 0.0541]
		0.01	0.0169	[0.0145, 0.0195]	0.0109	[0.0090, 0.0130]
	20	0.10	0.1128	[0.1067, 0.1191]	0.0996	[0.0938, 0.1055]
		0.05	0.0587	[0.0542, 0.0634]	0.0497	[0.0455, 0.0540]
		0.01	0.0146	[0.0123, 0.0170]	0.0101	[0.0082, 0.0122]
0.25	5	0.10	0.1643	[0.1571, 0.1716]	0.1010	[0.0952, 0.1070]
		0.05	0.0994	[0.0936, 0.1053]	0.0515	[0.0473, 0.0559]
		0.01	0.0298	[0.0266, 0.0332]	0.0103	[0.0084, 0.0124]
	10	0.10	0.1275	[0.1210, 0.1341]	0.0984	[0.0926, 0.1043]
		0.05	0.0701	[0.0652, 0.0752]	0.0497	[0.0455, 0.0540]
		0.01	0.0167	[0.0143, 0.0193]	0.0094	[0.0076, 0.0114]
	20	0.10	0.1101	[0.1040, 0.1163]	0.0966	[0.0909, 0.1025]
		0.05	0.0580	[0.0535, 0.0627]	0.0467	[0.0427, 0.0509]
		0.01	0.0121	[0.0101, 0.0143]	0.0090	[0.0072, 0.0109]

Our current Web programs report both asymptotic results and simulation results. As we have noted, simulation works well even for small samples. However, for larger samples, simulations can become quite time consuming. (For 10 samples of size 60, the hypothesis test program takes approximately 6.9 s to report. The confidence interval program takes about 7.3 s to report.) Hence the need for asymptotic results. We are currently engaged in performing a wide-ranging set of size/power studies. These studies suggest that for $n_j > 30$, the two approaches essentially coincide. After we have become convinced that this holds generally, we will modify the Web program so that simulations are not performed for larger sample sizes. Instead, for these larger sample sizes, only the asymptotic results will be reported. This will improve the program's performance.

Table 5

Confidence interval on a coefficient of variation shared by k normally distributed populations (based on Theorem A.6), $k = 2$, $n = n_1 = n_2$

			Asymptotic procedure		Simulation procedure	
COV	n	Nominal coverage	Estimate[b] of coverage	95% CI[b] on coverage	Estimate[b] of coverage	95% CI[b] on coverage
0.05	5	0.90	0.8134	[0.8057, 0.8210]	0.9026	[0.8967, 0.9083]
		0.95	0.8889	[0.8827, 0.8950]	0.9506	[0.9463, 0.9548]
		0.99	0.9653	[0.9616, 0.9688]	0.9912	[0.9893, 0.9929]
	10	0.90	0.8574	[0.8505, 0.8642]	0.8984	[0.8924, 0.9042]
		0.95	0.9197	[0.9143, 0.9249]	0.9496	[0.9452, 0.9538]
		0.99	0.9799	[0.9771, 0.9826]	0.9893	[0.9872, 0.9912]
	20	0.90	0.8789	[0.8724, 0.8852]	0.8973	[0.8913, 0.9032]
		0.95	0.9331	[0.9281, 0.9379]	0.9477	[0.9433, 0.9520]
		0.99	0.9848	[0.9823, 0.9871]	0.9887	[0.9865, 0.9907]
0.15	5	0.90	0.8146	[0.8069, 0.8222]	0.9012	[0.8953, 0.9070]
		0.95	0.8883	[0.8821, 0.8944]	0.9499	[0.9455, 0.9541]
		0.99	0.9650	[0.9613, 0.9685]	0.9900	[0.9880, 0.9919]
	10	0.90	0.8596	[0.8527, 0.8663]	0.9010	[0.8951, 0.9068]
		0.95	0.9238	[0.9185, 0.9289]	0.9517	[0.9474, 0.9558]
		0.99	0.9826	[0.9799, 0.9851]	0.9909	[0.9889, 0.9927]
	20	0.90	0.8853	[0.8790, 0.8915]	0.9035	[0.8976, 0.9092]
		0.95	0.9391	[0.9343, 0.9437]	0.9516	[0.9473, 0.9557]
		0.99	0.9875	[0.9852, 0.9896]	0.9908	[0.9888, 0.9926]
0.25	5	0.90	0.8163	[0.8086, 0.8238]	0.8985	[0.8925, 0.9043]
		0.95	0.8883	[0.8821, 0.8944]	0.9479	[0.9435, 0.9522]
		0.99	0.9658	[0.9621, 0.9693]	0.9904	[0.9884, 0.9922]
	10	0.90	0.8603	[0.8534, 0.8670]	0.8987	[0.8927, 0.9045]
		0.95	0.9231	[0.9178, 0.9282]	0.9512	[0.9469, 0.9553]
		0.99	0.9804	[0.9776, 0.9830]	0.9890	[0.9869, 0.9910]
	20	0.90	0.8888	[0.8826, 0.8949]	0.9070	[0.9012, 0.9126]
		0.95	0.9422	[0.9375, 0.9467]	0.9540	[0.9498, 0.9580]
		0.99	0.9877	[0.9854, 0.9898]	0.9911	[0.9892, 0.9928]

4. Summary

We have developed asymptotic theory that permits us to address three normal distribution coefficient of variation estimation or testing problems: obtain a confidence interval on the ratio of two coefficients of variation, perform a test of the hypothesis that the coefficients of variation associated with k populations are all equal, and obtain a confidence interval on a coefficient of variation that is shared by k populations. We have developed Web-based computer programs that implement these large sample techniques, and also provide simulation results that are valid for small samples. These programs can be accessed at the following web addresses: http://www1.fpl.fs.fed.us/covratio.html, http://www1.fpl.fs.fed.us/covtestk.html, and http://www1.fpl.fs.fed.us/covconfk.html.

Appendix A—The Theorems

A1. *Confidence Interval on the Ratio of Two Coefficients Of Variation*

We assume that we have n_1 observations, $x_{11}, \ldots, x_{n_1 1}$, from a $\mathrm{N}(\mu_1, \sigma_1^2)$ population, and n_2 observations, $x_{12}, \ldots, x_{n_2 2}$, from a $\mathrm{N}(\mu_2, \sigma_2^2)$ population, and that $\mu_1, \mu_2 > 0$. We assume that all of these observations are statistically independent. Let $n \equiv n_1 + n_2$. We further assume that $n_1/n \to \lambda_1 > 0$ and $n_2/n \to \lambda_2 > 0$ as $n \to \infty$. We denote the coefficient of variation of the first population by

$$c \equiv \sigma_1/\mu_1$$

so

$$\mu_1 = \sigma_1/c.$$

We denote the ratio of the coefficient of variation of the second population to the coefficient of variation of the first population by r. Thus

$$r \equiv (\sigma_2/\mu_2)/c$$

and

$$\mu_2 = \sigma_2/(rc).$$

Then, we have the following theorem.

Theorem A.1.

$$\sqrt{n}\left(\begin{pmatrix} \hat{\sigma}_1 \\ \hat{\sigma}_2 \\ \hat{c} \\ \hat{r} \end{pmatrix} - \begin{pmatrix} \sigma_1 \\ \sigma_2 \\ c \\ r \end{pmatrix}\right) \xrightarrow{D} \mathrm{N}(\mathbf{0}, I(\boldsymbol{\theta})^{-1})$$

where $\boldsymbol{\theta} \equiv (\sigma_1, \sigma_2, c, r)^T$,

$$\hat{\sigma}_j \equiv \sqrt{\sum_{i=1}^{n_j} (x_{ij} - \bar{x}_{\cdot j})^2 / n_j} \quad \textit{for } j = 1, 2$$

$$\bar{x}_{\cdot j} \equiv \sum_{i=1}^{n_j} x_{ij} / n_j \quad \textit{for } j = 1, 2$$

$$\hat{c} \equiv \hat{\sigma}_1 / \bar{x}_{\cdot 1}$$

$$\hat{r} \equiv (\hat{\sigma}_2 / \bar{x}_{\cdot 2}) / \hat{c}$$

and Fisher's information matrix is given by

$$I(\boldsymbol{\theta}) = \lambda_1 I_1(\boldsymbol{\theta}) + \lambda_2 I_2(\boldsymbol{\theta})$$

where

$$I_1(\boldsymbol{\theta}) = \begin{pmatrix} (2 + 1/c^2)/\sigma_1^2 & 0 & -1/(c^3\sigma_1) & 0 \\ 0 & 0 & 0 & 0 \\ -1/(c^3\sigma_1) & 0 & 1/c^4 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$I_2(\boldsymbol{\theta}) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & (2 + 1/(rc)^2)/\sigma_2^2 & -1/(r^2c^3\sigma_2) & -1/(r^3c^2\sigma_2) \\ 0 & -1/(r^2c^3\sigma_2) & 1/(r^2c^4) & 1/(r^3c^3) \\ 0 & -1/(r^3c^2\sigma_2) & 1/(r^3c^3) & 1/(r^4c^2) \end{pmatrix}.$$

Proof. The proof is a standard application of Theorem 6.1 of Ch. 6 of Lehmann (1983). In Appendix A of Verrill and Johnson (2007), we establish the conditions needed to invoke Lehmann's theorem. □

Based on Theorem A.1, an approximate $1 - \alpha$ confidence interval on r is given by

$$\hat{r} \pm z_{\alpha/2} \sqrt{\hat{d}_{44}/n}$$

where $z_{\alpha/2}$ is the appropriate critical value from a standard normal distribution, $\hat{d}_{44}$ is the 4th diagonal element of $I(\hat{\boldsymbol{\theta}})^{-1}$, and $I(\hat{\boldsymbol{\theta}})$ is $I(\boldsymbol{\theta})$ with σ_1, σ_2, c, and r replaced by $\hat{\sigma}_1$, $\hat{\sigma}_2$, $\hat{c}$, and $\hat{r}$.

A2. *Likelihood Ratio Based Confidence Interval on the Ratio of two Coefficients of Variation*

In Sec. A1 we obtained a confidence interval on the ratio of two coefficients of variation by establishing the asymptotic normality of the estimated parameter vector. In this section we take a likelihood ratio approach to this problem.

We make the same assumptions as those made in Sec. A1. Then, in the notation of Sec. 18.3 of Appendix H of Verrill and Johnson (2007), we have the following theorem.

Theorem A.2. *Provided that the ratio of coefficients of variation, r, equals* r_0,

$$2(\ln L(\hat{\boldsymbol{\theta}}_n) - \ln L(\mathbf{g}(\hat{\mathbf{v}}_n))) \xrightarrow{D} \chi_1^2$$

where $\boldsymbol{\theta} \equiv (\sigma_1, \sigma_2, c, r)^T$, $\boldsymbol{v} \equiv (\sigma_1, \sigma_2, c)^T$, *up to a constant*

$$\ln L(\boldsymbol{\theta}) = -n_1 \ln(\sigma_1) - \sum_{i=1}^{n_1}(x_{i1} - \sigma_1/c)^2/(2\sigma_1^2) - n_2 \ln(\sigma_2) - \sum_{i=1}^{n_2}(x_{i2} - \sigma_2/(rc))^2/(2\sigma_2^2)$$

$\hat{\boldsymbol{\theta}}_n = (\hat{\sigma}_1, \hat{\sigma}_2, \hat{c}, \hat{r})^T$ *is the solution of the unconstrained likelihood equations described in connection with Theorem* A.1,

$$\mathbf{g}(\boldsymbol{v}) = \begin{pmatrix} g_1(v_1, v_2, v_3) \\ g_2(v_1, v_2, v_3) \\ g_3(v_1, v_2, v_3) \\ g_4(v_1, v_2, v_3) \end{pmatrix} = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \\ r_0 \end{pmatrix} = \begin{pmatrix} \sigma_1 \\ \sigma_2 \\ c \\ r_0 \end{pmatrix}$$

and $\hat{\mathbf{v}}_n$ *is the solution to the likelihood equations obtained in Appendix B of the current article.*

Proof. Because two probability density functions are involved, Theorem A.2 is a slight extension of a standard likelihood ratio result. The details of the proof are provided in Appendix B of Verrill and Johnson (2007).

We have written a FORTRAN program that uses Theorem A.2 to perform a test of the hypothesis that $r = r_0$. The program uses this test to obtain a confidence interval for r—those r_0 that are not rejected at a α significance level constitute a $1 - \alpha$ confidence interval for r. A user of the program need only supply n_1, n_2, $\bar{x}_{.1}$, $\bar{x}_{.2}$, $\sqrt{\sum_{i=1}^{n_1}(x_{i1} - \bar{x}_{.1})^2/(n_1 - 1)}$, and $\sqrt{\sum_{i=1}^{n_2}(x_{i2} - \bar{x}_{.2})^2/(n_2 - 1)}$. The program can be run over the Web at http://www1.fpl.fs.fed.us/covratio.html.

A3. *Confidence Interval on a Coefficient of Variation That Is Shared by Two Normally Distributed Populations*

We assume that we have n_1 observations, $x_{11}, \ldots, x_{n_1 1}$, from a $\mathrm{N}(\mu_1, \sigma_1^2)$ population, and n_2 observations, $x_{12}, \ldots, x_{n_2 2}$, from a $\mathrm{N}(\mu_2, \sigma_2^2)$ population, and that $\mu_1, \mu_2 > 0$. We assume that all of these observations are statistically independent. Let $n \equiv n_1 + n_2$. We further assume that $n_1/n \to \lambda_1 > 0$ and $n_2/n \to \lambda_2 > 0$ as $n \to \infty$. We denote the shared coefficient of variation of the two populations by

$$c = \sigma_1/\mu_1 = \sigma_2/\mu_2$$

Then, we have the following theorem.

Theorem A.3.

$$\sqrt{n}\left(\begin{pmatrix}\hat{\sigma}_1\\ \hat{\sigma}_2\\ \hat{c}\end{pmatrix} - \begin{pmatrix}\sigma_1\\ \sigma_2\\ c\end{pmatrix}\right) \xrightarrow{D} \mathrm{N}(\mathbf{0}, I(\boldsymbol{\theta})^{-1})$$

where $\boldsymbol{\theta} \equiv (\sigma_1, \sigma_2, c)^T$, $\hat{\sigma}_1$, $\hat{\sigma}_2$, $\hat{c}$ are derived in Appendix B (set r_0 in Appendix B to 1) of the current article,

$$I(\boldsymbol{\theta}) = \lambda_1 I_1(\boldsymbol{\theta}) + \lambda_2 I_2(\boldsymbol{\theta})$$

and

$$I_1(\boldsymbol{\theta}) = \begin{pmatrix}(2+1/c^2)/\sigma_1^2 & 0 & -1/(c^3\sigma_1)\\ 0 & 0 & 0\\ -1/(c^3\sigma_1) & 0 & 1/c^4\end{pmatrix}$$

$$I_2(\boldsymbol{\theta}) = \begin{pmatrix}0 & 0 & 0\\ 0 & (2+1/c^2)/\sigma_2^2 & -1/(c^3\sigma_2)\\ 0 & -1/(c^3\sigma_2) & 1/c^4\end{pmatrix}.$$

Proof. The proof is a standard application of Theorem 6.1 of Ch. 6 of Lehmann (1983). In Appendix C of Verrill and Johnson (2007), we establish the conditions needed to invoke Lehmann's theorem. □

Based on Theorem A.3, an approximate $1-\alpha$ confidence interval on c is given by

$$\hat{c} \pm z_{\alpha/2}\sqrt{\hat{d}_{33}/n}$$

where $z_{\alpha/2}$ is the appropriate critical value from a standard normal distribution, $\hat{d}_{33}$ is the third diagonal element of $I(\hat{\boldsymbol{\theta}})^{-1}$, and $I(\hat{\boldsymbol{\theta}})$ is $I(\boldsymbol{\theta})$ with σ_1, σ_2, and c replaced by $\hat{\sigma}_1$, $\hat{\sigma}_2$, and $\hat{c}$. In Appendix I of Verrill and Johnson (2007), we establish that $d_{33} = c^4 + c^2/2$.

A4. *Likelihood Ratio Test of the Hypothesis That k Normally Distributed Populations Share the Same Coefficient of Variation*

We assume that we have n_1 observations, $x_{11}, \ldots, x_{n_1 1}$, from a $\mathrm{N}(\mu_1, \sigma_1^2)$ population, n_2 observations, $x_{12}, \ldots, x_{n_2 2}$, from a $\mathrm{N}(\mu_2, \sigma_2^2)$ population, ..., and n_k observations, $x_{1k}, \ldots, x_{n_k k}$, from a $\mathrm{N}(\mu_k, \sigma_k^2)$ population, and that $\mu_1, \ldots, \mu_k > 0$. We assume that all of these observations are statistically independent. Let $n \equiv n_1 + \cdots + n_k$. We further assume that $n_j/n \to \lambda_j > 0$ as $n \to \infty$ for $j = 1, \ldots, k$. We denote the shared coefficient of variation of the k populations by

$$c = \sigma_1/\mu_1 = \cdots = \sigma_k/\mu_k$$

Then, in the notation of Sec. 18.3 of Appendix H of Verrill and Johnson (2007), we have the following theorem.

Theorem A.4. *Provided that* $\sigma_1/\mu_1 = \cdots = \sigma_k/\mu_k$,

$$2(\ln L(\boldsymbol{\theta}_n) - \ln L(\mathbf{g}(\boldsymbol{v}_{n,\mathrm{Newt}}))) \xrightarrow{D} \chi^2_{k-1}$$

where $\boldsymbol{\theta} \equiv (\mu_1, \sigma_1, \ldots, \mu_k, \sigma_k)^T$, $\boldsymbol{v} \equiv (\sigma_1, \ldots, \sigma_k, c)^T$ *[note that in this section the parameter vector is given by* $\boldsymbol{\theta} \equiv (\mu_1, \sigma_1, \ldots, \mu_k, \sigma_k)^T$ *while in Sec. A3, A5, and A6* $\boldsymbol{\theta} \equiv (\sigma_1, \ldots, \sigma_k, c)^T$*], up to a constant*

$$\ln L(\boldsymbol{\theta}) = \sum_{j=1}^{k} \left(-n_j \ln(\sigma_j) - \sum_{i=1}^{n_j} (x_{ij} - \mu_j)^2/(2\sigma_j^2) \right).$$

$\hat{\boldsymbol{\theta}}_n$ *is the standard solution of the unconstrained likelihood equations, that is,*

$$\hat{\boldsymbol{\theta}}_n = \begin{pmatrix} \bar{x}_{\cdot 1} \\ s_1 \\ \vdots \\ \bar{x}_{\cdot k} \\ s_k \end{pmatrix}$$

where

$$s_j = \sqrt{\sum_{i=1}^{n_j} (x_{ij} - \bar{x}_{\cdot j})^2/n_j}$$

$$\mathbf{g}(\boldsymbol{v}) = \begin{pmatrix} v_1/v_{k+1} \\ v_1 \\ \vdots \\ v_k/v_{k+1} \\ v_k \end{pmatrix} = \begin{pmatrix} \sigma_1/c \\ \sigma_1 \\ \vdots \\ \sigma_k/c \\ \sigma_k \end{pmatrix}.$$

$\boldsymbol{v}_{n,\mathrm{Newt}}$ *is the Newton estimator of* $(\sigma_1, \ldots, \sigma_k, c)^T$ *given by*

$$\boldsymbol{v}_{n,\mathrm{Newt}} = -\left[\frac{\partial^2 \ln L}{\partial v_l v_m}\right]^{-1} \Big|_{\boldsymbol{v}_{n,c}} \begin{pmatrix} \partial \ln L/\partial v_1 \\ \vdots \\ \partial \ln L/\partial v_{k+1} \end{pmatrix} \Big|_{\boldsymbol{v}_{n,c}} + \boldsymbol{v}_{n,c} \tag{1}$$

where $\boldsymbol{v}_{n,c}$ *is any* $\sqrt{n}$*-consistent estimator of* $(\sigma_1, \ldots, \sigma_k, c)^T$ *[such as* $(s_1, \ldots, s_k, \hat{c})^T$ *where* $1/\hat{c} = \left(\sum_{j=1}^{k} n_j \bar{x}_{\cdot j}/s_j\right)/(n_1 + \cdots + n_k)$*]. The partial derivatives in Eq.* (1) *are listed in Appendix C of Verrill and Johnson* (2007)*, and a simple technique for solving the equation is provided in Appendix I of Verrill and Johnson* (2007).

Proof. Because k probability density functions are involved, and because we are dealing with a Newton one-step estimator, Theorem A.4 is a slight extension of a standard likelihood ratio result. The details of the proof are provided in Appendix E of Verrill and Johnson (2007). □

We have written a FORTRAN program that uses Theorem A.4 to perform a test of the hypothesis that $\sigma_1/\mu_1 = \cdots = \sigma_k/\mu_k$. The user need only supply $n_1, \ldots, n_k$, $\bar{x}_{\cdot 1}, \ldots, \bar{x}_{\cdot k}$, $\sqrt{\sum_{i=1}^{n_1}(x_{i1} - \bar{x}_{\cdot 1})^2/(n_1 - 1)}, \ldots, \sqrt{\sum_{i=1}^{n_k}(x_{ik} - \bar{x}_{\cdot k})^2/(n_k - 1)}$. The program can be run over the Web at http://www1.fpl.fs.fed.us/covtestk.html.

We note that in this program, we start with the $\sqrt{n}$-consistent estimator given by $(s_1, \ldots, s_k, \hat{c})$, perform a limited number of backtracking Newton steps (which still leaves us with a $\sqrt{n}$-consistent estimator), and then do a final full Newton step.

A5. *Confidence Interval on a Coefficient of Variation That Is Shared by k Normally Distributed Populations*

We make the same assumptions as those made in Sec. A4. However, here the parameter vector is given by $\boldsymbol{\theta} \equiv (\sigma_1, \ldots, \sigma_k, c)^T$ (as opposed to the vector $(\mu_1, \sigma_1, \ldots, \mu_k, \sigma_k)^T$ of Sec. A4).

Then, we have the following theorem.

Theorem A.5.

$$\sqrt{n}(\boldsymbol{\theta}_{n,\text{Newt}} - \boldsymbol{\theta}) \xrightarrow{D} \text{N}(\mathbf{0}, I(\boldsymbol{\theta})^{-1})$$

where $\boldsymbol{\theta}_{n,\text{Newt}}$ is the Newton estimator of $(\sigma_1, \ldots, \sigma_k, c)^T$ given by

$$\boldsymbol{\theta}_{n,\text{Newt}} \equiv -\left[\frac{\partial^2 \ln L}{\partial\theta_l\theta_m}\right]^{-1}\Big|_{\boldsymbol{\theta}_{n,c}} \begin{pmatrix} \partial \ln L/\partial\theta_1 \\ \vdots \\ \partial \ln L/\partial\theta_{k+1} \end{pmatrix}\Big|_{\boldsymbol{\theta}_{n,c}} + \boldsymbol{\theta}_{n,c}, \tag{2}$$

$\boldsymbol{\theta}_{n,c}$ is any $\sqrt{n}$-consistent estimator of $(\sigma_1, \ldots, \sigma_k, c)^T$ [such as $(s_1, \ldots, s_k, \hat{c})^T$ where $1/\hat{c} = \left(\sum_{j=1}^{k} n_j\bar{x}_{\cdot j}/s_j\right)/(n_1 + \cdots + n_k)$], and

$$I(\boldsymbol{\theta}) = \sum_{j=1}^{k} \lambda_j I_j(\boldsymbol{\theta})$$

where the j, jth element of $I_j(\boldsymbol{\theta})$ is $(2 + 1/c^2)/\sigma_j^2$, the $j, k+1$ th and $k+1, j$ th elements of $I_j(\boldsymbol{\theta})$ are $-1/(c^3\sigma_j)$, the $k+1, k+1$th element is $1/c^4$, and the remaining elements are 0. The partial derivatives in Eq. (2) are listed in Appendix C of Verrill and Johnson (2007), and a simple technique for solving the equation is provided in Appendix I of Verrill and Johnson (2007).

Proof. Because k probability density functions are involved, and because we are dealing with a Newton one-step estimator, Theorem A.5 is a slight extension of a standard efficient likelihood estimator result. The details of the proof are provided in Appendix F of Verrill and Johnson (2007). □

Based on Theorem A.5, an approximate $1 - \alpha$ confidence interval on c is given by

$$\hat{c} \pm z_{\alpha/2}\sqrt{\hat{d}_{k+1,k+1}/n}$$

where $\hat{c}$ is the $k+1$th element of $\boldsymbol{\theta}_{n,\text{Newt}}$, $z_{\alpha/2}$ is the appropriate critical value from a standard normal distribution, $\hat{d}_{k+1,k+1}$ is the $k+1$th diagonal element of $I(\hat{\boldsymbol{\theta}})^{-1}$, and $I(\hat{\boldsymbol{\theta}})$ is $I(\boldsymbol{\theta})$ with $(\sigma_1, \dots, \sigma_k, c)^T$ replaced by $\boldsymbol{\theta}_{n,\text{Newt}}$. In Appendix I of Verrill and Johnson (2007), we establish that $d_{k+1,k+1} = c^4 + c^2/2$.

A6. *Likelihood Ratio Based Confidence Interval on a Coefficient of Variation That Is Shared by k Normally Distributed Populations*

In Sec. A5, we obtained a confidence interval on the coefficient of variation by establishing the asymptotic normality of the Newton one-step estimator of the parameter vector. In this section we take a likelihood ratio approach to this problem.

We make the same assumptions as those made in Sec. A5. Then, in the notation of Sec. 18.3 of Appendix H of Verrill and Johnson (2007), we have the following theorem.

Theorem A.6. *Provided that* $c = c_0$,

$$2(\ln L(\boldsymbol{\theta}_{n,\text{Newt}}) - \ln L(\mathbf{g}(\hat{\mathbf{v}}_n))) \xrightarrow{D} \chi_1^2$$

where $\boldsymbol{\theta} \equiv (\sigma_1, \dots, \sigma_k, c)^T$, $\mathbf{v} \equiv (\sigma_1, \dots, \sigma_k)^T$, *up to a constant*

$$\ln L(\boldsymbol{\theta}) = \sum_{j=1}^{k} \left(-n_j \ln(\sigma_j) - \sum_{i=1}^{n_j} (x_{ij} - \sigma_j/c)^2/(2\sigma_j^2) \right).$$

$\boldsymbol{\theta}_{n,\text{Newt}}$ *is the Newton estimator of* $(\sigma_1, \dots, \sigma_k, c)^T$ *given by*

$$\boldsymbol{\theta}_{n,\text{Newt}} \equiv -\left[\frac{\partial^2 \ln L}{\partial\theta_l \theta_m}\right]^{-1} \Big|_{\boldsymbol{\theta}_{n,c}} \begin{pmatrix} \partial \ln L/\partial\theta_1 \\ \vdots \\ \partial \ln L/\partial\theta_{k+1} \end{pmatrix} \Big|_{\boldsymbol{\theta}_{n,c}} + \boldsymbol{\theta}_{n,c} \tag{3}$$

where $\boldsymbol{\theta}_{n,c}$ *is any* $\sqrt{n}$*-consistent estimator of* $(\sigma_1, \dots, \sigma_k, c)^T$ [*such as* $(s_1, \dots, s_k, \hat{c})^T$ *where* $1/\hat{c} = \left(\sum_{j=1}^{k} n_j \bar{x}_{\cdot j}/s_j\right)/(n_1 + \cdots + n_k)$], *and* $\hat{\mathbf{v}}_n$ *is the solution of the constrained likelihood equations obtained in Appendix G of Verrill and Johnson* (2007). (*The constraint is* $c = c_0$.)

The partial derivatives in Eq. (3) are listed in Appendix C of Verrill and Johnson (2007), and a simple technique for solving the equation is provided in Appendix I of Verrill and Johnson (2007).

Proof. Because k probability density functions are involved, and because we are dealing with a Newton one-step estimator, Theorem A.6 is a slight extension of a standard likelihood ratio result. The details of the proof are provided in Appendix G of Verrill and Johnson (2007). □

We have written a FORTRAN program that uses Theorem A.6 to perform a test of the hypothesis that $c = c_0$. The program uses this test to obtain a confidence interval for c — those c_0 that are not rejected at a α significance level constitute a $1-\alpha$ confidence interval for c. A user of the program need only supply $n_1,\ldots,n_k$, $\bar{x}_{\cdot 1},\ldots,\bar{x}_{\cdot k}$, $\sqrt{\sum_{i=1}^{n_1}(x_{i1}-\bar{x}_{\cdot 1})^2/(n_1-1)},\ldots,\sqrt{\sum_{i=1}^{n_k}(x_{ik}-\bar{x}_{\cdot k})^2/(n_k-1)}$. The program can be run over the Web at http://www1.fpl.fs.fed.us/covconfk.html.

We note that in this program, we start with the $\sqrt{n}$-consistent estimator given by $(s_1,\ldots,s_k,\hat{c})$, perform a limited number of backtracking Newton steps (which still leaves us with a $\sqrt{n}$-consistent estimator), and then do a final full Newton step.

Appendix B—The Solution to the Likelihood Equations in the Two Population Case

Recall that $c \equiv \sigma_1/\mu_1$ and $r_0 \equiv (\sigma_2/\mu_2)/c$.

Up to a constant, the log likelihood in this case is

$$\ln L = -n_1\ln(\sigma_1) - \sum_{i=1}^{n_1}(x_{i1}-\sigma_1/c)^2/\left(2\sigma_1^2\right) - n_2\ln(\sigma_2) - \sum_{i=1}^{n_2}(x_{i2}-\sigma_2/(r_0c))^2/\left(2\sigma_2^2\right).$$

We have

$$\frac{\partial \ln L}{\partial c} = -\sum_{i=1}^{n_1}(x_{i1}-\sigma_1/c)/(c^2\sigma_1) - \sum_{i=1}^{n_2}(x_{i2}-\sigma_2/(r_0c))/\left(r_0c^2\sigma_2\right).$$

Setting $\frac{\partial \ln L}{\partial c}=0$, we obtain

$$\sum_{i=1}^{n_1}(x_{i1}-\sigma_1/c)/\sigma_1 + \sum_{i=1}^{n_2}(x_{i2}-\sigma_2/(r_0c))/(r_0\sigma_2) = 0$$

or

$$\left(n_1\bar{x}_{\cdot 1}/\sigma_1 + n_2\bar{x}_{\cdot 2}/(r_0\sigma_2)\right)/\left(n_1+n_2/r_0^2\right) = 1/c. \tag{4}$$

Next,

$$\frac{\partial \ln L}{\partial \sigma_1} = -n_1/\sigma_1 + \sum_{i=1}^{n_1}(x_{i1}-\sigma_1/c)/(c\sigma_1^2) + \sum_{i=1}^{n_1}(x_{i1}-\sigma_1/c)^2/\sigma_1^3.$$

Setting $\frac{\partial \ln L}{\partial \sigma_1}=0$, we obtain

$$\begin{aligned}
\sigma_1^2 &= \sum_{i=1}^{n_1}(x_{i1}-\sigma_1/c)(\sigma_1/c)/n_1 + \sum_{i=1}^{n_1}(x_{i1}-\sigma_1/c)^2/n_1 \\
&= (\bar{x}_{\cdot 1}-\sigma_1/c)(\sigma_1/c) + \sum_{i=1}^{n_1}{x_{i1}}^2/n_1 - 2\bar{x}_{\cdot 1}\sigma_1/c + (\sigma_1/c)^2 \\
&= -\bar{x}_{\cdot 1}\sigma_1/c + \sum_{i=1}^{n_1}{x_{i1}}^2/n_1
\end{aligned}$$

so

$$\sum_{i=1}^{n_1}x_{i1}^2/n_1 - \sigma_1^2 = \bar{x}_{\cdot 1}\sigma_1/c$$

or (assuming that $\bar{x}_{\cdot 1} \neq 0$)

$$\left(\sum_{i=1}^{n_1} x_{i1}^2/n_1 - \sigma_1^2\right)/(\bar{x}_{\cdot 1}\sigma_1) = 1/c. \tag{5}$$

Similarly, setting $\frac{\partial \ln L}{\partial \sigma_2} = 0$, we obtain (assuming that $\bar{x}_{\cdot 2} \neq 0$)

$$\left(\sum_{i=1}^{n_2} x_{i2}^2/n_2 - \sigma_2^2\right)/(\bar{x}_{\cdot 2}\sigma_2/r_0) = 1/c. \tag{6}$$

From Eqs. (4)–(6), we have

$$\left(\sum_{i=1}^{n_1} x_{i1}^2/n_1 - \sigma_1^2\right)/(\bar{x}_{\cdot 1}\sigma_1) = (n_1\bar{x}_{\cdot 1}/\sigma_1 + n_2\bar{x}_{\cdot 2}/(r_0\sigma_2))/\big(n_1 + n_2/r_0^2\big) \tag{7}$$

and

$$\left(\sum_{i=1}^{n_2} x_{i2}^2/n_2 - \sigma_2^2\right)/(\bar{x}_{\cdot 2}\sigma_2/r_0) = (n_1\bar{x}_{\cdot 1}/\sigma_1 + n_2\bar{x}_{\cdot 2}/(r_0\sigma_2))/\big(n_1 + n_2/r_0^2\big). \tag{8}$$

Now define

$$y \equiv \sigma_1/\sigma_2$$

so

$$\sigma_2 = \sigma_1/y. \tag{9}$$

Equation (7) becomes

$$\sum_{i=1}^{n_1} x_{i1}^2/n_1 - \sigma_1^2 = \big(n_1\bar{x}_{\cdot 1}^2 + n_2\bar{x}_{\cdot 1}\bar{x}_{\cdot 2}y/r_0\big)/\big(n_1 + n_2/r_0^2\big)$$

or

$$\sum_{i=1}^{n_1} x_{i1}^2/n_1 - n_1\bar{x}_{\cdot 1}^2/\big(n_1 + n_2/r_0^2\big) - (n_2\bar{x}_{\cdot 1}\bar{x}_{\cdot 2}y/r_0)/\big(n_1 + n_2/r_0^2\big) = \sigma_1^2. \tag{10}$$

Equation (8) becomes

$$\sum_{i=1}^{n_2} x_{i2}^2/n_2 - \sigma_1^2/y^2 = \big(n_1\bar{x}_{\cdot 1}\bar{x}_{\cdot 2}/(r_0 y) + n_2\bar{x}_{\cdot 2}^2/r_0^2\big)/\big(n_1 + n_2/r_0^2\big)$$

or

$$y^2\left(\sum_{i=1}^{n_2} x_{i2}^2/n_2 - \big(n_2\bar{x}_{\cdot 2}^2/r_0^2\big)/\big(n_1 + n_2/r_0^2\big)\right) - y(n_1\bar{x}_{\cdot 1}\bar{x}_{\cdot 2}/r_0)/\big(n_1 + n_2/r_0^2\big) = \sigma_1^2. \tag{11}$$

From Eqs. (10) and (11) we have

$$ay^2 + by + d = 0$$

(we use d here rather than the standard c because we have already used c to denote the coefficient of variation) where

$$a \equiv \sum_{i=1}^{n_2} x_{i2}^2/n_2 - n_2\left(\bar{x}_{\cdot 2}^2/r_0^2\right)/\left(n_1 + n_2/r_0^2\right) \tag{12}$$

$$b \equiv \left((n_2 - n_1)\bar{x}_{\cdot 1}\bar{x}_{\cdot 2}/r_0\right)/\left(n_1 + n_2/r_0^2\right) \tag{13}$$

$$d \equiv -\left(\sum_{i=1}^{n_1} x_{i1}^2/n_1 - n_1\bar{x}_{\cdot 1}^2/\left(n_1 + n_2/r_0^2\right)\right). \tag{14}$$

The possible solutions for $y = \sigma_1/\sigma_2$ are, of course, $\left(-b \pm \sqrt{b^2 - 4ad}\right)/(2a)$.

Now note that

$$a = \sum_{i=1}^{n_2} x_{i2}^2/n_2 - n_2\left(\bar{x}_{\cdot 2}^2/r_0^2\right)/\left(n_1 + n_2/r_0^2\right) \geq \sum_{i=1}^{n_2} x_{i2}^2/n_2 - \bar{x}_{\cdot 2}^2 \geq 0$$

with $a = 0$ only if all of the x_{i2}'s are zero. Similarly, $d \leq 0$ with equality only if all the x_{i1}'s equal zero. Thus, unless all the x_{i1}'s equal 0 or all the x_{i2}'s equal 0,

$$-4ad > 0 \quad \text{and} \quad \sqrt{b^2 - 4ad} > |b|.$$

So if $b > 0$,

$$\left(-b - \sqrt{b^2 - 4ad}\right)/(2a) < 0$$

and

$$\left(-b + \sqrt{b^2 - 4ad}\right)/(2a) > (-|b| + |b|)/(2a) = 0.$$

If $b < 0$,

$$\left(-b - \sqrt{b^2 - 4ad}\right)/(2a) < (|b| - |b|)/(2a) = 0$$

and

$$\left(-b + \sqrt{b^2 - 4ad}\right)/(2a) > 0.$$

Thus, $\left(-b + \sqrt{b^2 - 4ad}\right)/(2a)$ is the unique solution for $y = \sigma_1/\sigma_2$. The estimates of σ_1^2 and σ_2^2 can then be obtained from Eqs. (10) and (9). (See Appendix D of Verrill and Johnson, 2007 for a proof that the estimate of σ_1^2 obtained from Eq. (10) is positive.)

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