Composite Estimation for Forest Inventories Using Multiple Value Groups and Strata

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ABSTRACT. *Due to economic and political pressures placed on timber production, information from timber inventories needs to be broken down into information on particular value groups, which are groupings based on individual tree characteristics. Thus estimates of totals and variances need to be computed for individual value groups, all value groups combined, and any collection of value groups. A method for reporting these estimates using a variance-covariance matrix is given. This is followed by two examples of the estimation of totals and variances for value groups applied to a population comprising two strata. Poisson (3P) and sample-tree sampling are used in the first and second strata, respectively. These examples are included to highlight important estimation considerations. A worked example based on an artificial population is also given. The population for the worked example is divided into two strata, with three value groups in each stratum. 3P and sample-tree sampling are used in the first and second strata. South. J. Appl. For. 20(2):103–109.*

For many timber inventories, materials being sold are broken down into value groups, which are groupings based on quality factors such as species, tree size, tree grade, or some combination of these factors. For timber users, some value groups are much more important than others. For example, a mixed hardwood–softwood stand could contain three value groups that consist of high value hardwoods, softwood sawtimber, and pulpwood. A bidder who has little use for pulpwood would be interested in an estimate of the total volume for the other two value groups. Some measure of the precision of the value group estimates is also important. This measure of precision lets the user know whether the true total volume is likely to differ substantially from the estimate of total volume in the value group of interest. In the previous example, the same bidder may want to lower his or her bid if the estimate for the hardwood value group is likely to be inaccurate.

To improve the efficiency of value group estimates and report more accurate information about available timber, the USDA Forest Service is implementing the following changes to timber inventories.

1. Increase stratification to improve the precision of estimates.
2. In each stratum, use sampling methods that will give precise estimates of total volume for the minimum cost and make the best use of available sampling resources.
3. Report a degree of precision on individual value groups and any combination of value groups.

To achieve these three goals, estimates of total volume (indicated by $\hat{Y}$) and a degree of precision [indicated by var $(\hat{Y})$] need to be calculated in each stratum for individual value groups, all value groups combined, and any combination of value groups. These estimates then must be combined across all strata. The method given in this paper allows $\hat{Y}$ and var$(\hat{Y})$ to be calculated for all possible value group combinations with the fewest number of calculations.

Statistical Background

Let M = the number of value groups and L = the number of strata. An obstacle in reporting variances for any collection of value groups is that the number of possible combinations of value groups, which is

$$\sum_{i=1}^{M}\binom{M}{i}$$

becomes very large even when the number of value groups is relatively small. For example, if a population contains 7 value groups, 127 different variance estimates would be needed. Rather than reporting variances for all possible combinations of value groups, we propose the following. Let $C = \{j: j \in 1, \ldots M\}$ be any set of value group indices. Then the variance for any collection of value groups is given by

$$\operatorname{var}\left(\sum_{j\in C}\hat{Y}_{.j}\right)=\sum_{j\in C}\operatorname{var}\left(\hat{Y}_{.j}\right)+\sum\sum_{j'\neq j}\operatorname{cov}\left(\hat{Y}_{.j'},\hat{Y}_{.j}\right) \quad (1)$$

(Mood et al. 1974, p. 178)

where

$$\hat{Y}_{.j}=\sum_{l=1}^{L}\hat{Y}_{lj}$$

is the sum for value group j across all L strata. Thus the simplest method for reporting variances for any collection of value groups is to report a variance-covariance matrix and to let the user compute variance estimates only for the value groups of interest using (1). For example, if $M = 3$ we report

$$\begin{pmatrix}\operatorname{var}(\hat{Y}_{.1}) & \operatorname{cov}(\hat{Y}_{.1},\hat{Y}_{.2}) & \operatorname{cov}(\hat{Y}_{.1},\hat{Y}_{.3})\\ \operatorname{cov}(\hat{Y}_{.2},\hat{Y}_{.1}) & \operatorname{var}(\hat{Y}_{.2}) & \operatorname{cov}(\hat{Y}_{.2},\hat{Y}_{.3})\\ \operatorname{cov}(\hat{Y}_{.3},\hat{Y}_{.1}) & \operatorname{cov}(\hat{Y}_{.3},\hat{Y}_{.2}) & \operatorname{var}(\hat{Y}_{.3})\end{pmatrix} \quad (2)$$

The disadvantage of using (1) is that covariances may not be readily available for some sampling designs (point-3P sampling for example). However, these covariances can be obtained using a number of methods, the simplest being

$$\operatorname{cov}(\hat{Y}_{.j'},\hat{Y}_{.j})=\frac{\operatorname{var}(\hat{Y}_{.(j\&j')})-\operatorname{var}(\hat{Y}_{.j'})-\operatorname{var}(\hat{Y}_{.j})}{2} \quad (3)$$

where $\operatorname{var}(\hat{Y}_{.(j\&j')})=\operatorname{var}(\hat{Y}_{.j'}+\hat{Y}_{.j})$. Computation of the variance for a single value group, i.e. $\operatorname{var}(\hat{Y}_{.j})$, is generally a straightforward process as discussed later. The only additional information required to calculate the covariances in (3) above is $\operatorname{var}(\hat{Y}_{.(j\&j')})$. This requires at most $L\binom{M}{2}$ additional computations. In practice, the actual number of computations should be smaller since $\operatorname{var}(\hat{Y}_{l(j\&j')}) = \operatorname{var}\left(\hat{Y}_{lj}\right)+\operatorname{var}\left(\hat{Y}_{lj'}\right)$ whenever the samples used to calculate $\hat{Y}_{lj}$ and $\hat{Y}_{lj'}$, are independent in stratum l.

Alternative options for generating covariances are bootstrapping and jackknifing as well as direct calculation. Certain practical considerations make these alternatives less attractive. Some of the items that were considered before choosing the method were:

1. The data structures used in the current NFS national cruise computer programs do not allow bootstrapping or jackknifing. Incorporating these would require a major revision of the programs.
2. Covariance estimators are rarely given in the literature; so, before implementation, they would have to be derived for all of the current sampling designs as well as any new designs.

An Example

Required notation:

- L = the number of strata.
- g = the number of sample groups for the sample-tree method.
- M = the number of value groups.
- i = tree subscript.
- j, j' = value group subscript, $j, j' = 1 \ldots M$.
- k = sample group subscript, $k = 1 \ldots g$.
- l = strata subscript, $l = 1 \ldots L$.
- X_l = total of the sampling covariate in stratum l.
- $\hat{Y}_{l.}$ = estimated total of the variable of interest in stratum l
- $\hat{Y}_{.j}$ = estimated total of the variable of interest for value group j summed across all strata.
- $\hat{Y}_{lj}$ = estimated total of the variable of interest for value group j in stratum l.
- y_{li} = variable of interest for tree i in stratum l.
- x_{li} = covariate for tree i in stratum l.
- π_{li} = the probability of selection for tree i in stratum l
- f_k = sampling frequency for sample group k in sample-tree sampling.
- $n(e)_l$ = expected sample size in stratum l.
- n_l = sample size in stratum l.
- n_{lj} = actual sample size in stratum l for value group j
- $n_{l(k)}$ = actual sample size in stratum l in sample group k
- N_l = number of units in stratum l.

We use two strata ($L = 2$) and an unspecified number of value groups. In the first stratum 3P sampling is used (Schreuder et al. 1993) and in the second stratum a systematic sampling method, where every fth tree is selected, is used. This method is referred to as *sample-tree sampling* (USDA Forest Service Timber Cruising Handbook 1993). While this method is not commonly used in all regions of the United States, it is ideal for illustrating sample independence consideration, which will be discussed later.

The desired output for the example is the variance-covariance matrix given in (2). In the following subsections, the theory for estimating the variance for value group j ($\text{var}(\hat{Y}_{lj})$) and for any two value groups j and j' ($\text{var}(\hat{Y}_{l(j\&j')})$) in each stratum l is given (additional information on the estimation of these variances can be found in Cochran 1977, p. 35–38, 140–144). We also discuss the independence of samples in the two sampling methods and contrast them.

Estimation for 3P Sampling

Poisson sampling was used in the first stratum ($l = 1$). An efficient estimator of the population total in the first stratum is the adjusted Poisson sampling estimator:

$$\hat{Y}_{1.} = \frac{n(e)_1}{n_1} \sum_{i=1}^{n_1} \frac{y_{1i}}{\pi_{1i}} \tag{4}$$

where the probability of selection is $\pi_{1i} = n(e)_1 x_{1i}/X_1$.[1] To estimate value group totals, a new indicator variable needs to be defined. Let

$$y'_{1i} = \begin{pmatrix} y_{1i}, \text{if tree } i \text{ belongs to value group } j, \\ 0, \text{ otherwise} \end{pmatrix} \tag{5}$$

The estimate of the total for value group j is

$$\hat{Y}_{1j} = \frac{n(e)_1}{n_1} \sum_{i=1}^{n_1} \frac{y'_{1i}}{\pi_{1i}} \tag{6}$$

An approximate estimator of the variance for the stratum total $\hat{Y}_{1.}$ is

$$\text{var}(\hat{Y}_{1.}) = \sum_{i=1}^{n_1} \left(\frac{n(e)_1 y_{1i}}{\pi_{1i}} - \hat{Y}_{1.} \right)^2 / \left(n_1(n_1 - 1)\right) \tag{7}$$

(Grosenbaugh 1974).

Substituting y'_{1i} for y_{1i} and $\hat{Y}_{1j}$ for $\hat{Y}_{1.}$ in (7) gives the estimated variance for value group j, i.e., $\text{var}(\hat{Y}_{1j})$.

To estimate the population total for any two value groups j and j', define the new indicator variable

$$y''_{1i} = \begin{pmatrix} y_{1i}, \text{ if tree } i \text{ belongs to value group } j \text{ or } j' \\ 0, \text{ otherwise} \end{pmatrix}$$

[1] $\frac{1}{\pi_{li}}$ is the same as the expansion factor.

Then the estimator for the population total for value groups j and j' is given by

$$Y_{1(j\&j')} = \frac{n(e)_1}{n_1} \sum_{i=1}^{n_1} \frac{y''_{1i}}{\pi_{1i}} \tag{8}$$

Variance estimators for the sum of the estimated totals for any two value groups j and j' are given by

$$\text{var}(\hat{Y}_{1(j\&j')}) = \sum_{i=1}^{n_1} \left(\frac{n(e)_1 y''_{1i}}{\pi_{1i}} \right) - (\hat{Y}_{1(j\&j')})^2 / \left(n_1(n_1 - 1)\right) \tag{9}$$

Note that when 3P sampling is used, $\text{var}(\hat{Y}_{1(j\&j')}) \neq \text{var}(\hat{Y}_{1j}) + \text{var}(\hat{Y}_{1j'})$ because the value group estimators are not independent. The correlation between $\hat{Y}_{1j}$ and $\hat{Y}_{1j'}$ will be negative because an increase in one value group estimate has a tendency to decrease the estimate in the other value group.

Estimation for Sample-Tree Sampling

Sample-tree sampling is used in the second stratum. For this systematic sampling method, individual value groups or collections of value groups are assigned to one of g groups, referred to as sample groups, and the trees in that sample group are assigned a sampling frequency f_k, $k = 1, \ldots g$. Thus if tree i belongs to sample group k, the probability of selection is given by $\pi_{2i} = 1/f_k$. The estimator of the stratum total is given by

$$\hat{Y}_{2.} = \sum_{i=1}^{n_2} \frac{y_{2i}}{\pi_{2i}} \tag{10}$$

Estimators for the total of value group j and value groups j and j' combined are

$$\hat{Y}_{2j} = \sum_{i=1}^{n_2} \frac{y'_{2i}}{\pi_{2i}} \tag{11}$$

and

$$\hat{Y}_{2(j\&j')} = \sum_{i=1}^{n_2} \frac{y''_{2i}}{\pi_{2i}} \tag{12}$$

where

$$y'_{2i} = \begin{pmatrix} y_{2i}, \text{ if tree } i \text{ belongs to value group } j \\ 0, \text{ otherwise} \end{pmatrix} \tag{13}$$

and

$$y''_{2i} = \begin{pmatrix} y_{2i}, \text{ if tree } i \text{ belongs to value group } j \text{ or } j' \\ 0, \text{ otherwise} \end{pmatrix}$$

The variance estimator of the total for all value groups combined in the stratum is given by

$$\text{var}(\hat{Y}_{2.}) = \left(1 - \frac{n_2}{N_2}\right) \sum_{i=1}^{n_2} \left(\frac{n(e)_2 y_{2i}}{\pi_{2i}} - \hat{Y}_{2.} \right)^2 / (n_2(n_2 - 1)) \quad (15)$$

Variance estimators for value group totals are more complicated due to the independence between sample groups and dependence within sample groups. For any sample group that contains only one value group, j, the outcome of the sample is independent from the outcome of the samples collected in the other sample groups. Thus, if the sample group only contains one value group, the sample can be viewed as an independent sample of size $n_{2j} = n_{2(k)}$. In this situation the sample group and value group sample sizes are equal. Hence the variance estimator when value group j is the only value group in a sample group is given by

$$\text{var}(\hat{Y}_{2j}) = \left(1 - \frac{n_2}{N_2}\right) \sum_{i=1}^{n_{2j}} \left(\frac{n_{2j} y'_{2i}}{\pi_{i1}} - \hat{Y}_{2j.} \right)^2 \quad (16)$$

If sample group k contains more than one value group, the variance estimator for value group j is given by

$$\text{var}(\hat{Y}_{2j}) = \left(1 - \frac{n_{2(k)}}{N_2}\right) \frac{\sum_{i=1}^{n_{2(k)}} y^{*2}_{2i} - \left(\sum_{i=1}^{n_{2j}} y^*_{2i}\right)^2}{\left(n_{2(k)}(n_{2(k)} - 1)\right)} \quad (17)$$

where

$$y^*_{2i} = \begin{pmatrix} y_{2i}, \text{ if tree } i \text{ belongs to value group } k \\ 0, \text{ otherwise} \end{pmatrix} \quad (18)$$

Two cases need to be considered when estimating the totals for any two value groups. If value groups j and j' are in different sample groups, then $\hat{Y}_{2j}$ and $\hat{Y}_{2j'}$ are independent random variables and

$$\text{var}(\hat{Y}_{2(j\&j')}) = \text{var}(\hat{Y}_{2j}) + \text{var}(\hat{Y}_{2j'}) \quad [19]$$

Variances estimators for the estimated totals of any two value groups j and j' that are both in the same sample group are given by

$$\text{var}(\hat{Y}_{2(j\&j')}) = \left(1 - \frac{n_{2(k)}}{N_2}\right) \frac{\sum_{i=1}^{n_{2(k)}} y''^{2}_{2i} - \left(\sum_{i=1}^{n_{2(k)}} y''_{2i}\right)^2}{\left(n_{2(k)}(n_{2(k)} - 1)\right)} \quad (20)$$

because $\hat{Y}_{2j}$ and $\hat{Y}_{2j'}$ are not independent.

Combining Strata Estimates

Estimates in each of the L strata are combined to form the variance-covariance matrix. Since strata are sampled independently, the diagonal elements of the variance-covariance matrix are computed by summing across the strata, i.e.

$$\text{var}(\hat{Y}_{.j}) = \sum_{l=1}^{L} \text{var}(\hat{Y}_{lj}), j = 1, \ldots M \quad (21)$$

To calculate the off-diagonal elements of the variance-covariance matrix, Equation (3) is used with the variance for the estimated totals of any two value groups j and j', given by

$$\text{var}(\hat{Y}_{.(j\&j')}) = \sum_{l=1}^{L} \text{var}(\hat{Y}_{i(j\&j')})$$

A Worked Example

To illustrate the proposed method, one sample was drawn from each stratum in an artificial population, and the value group estimators for total volume and their variances were calculated. The artificial population contains data from $N = 1600$ loblolly pine trees (*Pinus taeda* L.). The data comprise diameter squared times height (x_i) and total tree volume (y_i) measurements for each tree. For testing purposes, one of the three value group codes was randomly assigned to each tree and the population was divided into two strata with $N_1 = 800 = N_2$. The 800 largest trees were assigned to the first stratum where 3P sampling was used. The expected sample size for 3P sampling was $n(e)_1 = 10$. The 800 smallest trees were assigned to the second stratum because they were more homogeneous, and the sample-tree method is most effective in this situation. This stratum was divided into two sample groups. The first sample group contained only value group 1 trees and had a sampling frequency of $f_1 = 40$. The second sample group contained value groups 2 and 3 and had a sampling frequency of $f_2 = 50$.

3P Sampling Estimation

The trees selected by 3P sampling in the first stratum are listed in Table 1. In addition, the indicator variables given in (5) are also listed. The expected and actual sample sizes are $n(e)_1 = 10$ and $n_1 = 13$. The estimates of the total volume for each stratum and all pairs of strata are calculated using (4), (6), and (8), respectively. The variance estimates for all possible stratum estimates are calculated using Equations (7) and (9). The estimated totals are

$\hat{Y}_{1.} = 9930.0$

$\hat{Y}_{11} = 2316.2$

Table 1. Sample values selected from stratum 1 of the test population using 3P sampling. $y'_{1i}, j=1$ is the value of y'_{1i} in value group 1

Tree number	Value group	x_{1i}	y_{1i}	π_{1i}	$y'_{1i}, j=1$	$y'_{1i}, j=2$	$y'_{1i}, j=3$
1	2	2756.25	4.28	0.005511	0	4.28	0
2	1	2866.88	4.86	0.005732	4.86	0	0
3	3	3739.77	6.61	0.007477	0	0	6.61
4	3	3794.56	9.00	0.007587	0	0	9.00
5	1	6450.28	15.11	0.012896	15.11	0	0
6	3	6629.24	13.84	0.013254	0	0	13.84
7	3	8328.32	17.41	0.016651	0	0	17.41
8	2	8760.96	12.28	0.017516	0	12.28	0
9	2	10813.44	22.77	0.021620	0	22.77	0
10	1	11727.81	23.25	0.023448	23.25	0	0
11	3	12608.32	28.19	0.025209	0	0	28.19
12	3	12736.08	29.33	0.025464	0	0	29.33
13	2	13320.19	24.95	0.026632	0	24.95	0

$\hat{Y}_{12} = 2667.5$

$\hat{Y}_{13} = 4946.2$

$\hat{Y}_{1(1\&2)} = 4983.7$

$\hat{Y}_{1(1\&3)} = 7262.5$

$\hat{Y}_{1(2\&3)} = 7613.8$

with variance estimates

$\text{var}(\hat{Y}_{1.}) = 181184.1$

$\text{var}(\hat{Y}_{11}) = 1499209.4$

$\text{var}(\hat{Y}_{12}) = 1359952.0$

$\text{var}(\hat{Y}_{13}) = 2376776.5$

$\text{var}(\hat{Y}_{1(1\&2)}) = 1846138.1$

$\text{var}(\hat{Y}_{1(1\&3)}) = 1997571.0$

$\text{var}(\hat{Y}_{1(2\&3)}) = 1573412.8$

Note that for the $\text{var}(\hat{Y}_{1j})$ and $\text{var}(\hat{Y}_{1(j\&j')})$ estimates, the sum goes from $i = 1, \ldots n_1$. Thus the summation term for the variance estimate of value group 1, i.e.

$$\left(\frac{n(e)_1 y'_{1i}}{\pi_{1i}} - \hat{Y}_{1j}\right)^2$$

becomes $(\hat{Y}_{1j})^2$ whenever $(y'_{1i}) = 0$. The same is true for (9) whenever $(y''_{1i}) = 0$.

Sample-Tree Estimation

The trees selected in the second stratum using sample-tree sampling are listed in Table 2. The value group indicator variables given in (13) and the sample group indicator variables given by (18) are also listed. The sampling frequency for value group 1 was $f_1 = 40$. Hence, every fortieth tree with

Table 2. Sample values selected from stratum 2 of the test population using sample-tree sampling. $y'_{2i}, j=1$ is the value of y'_{2i} in value group 1. $y*_{2i}, k=1$ is the value of $y*_{2i}$ in sample group 1.

Tree number	Value group	y_{2i}	π_{2i}	$y'_{2i}, j=1$	$y'_{2i}, j=2$	$y'_{2i}, j=3$	$y^*_{2i}, k=1$	$y^*_{2i}, k=2$
1	3	3.87	0.0200	0	0	3.87	0	3.87
2	1	3.92	0.0250	3.92	0	0	3.92	0
3	2	2.19	0.0200	0	2.19	0	0	2.19
4	2	1.40	0.0200	0	1.40	0	0	1.40
5	1	2.00	0.0250	2.00	0	0	2.00	0
6	2	3.52	0.0200	0	3.52	0	0	3.52
7	1	2.75	0.0250	2.75	0	0	2.75	0
8	2	1.00	0.0200	0	1.00	0	0	1.00
9	3	1.15	0.0200	0	0	1.15	0	1.15
10	1	2.48	0.0250	2.48	0	0	2.48	0
11	2	1.15	0.0200	0	1.15	0	0	1.15
12	1	1.27	0.0250	1.27	0	0	1.27	0
13	2	3.75	0.0200	0	3.75	0	0	3.75
14	3	5.52	0.0200	0	0	5.52	0	5.52
15	1	1.37	0.0250	1.37	0	0	1.37	0
16	3	1.60	0.0200	0	0	1.60	0	1.60

value group code 1 was selected. This resulted in a sample group and value group sample of size $n_{2(1)} = n_{21} = 6$. Trees with value group codes 2 or 3 were assigned to sample group 2 and selected with sampling frequency $f_2 = 50$. Thus every fiftieth tree encountered that was either value group 2 or 3 was selected. This resulted in a sample group size of $n_{2(2)} = 10$ with value group sample sizes of $n_{22} = 6$ and $n_{23} = 4$ for value groups 2 and 3 respectively. Estimates of the total volume, total volume in each value group, and for each pair of value groups, given by (10), (11) and (12) respectively, are

$$\hat{Y}_{1.} = 1964.0$$

$$\hat{Y}_{11} = 629.7$$

$$\hat{Y}_{12} = 596.5$$

$$\hat{Y}_{13} = 762.2$$

$$\hat{Y}_{1(1\&2)} = 1202.1$$

$$\hat{Y}_{1(1\&3)} = 1158.6$$

$$\hat{Y}_{1(2\&3)} = 1257.5$$

The variance estimate for all value groups combined ($\mathrm{var}(\hat{Y}_{2.})$) is calculated using (15). Equation (16) is used to calculate the variance in the first value group ($\mathrm{var}(\hat{Y}_{21})$), because the first sample group only contains trees from the first value group. To calculate the variance estimates for value groups 2 and 3 ($\mathrm{var}(\hat{Y}_{2j}), j = 2,3$), (17) is used. Samples drawn in different value groups can be assumed independent. Thus, $\mathrm{var}(\hat{Y}_{2(1\&2)})$ and $\mathrm{var}(\hat{Y}_{2(1\&3)})$ are calculated using (19). The variance estimate, $\mathrm{var}(\hat{Y}_{2(2\&3)})$, is calculated using (20), because both values groups 2 and 3 are assigned to sample group 2 and are therefore not independent samples. The variance estimates are

$$\mathrm{var}(\hat{Y}_{1.}) = 81449.6$$

$$\mathrm{var}(\hat{Y}_{11}) = 12108.5$$

$$\mathrm{var}(\hat{Y}_{12}) = 56109.1$$

$$\mathrm{var}(\hat{Y}_{13}) = 104319.7$$

$$\mathrm{var}(\hat{Y}_{1(1\&2)}) = 68217.7$$

$$\mathrm{var}(\hat{Y}_{1(1\&3)}) = 116428.3$$

$$\mathrm{var}(\hat{Y}_{1(2\&3)}) = 65710.9$$

Combining the Strata Estimates

To estimate the totals in any value group, the estimates are added across the strata. For example, the total across both strata for value group 1 is

$$\hat{Y}_{.1} = \hat{Y}_{11} + \hat{Y}_{21}$$

Any other combination of value groups can be calculated in a similar manner. All the possible combinations of value group volume estimates for the two samples are

$$\hat{Y}_{..} = 11894.0$$

$$\hat{Y}_{.1} = 2945.9$$

$$\hat{Y}_{.2} = 3264.1$$

$$\hat{Y}_{.3} = 5708.4$$

$$\hat{Y}_{.(1\&2)} = 6185.8$$

$$\hat{Y}_{.(1\&3)} = 8421.1$$

$$\hat{Y}_{.(2\&3)} = 8871.3$$

The diagonal elements of the variance-covariance matrix are also generated by adding the across the strata using (21) For example, the variance for value group 1 is

$$\mathrm{var}(\hat{Y}_{.1}) = \mathrm{var}(\hat{Y}_{11}) + \mathrm{var}(\hat{Y}_{21})$$

The covariances, which are the off-diagonal elements of (2), require the variances for every pair of value group estimates and are generated by adding across the strata as before. Thus, the variance estimate for value groups 1 and 2 is given by

$$\mathrm{var}(\hat{Y}_{.(1\&2)}) = \mathrm{var}(\hat{Y}_{1(1\&2)}) + \mathrm{var}(\hat{Y}_{2(1\&2)})$$

To conclude this example, using (3) to generate the covariance for value groups 1 and 2 combined gives

$$\mathrm{cov}(\hat{Y}_{.1}, \hat{Y}_{.2}) = \frac{\mathrm{var}(\hat{Y}_{.(1\&2)} - \mathrm{var}(\hat{Y}_{.1}) - \mathrm{var}(\hat{Y}_{.2})}{2}$$

The variance-covariance matrix for the two samples is

$$\begin{pmatrix} 1511317.9 & -506511.6 & -939207.4 \\ -506511.6 & 1416061.2 & -1129016.9 \\ -939207.4 & -1129016.9 & 2481096.3 \end{pmatrix}$$

Conclusions

The method discussed for estimating the variance of an arbitrary collection of value groups is a combination of well-documented statistical results that minimizes the number of calculations and the amount of information that must be reported This method is also be the easiest method to incorporate into the current Forest Service and industry timber inventories.

Literature Cited

COCHRAN, W.G. 1977. Sampling techniques. Ed. 3. Wiley, New York. 428 p.

GROSENBAUGH, L R 1974 STX 3-3-73 Tree content and value estimation using various sample designs, dendrology methods, and V-S-L conversion coefficients. USDA For. Serv. Res. Pap. SE-117. 112 p.

MOOD A.M., GRAYBILL, F.A., AND BOES, D.C. 1974. Introduction to the theory of statistics. Ed. 3. McGraw Hill, New York. 564 p.

USDA FOREST SERVICE. 1993. Timber cruising handbook. FSH 2409.12.

SCHREUDER, H.T., T.G. GREGOIRE, AND G. B. WOOD, 1993. Sampling methods for multiresource forest inventory. Wiley, New York. 446 p.