COMPARING METHODS TO ESTIMATE REINEKE'S MAXIMUM SIZE-DENSITY RELATIONSHIP SPECIES BOUNDARY LINE SLOPE

Curtis L. VanderSchaaf and Harold E. Burkhart[1]

Abstract—Maximum size-density relationships (MSDR) provide natural resource managers useful information about the relationship between tree density and average tree size. Obtaining a valid estimate of how maximum tree density changes as average tree size changes is necessary to accurately describe these relationships. This paper examines three methods to estimate the slope of the MSDR species boundary line across a range of site qualities: ordinary least squares, first-difference model, and the linear mixed effects model.

INTRODUCTION

According to the theoretical reasoning behind stand density index (SDI), all stands should eventually approach and track along the same Maximum Size-Density Relationship (MSDR) boundary for a particular species/region combination (Reineke 1933, Drew and Flewelling 1977, Williams 1996, Harms and others 2000). The MSDR is commonly expressed in terms of SDI as seen in equation (1):

$$SDI = TPA*(QMD/10)^b \quad (1)$$

where:
TPA = trees per acre
QMD = quadratic mean diameter (inches)
b = MSDR boundary coefficient

Generally, for the model form used in equation (1), b is assumed to be near 1.6. Usually, b is obtained by regressing LnTPA against LnQMD which gives a negative b. Williams (1996) shows how equation (1) is derived from the LnTPA-LnQMD regression and how b becomes positive when used in equation (1). Although this study pertains to estimating the b of Reineke's MSDR, this analysis also applies to the well-known Self-thinning rule (Westoby 1984, Weller 1987) which uses average volume or average biomass rather than average diameter.

Reineke (1933) conceptualized the MSDR boundary as a linear asymptote on the log-log scale that all stands would eventually approach and track along for a particular species/region combination. Since that time ecologists and foresters have proposed modifications to the theory behind the self thinning concept. For the remainder of this paper, the terminology of Weller (1990) will be used:

1. Individual stand MSDR boundaries will be referred to as MSDR dynamic thinning line boundaries. We assume a plot to be representative of a stand.
2. The MSDR species boundary line I shall be defined as a static upper limit of maximum tree density-average tree size relationships (or conversely maximum average tree size-tree density relationships) that applies to all stands of a certain species within a particular geographical area.

For the MSDR species boundary line I, "static" refers to the fact that mid-rotation and regeneration management techniques, site quality, genetics, etc., have no impact on this boundary line as opposed to MSDR dynamic thinning lines which can be affected by genetic stock and silvicultural treatments (VanderSchaaf and Burkhart 2007, Weller 1990). The MSDR species boundary line I is a line of constant slope connecting maximum tree densities across a range of average tree sizes regardless of site quality, planting density, genetics, thinnings, etc. Maximum tree densities across the range of average tree sizes can be a conglomeration from many stands and the maximum tree densities are not necessarily obtained from an individual stand (VanderSchaaf and Burkhart 2007). Conversely, the axes can be rotated such that the MSDR species boundary line I is a line of constant slope that connects the maximum average tree size for a given TPA regardless of site quality, planting density, genetics, thinnings, etc.

VanderSchaaf and Burkhart (2007) proposed a second definition of the MSDR species boundary line—what they called the MSDR species boundary line II. The slope of the MSDR species boundary line II is considered to be the population average of all MSDR dynamic thinning line slopes for a particular species. Due to the impacts of planting density on MSDR dynamic thinning line boundary levels, the ordinary least squares (OLS) estimate of b was found to be sensitive to the range of planting densities included in the model fitting dataset. Since all observations are considered independent, OLS accounted for differences in dynamic thinning line boundary levels by adjusting both the species boundary line intercept and slope terms.

Despite the belief that MSDR dynamic thinning line boundary levels are independent of site quality (Reineke 1933), studies have found site quality can impact boundary levels for conifer species (Hynynen 1993, Pittman and Turnblom 2003) including loblolly pine (*Pinus taeda* L., Strub and Bredenkamp 1985, Harrison and Daniels 1988). These studies imply the greatest TPA that can occur for a particular QMD is larger on higher quality sites, or that not all MSDR dynamic thinning line boundaries occur along the MSDR

[1]Arkansas Forest Resources Center, University of Arkansas at Monticello, Monticello, AR 71656; Department of Forestry, Virginia Polytechnic Institute and State University, Blacksburg, VA 24061, respectively.

Citation for proceedings: Stanturf, John A., ed. 2010. Proceedings of the 14th biennial southern silvicultural research conference. Gen. Tech. Rep. SRS-121. Asheville, NC: U.S. Department of Agriculture, Forest Service, Southern Research Station. 614 p.

species boundary line. If differences in the boundary level due to site quality are not accounted for when estimating the b of MSDR species boundary lines, the estimated b may not be reflective of how individual stands self-thin during the MSDR dynamic thinning line stage of stand development (VanderSchaaf and Burkhart 2007). See figure 1.

Often, the MSDR species boundary line is used as a limit for stand density in models (Oliver and Powers 1978, Puettmann and others 1993, Smith and Hann 1984). Monserud and others (2005) describe the utility of limiting stand density using self-thinning constraints. When using self-thinning constraints, stand development is predicted using equations independent of the self-thinning constraint until the stand reaches the upper boundary. Stand tree density or average tree size is then constrained such that the stand trajectory tracks along this upper boundary either for a limited amount of time before diverging from the boundary or the stand tracks along this upper boundary into infinity. There are several statistical methods to estimate b that includes OLS, first-difference models, and mixed models. When growth and yield models are constrained by MSDR species boundary lines using an estimated b, an improper estimated value of b can produce serious errors in estimation of stand density development, stand yield, and ultimately economic decisions.

The objectives of this study were to: (1) ascertain whether different statistical methods used to estimate the MSDR species boundary line b result in varying values, and (2) determine whether certain statistical methods produce more stable estimates of the MSDR species boundary line II b as the range of site quality in the fitting dataset varies.

METHODS

Data

Tree and stand measurements were obtained from a long-term loblolly pine plantation thinning study maintained by the Loblolly Pine Growth and Yield Research Cooperative at Virginia Polytechnic Institute and State University. Permanent research plots were established from 1980 to 1982 on 186 sites located throughout the Southeastern United States (Burkhart and others 1985). At each study site location, three plots were established; one was left unthinned and the two other plots were thinned. For this current analysis, only the unthinned plots were used to eliminate the potential affects of mid-rotation thinning on MSDRs. Plots were remeasured every three years and the number of remeasurements differed across sites due to factors such as clearcutting, insect or disease damage, etc. For sites where planting density is known, it ranged from 570 to 1223 seedlings per acre. Site quality was measured as site index at base age 25. To estimate site index for an individual plot, the average height of the tallest half of the surviving trees for the measurement age closest to age 25 was placed into a site index equation developed by Burkhart and others (1987). Site index ranged from 41 to 76 feet and averaged 56.

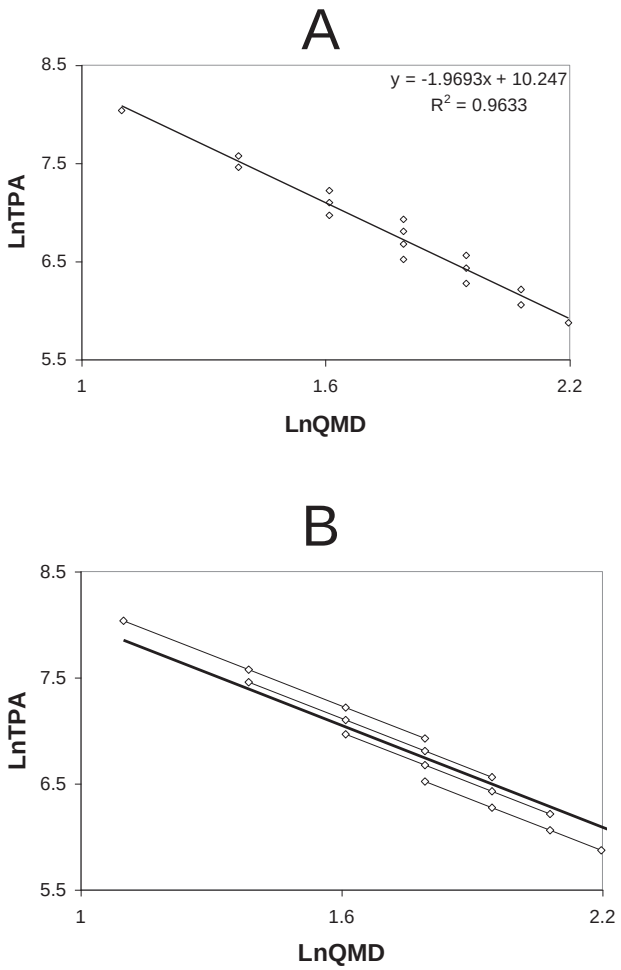

Figure 1—LnTPA over LnQMD for four theoretical stands where each stand has a MSDR dynamic thinning line slope of -1.6. Lighter lines connect observations from the same stand in Figure 1-B. One stand has a Reineke SDI of 450, a second stand has a Reineke SDI of 400, a third stand has a Reineke SDI of 350, and a fourth stand has a Reineke SDI of 300, differences in Reineke's maximum SDI are attributed to differences in site quality. The LnQMD of observations occurring along MSDR dynamic thinning lines also differ by site quality. In Figure 1-A, all observations are assumed independent of one another. When using OLS to estimate the MSDR species boundary line slope, an estimate of -1.9693 is obtained because OLS ignores differences in boundary levels, or correlation among observations from an individual MSDR dynamic thinning line. When using a mixed effects model, a slope of -1.6 is estimated since the technique accounts for differences in MSDR dynamic thinning line boundary levels (or accounts for correlation among observations from the same MSDR dynamic thinning line) when estimating the MSDR species boundary line slope.

Stages of Self-Thinning in Individual Stands

Within the overall self-thinning stage of stand development, or when density-dependent mortality is occurring, there is generally considered to be three stages of stand development (VanderSchaaf and Burkhart 2007):

1. The self-thinning stage of stand development is initially composed of a curved approach to the MSDR dynamic thinning line. During this initial component of self-thinning, mortality is less than the mortality at maximum competition and thus the trajectory has a concave shape (del Rio and others 2001).
2. With increases in individual tree size and the death of other trees, eventually the growth trajectory becomes linear where an increase in QMD is assumed to be an exact function of the maximum value of Reineke's SDI, the change in TPA, and the MSDR dynamic thinning line b. This stage is known as the MSDR dynamic thinning line stage of stand development (Weller 1990), or when a stand is fully-stocked (del Rio and others 2001).
3. Eventually, as trees die, the residual trees cannot continue to fully occupy canopy gaps and the trajectory diverges from the MSDR dynamic thinning line (Bredenkamp and Burkhart 1990, Cao and others 2000, Zeide 1995).

Over the entire range of self-thinning the relationship between LnQMD and LnTPA is curvilinear; however, this analysis deals only with the linear stage (or portion) of the trajectory (Cao and others 2000, del Rio and others 2001, Monserud and others 2005, VanderSchaaf 2006, Yang and Titus 2002, Zeide 1985).

Determining What Observations Occur Along MSDR Dynamic Thinning Line Boundaries

Graphs of LnQMD over LnTPA were constructed for each unthinned study plot to determine when stand trajectories had reached a MSDR dynamic thinning line boundary. In order to be consistent with the recommendations of Weller (1987, 1991), visual inspection of all plots was conducted to ensure that only those observations occurring along a MSDR dynamic thinning line boundary were included when estimating the MSDR species boundary line b coefficient (fig. 2A).

Slopes of individual MSDR dynamic thinning line boundaries were greatly affected by measurement ages that were diverging from MSDR dynamic thinning line boundaries—stage 3 of self-thinning. Therefore, all points diverging from individual MSDR dynamic thinning line boundaries were eliminated when determining the MSDR species boundary line b. If a plot had two consecutive points along the MSDR dynamic thinning line—a growth trajectory moving in a straight line (MSDR dynamic thinning line) to the left—the plot was included in the analysis. Data were obtained from 121 of the 186 sites. Measurement ages at which observations occurred along a MSDR dynamic thinning line

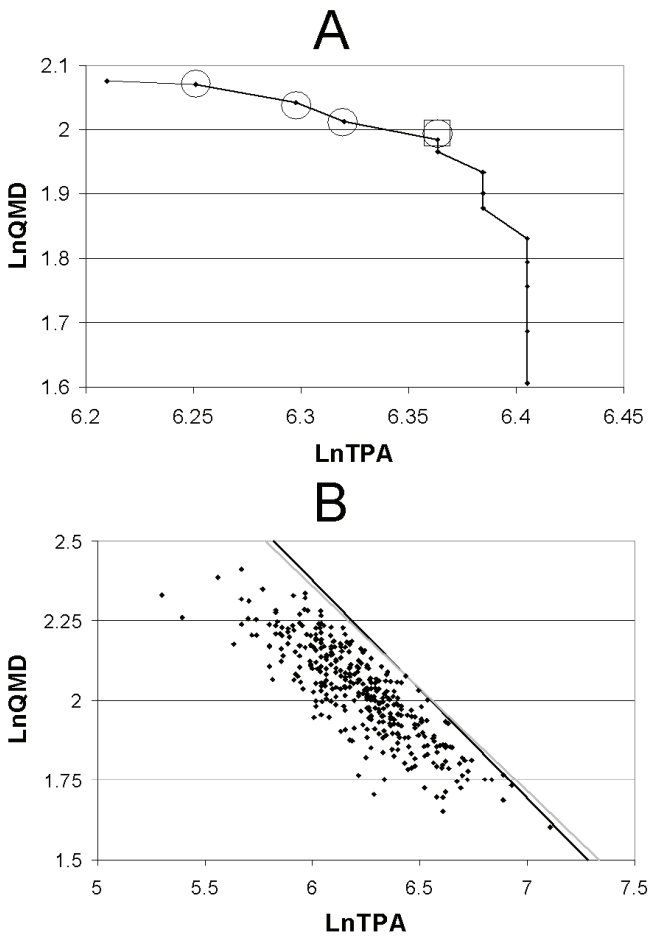

Figure 2—A. Loblolly pine growth trajectory for a planting density of 605 seedlings per acre. Vertical movements of the trajectory indicate the plantation is not self-thinning, while movements to the left and up indicates the plantation is self-thinning. A constant "linear" movement to the left and up indicates the trajectory has reached the MSDR dynamic thinning line boundary for that individual plantation. Therefore, the measurement ages with circles around them were used to estimate the b for the OLS and linear mixed effects analyses. For the first-difference model and the Mean slope calculation method, the observation with a square was omitted when determining b values between two successive measurement ages. The last measurement age was not included in the analysis because it is diverging from the MSDR dynamic thinning line boundary. B. Plot of all individual LnQMD-LnTPA observations (n = 365) occurring along MSDR dynamic thinning line boundaries. The figure also contains OLS (black line) and linear mixed effects (gray line) estimated MSDR species boundary lines that have b's of -1.4593 and -1.5537; respectively. Boundary lines were positioned above all observations.

boundary ranged from 19 to 42 years. For the observations occurring along MSDR dynamic thinning lines, TPA ranged from 200 to 1220 and averaged 500, QMD ranged from 5.0 to 11.2 inches and averaged 7.7, and basal area ranged from 93 to 255 square feet per acre and averaged 171. When using an exponent of 1.6 for equation (1), SDI ranged from 205 to 434 and averaged 329.

MSDR Species Boundary Line b Estimation Methods

Mean slope calculation method—Rather than using regression analysis techniques, an estimated value of the MSDR species boundary line b can be obtained by determining the arithmetic mean of all successive slopes from individual plots at the MSDR stage of stand development:

$$b = (LnTPA_{t+3} - LnTPA_t)/(LnQMD_{t+3} - LnQMD_t) \quad (2)$$

where:

$LnTPA_{t+3}$–LnTPA at the successive measurement (all plots were remeasured on 3 year intervals)
$LnTPA_t$–LnTPA at the current measurement (t)
$LnQMD_{t+3}$–LnQMD at the successive measurement
$LnQMD_t$–LnQMD at the current measurement (t)

Equation (2) quantifies the change in LnTPA given a change in LnQMD—the b in equation (1). The Mean slope calculation method is used in this paper only as a "reality check" for the estimated b's using the other statistical methods.

Ordinary least squares—Several researchers have used either linear or non-linear regression to estimate the MSDR species boundary line b (Bredenkamp and Burkhart 1990, Hynynen 1993, MacKinney and Chaiken 1935, Oliver and Powers 1978, Smith and Hann 1984, Williams 1996, Wittwer and others 1998, Yang and Titus 2002). For this paper, the OLS form will be used:

$$LnTPA = b_0 + b_1 LnQMD + \epsilon \quad (3)$$

where:
b_0, b_1 = parameters to be estimated where b_1 = b from equation (1).
ϵ = random error where it is assumed $\epsilon \sim N(0, \sigma^2 I)$.
When using OLS to estimate the MSDR species boundary line b, individual MSDR dynamic thinning line b's are ignored. OLS, and the commonly used form of non-linear regression (e.g., when not directly accounting for autocorrelation among observations) to estimate the MSDR species boundary line b do not account for correlations among observations from individual stands. For OLS, Proc Reg of the SAS Institute (SAS 1996) was used to estimate MSDR species boundary line b's.

First-difference model—Wittwer and others (1998) and Lynch and others (2004) used a first-difference approach to account for dependencies among observations from the same stand (or plot) when determining the MSDR species boundary line b. A first-difference model is expressed as:

$$LnTPA_{t+3} - LnTPA_t = b(LnQMD_{t+3} - LnQMD_t) + \epsilon \quad (4)$$

where:

b is a parameter to be estimated and is equivalent to the b from equation (1).
ϵ = random error where it is assumed $\epsilon \sim N(0)$.

In this study, the youngest observation occurring along an individual MSDR dynamic thinning line boundary was not included in the first-difference model analysis since the lag of LnTPA and lag of LnQMD did not exist along the MSDR dynamic thinning line boundary (fig. 2A). Thus, the sample size for this particular method is the same as the Mean slope calculation method and is less than the OLS and linear mixed effects model analyses. Parameter estimates were obtained using Proc MODEL of the SAS Institute (SAS 1988).

Linear mixed effects model—Although a first-difference model accounts for autocorrelation between successive measurements from the same plot, it fails to account for autocorrelation among many observations from the same plot. Linear parametric mixed effects analyses can account for correlation among many observations from the same plot (Lappi and Bailey 1988, Schabenberger and Pierce 2002, pg. 408). Hynynen (1993) used a mixed effects analysis to estimate the MSDR species boundary line b but did not report an OLS estimated value and thus it cannot be determined whether the two statistical methods produced similar estimates of the MSDR species boundary line b.

$$LnTPA = (b_0 + u_{0i}) + (b_1 + u_{1i})LnQMD + \epsilon \quad (5)$$

where:
u_{0i}, u_{1i} = are cluster-specific random effects to be predicted and assumed to be $N(0, \sigma_0^2)$ and $N(0, \sigma_1^2)$, respectively. A cluster is an individual plot (indexed by *i*).
ϵ = random error where it is assumed $\epsilon \sim N(0, \sigma^2 I)$.

For this analysis, the intercept (b_0) and b (b_1 is the MSDR species boundary line slope) terms from equation (3) were assumed to be random parameters (each individual plot, or cluster, has its own intercept and slope). Estimated parameters were obtained using Proc Mixed of the SAS Institute (SAS 1988). For all analyses, an "unstructured" random effects covariance-variance matrix (un) was used in Proc Mixed; thus the data were used to estimate the entire covariance-variance structure (σ_0^2, σ_1^2, σ_{01}).

Stability of the Estimated MSDR Species Boundary Line b in Relation to Site Quality

For this paper, stability refers to the extent to which parameter estimates do not change when the range of site indices in the fitting dataset changes. In order to see

how including different site indices can affect the estimated value of the MSDR species boundary line b for a particular statistical method, several analyses were conducted. First, observations from MSDR dynamic thinning lines of all site qualities were included to estimate the MSDR species boundary line b using the four statistical methods. Second, to determine if site index affects the estimated value of the MSDR species boundary line b, estimated b's were obtained from datasets containing all site indices greater than 50, 55, and 60 feet. These categorical values result in nearly equal decreases in sample sizes while also providing sufficient numbers of observations to estimate the MSDR species boundary line b.

RESULTS

Estimates of the MSDR species boundary line b are given in table 1. The OLS parameter estimates are the least stable as the range of site indices included in the dataset change. This instability apparently results from the lack of independence of observations. Based on residual and lag residual plots, the data used in this study have a strong presence of positive autocorrelation. When using the first-difference model the autocorrelation was reduced.

The linear mixed effects analysis produces relatively stable estimates of b as site index changes (b ranges only from -1.6639 to -1.5537 when including different site qualities). This statistical method seems to be robust against factors such as site quality that can change the MSDR dynamic thinning line boundary producing unstable MSDR species boundary line b estimates when using OLS (e.g., -1.7626 to -1.4593). The first-difference model also produced relatively stable estimates of b as site index changed (b ranges only from -1.6266 to -1.5792 when including different site qualities).

DISCUSSION

Some may argue this paper deals with estimating the b of the MSDR species boundary line and thus accounting for MSDR dynamic thinning lines is not necessary. If all we wanted to know was the maximum TPA for a given average tree size across all loblolly pine plantations in the Southeastern US then the OLS estimated b would be appropriate. However, beyond determining the maximum TPA for a given average tree size, it is often desired to quantify how, on average, maximum TPA changes for a given change in average tree size for individual plantations – this can also be thought of as the b of equation (1). As seen in table 1 and fig. 2B, the OLS estimated b provides a quantification of the upper boundary of the relationship between average tree size and tree density. However, this b fails to adequately describe

Table 1—Estimates of the MSDR species boundary line slope (b) using several statistical methods for the full dataset and for subsets of the data consisting of varying site indices

Site index	Estimation method	n	Intercept	Std. error	b	Std. error
>50	OLS	300	9.3981	0.1141	-1.5563	0.0554
	First	185	-	-	-1.6071	0.0237
	Mixed	300	9.4611	0.0597	-1.5858	0.0292
	Mean	185	-	-	-1.5669	0.0247
>55	OLS	186	9.2490	0.1614	-1.4850	0.0772
	First	113	-	-	-1.5831	0.0267
	Mixed	186	9.5078	0.0723	-1.6049	0.0352
	Mean	113	-	-	-1.5588	0.0284
>60	OLS	81	9.8398	0.2384	-1.7626	0.1117
	First	47	-	-	-1.6266	0.0440
	Mixed	81	9.6361	0.1106	-1.6639	0.0553
	Mean	47	-	-	-1.6037	0.0479
All	OLS	365	9.1848	0.0996	-1.4593	0.0490
	First	226	-	-	-1.5792	0.0212
	Mixed	365	9.3791	0.0539	-1.5537	0.0259
	Mean	226	-	-	-1.5522	0.0223

Std. error—standard error of the estimate, OLS—Ordinary Least Squares method, First—First-difference model method, Mixed—Linear mixed effects model method, Mean—Mean slope calculation method .

the expected population change (based on the Mean slope calculation method) in maximum TPA given a change in average tree size. Growth and yield models are often constrained using the MSDR species boundary line or the Self-thinning rule. In many cases, based on the data used in this analysis and analyses conducted by VanderSchaaf and Burkhart (2007), the estimated OLS b would, on average, incorrectly predict stand development. A few MSDR dynamic thinning lines may have a b close to the estimated OLS b though.

Why is the OLS Estimated b Not Sufficient For Describing MSDR Dynamic Thinning Lines?

Trajectories of stands with lower site indices generally fall short of the MSDR species boundary line when using a b near –1.6 if the MSDR species boundary line is established using greater site indices. It has been reported the maximum tree density obtainable for a given average tree size is less on lower quality sites (Strub and Bredenkamp 1985, Harrison and Daniels 1988, Hynynen 1993, Pittman and Turnblom 2003). Thus, MSDR dynamic thinning line boundaries of varying site indices may not always occur at the MSDR species boundary line when using a b near -1.6. For the data used in this study, MSDRs were found to vary across site index. To account for differences in MSDR dynamic thinning line boundaries, in addition to adjusting the intercept, OLS adjusts the MSDR species boundary line b (table 1 and fig. 1).

From a statistical point of view, the OLS estimated MSDR species boundary line b fails to adequately describe, on average, the b of MSDR dynamic thinning lines because it fails to account for correlation among observations from the same stand. By assuming all observations from MSDR dynamic thinning lines are independent, OLS does not estimate the MSDR species boundary line b taking into account that MSDR dynamic thinning line boundary levels differ in relation to site index (e.g., fig. 1). Perhaps many studies that have found a large difference between their estimated MSDR species boundary line b and Reineke's original value of -1.605 (or MacKinney and Chaiken's value of -1.707) should be reexamined using mixed models.

For the data used in this study and when using the mixed effects estimate of the MSDR species boundary line II b for the entire dataset (-1.5537), lower site indices were found to have a lower maximum SDI (SDI = 266.1065 +1.190794SI, R-square = 0.0321) similar to Strub and Bredenkamp (1985). As compared to planting density (VanderSchaaf and Burkhart 2007), site quality appears to have a less drastic affect on estimated MSDR species boundary line b's for loblolly pine plantations (table 1). This may potentially be due to the fact that site quality has less impact on MSDR dynamic thinning line boundary levels. However, planting density and genetics are confounding factors in this study. The data used by VanderSchaaf and Burkhart (2007) were obtained from sites basically having the same site quality and of the same genetic stock. Thus, site quality may have just as much impact as planting density on loblolly pine plantation MSDR dynamic thinning line boundary levels and estimated MSDR species boundary line b's.

When comparing the stability of the linear mixed effects model to the first-difference model, both methods produce relatively stable estimates of the MSDR species boundary line II b. In fact, unlike for planting density (VanderSchaaf and Burkhart 2007), the first-difference model appeared to be more stable than the mixed effects model. However, when comparing estimates from the two models to the Mean slope calculation method, both appear to give reasonable estimates. Actually, for most datasets, a linear mixed effects model will likely be superior because the estimated MSDR species boundary line II b value will be based on accounting for autocorrelation among all observations for a particular MSDR dynamic thinning line boundary. Although the Mean slope calculation method was used as the "reality check," in fact, the linear mixed effects method may be superior to both equations (2) and (4) when estimating the MSDR species boundary line II b. Based on the mixed effects analysis, the predicted MSDR dynamic thinning line b's ranged from -2.1748 to -1.1929 –- this range is similar to other reported studies (Pretzsch and Biber 2005).

CONCLUSIONS

Site quality can impact MSDR dynamic thinning line boundary levels causing instability in MSDR species boundary line b estimates. By using mixed models, more stable estimates can be obtained and the impact of site quality on the b estimate can be reduced. Based on the data used in this analysis and the linear mixed effects analysis for the entire dataset (n = 365), and the analyses by VanderSchaaf and Burkhart (2007), foresters some 70 years ago (MacKinney and Chaiken 1935, Reineke 1933) may well have determined the population average MSDR dynamic thinning line b for loblolly pine stands throughout the Southeastern US. Results from this study suggest quantifying MSDR dynamic thinning line boundaries and b's will provide a better description of maximum tree density–average tree size relationships for individual plantations.

ACKNOWLEDGMENTS

Jamie Schuler provided several useful comments. Financial support was received from members of the Loblolly Pine Growth and Yield Research Cooperative.

LITERATURE CITED

Bredenkamp, B.V.; Burkhart, H.E. 1990. An examination of spacing indices for *Eucalyptus grandis*. Canadian Journal Forest Research. 20: 1909-1916.

Burkhart, H.E.; Cloeren, D.C; Amateis, R.L. 1985. Yield relationships in unthinned loblolly pine plantations on cutover, site-prepared lands. Southern Journal of Applied Forestry. 9: 84-91.

Burkhart, H.E.; Farrar, K.D.; Amateis, R.L.; Daniels, R.F. 1987. Simulation of individual tree growth and stand development in loblolly pine plantations on cutover, site-prepared areas. FWS-1-87. Blacksburg, VA: Virginia Tech University, Department of Forestry. Revised 2001.

Cao, Q.V.; Dean, T.J.; Baldwin, V.C. 2000. Modeling the size-density relationship in direct-seeded slash pine stands. Forest. Science 46: 317-321.

del Rio, M.; Montero, G.; Bravo, F. 2001. Analysis of diameter-density relationships and self-thinning in non-thinned even-aged Scots pine stands. Forest Ecology and Management. 142: 79-87.

Drew, T.J.; Flewelling, J.W. 1977. Some recent Japanese theories of yield density relationships and their application to Monterey pine plantations. Forest Science. 23: 517-534.

Harms, W.R.; Whitesell, C.D.; DeBell, D.S. 2000. Growth and development of loblolly pine in a spacing trial planted in Hawaii. Forest Ecology and Management. 126: 13-24.

Harrison, W.C.; Daniels, R.F. 1988. A new biomathematical model for growth and yield of loblolly pine plantations. In: Ek, A.R.; Shifley, S.R.; Burk, T.E. (eds.). Forest Growth Modeling and Prediction, Vol. 1, Gen. Tech. Rep. NC-120. St. Paul, MN: U.S. Department of Agriculture, Forest Service, North Central Forest Experiment Station: 293-304.

Hynynen, J. 1993. Self-thinning models for even-aged stands of *Pinus sylvestris, Picea abies*, and *Betula penula*. Scandanavian Journal of Forest Research. 8: 326-336.

Lappi, J.; Bailey, R.L. 1988. A height prediction model with random stand and tree parameters: an alternative to traditional site index methods. Forest Science 34: 907-927.

Lynch, T.B.; Wittwer, R.F.; Stevenson, D.J. 2004. Estimation of Reineke and volume-based maximum size-density lines for shortleaf pine. In: Connor, K.F. (ed.). Proceedings of the twelfth biennial southern silvicultural research conference. Gen. Tech. Rep. SRS-GTR-71. Asheville, NC: U.S. Department of Agriculture, Forest Service, Southern Research Station: 226.

MacKinney, A.L.; Chaiken, L.E. 1935. A method of determining density of loblolly pine stands. Tech. Note 15. Asheville, NC: U.S. Department of Agriculture, Forest Service, Appalachian Forest Experiment Station. 3 p.

Monserud, R.A.; Ledermann, T.; Sterba, H. 2005. Are self-thinning constraints needed in a tree-specific mortality model? Forest Science. 50: 848-858.

Oliver, W.W.; Powers, R.F. 1978. Growth models for ponderosa pine: I. Yield of unthinned plantations in northern California. Res. Pap. PSW-133. Berkley, CA: U.S. Department of Agriculture, Forest Service, Pacific Southwest Forest and Range Experiment Station. 21 p.

Pittman, S.D.; Turnblom, E.C. 2003. A study of self-thinning using coupled allometric equations: implications for coastal Douglas-fir stand dynamics. Canadian Journal of Forest Research. 33: 1661-1669.

Pretzsch, H.; Biber, P. 2005. A re-evaluation of Reineke's rule and stand density index. Forest Science. 51: 304-320.

Puettmann, K.J.; Hann, D.W.; Hibbs, D.E. 1993. Evaluation of the size-density relationships for pure red alder and Douglas-fir stands. Forest Science. 39: 7-27.

Reineke, L.H. 1933. Perfecting a stand-density index for even-age forests. Journal of Agricultural Research. 46: 627-638.

SAS Institute, Inc. 1988. SAS/ETS user's guide. Version 6. 1st ed. Cary, N.C.: SAS Institute.

SAS Institute, Inc. 1996. SAS/STAT guide for personal computers. Version 6.12. Cary, N.C.: SAS Institute.

Schabenberger, O.; Pierce, F.J. 2002. Contemporary statistical models for the plant and soil sciences. Boca Raton, FL: CRC Press. 738 p.

Smith, N.J.; Hann, D.W. 1984. A new analytical model based on the -3/2 power rule of self-thinning. Canadian Journal of Forest Research. 14: 605-609.

Strub, M.R.; Bredenkamp, B.V. 1985. Carrying capacity and thinning response of *Pinus taeda* in the CCT experiments. South African Forestry Journal. 2: 6-11.

VanderSchaaf, C.L. 2006. Modeling maximum size-density relationships of loblolly pine (*Pinus taeda* L.) plantations. Blacksburg, VA: Virginia Polytechnic Institute and State University. 171 p. Ph.D. Dissertation.

VanderSchaaf, C.L.; Burkhart, H.E. 2007. Comparison of methods to estimate Reineke's maximum size-density relationship species boundary line slope. Forest Science. 53: 435-442.

Weller, D.E. 1987. A reevaluation of the -3/2 power rule of plant self-thinning. Ecological Monographs. 57: 23-43.

Weller, D.E. 1990. Will the real self-thinning rule please stand up? – A reply to Osawa and Sugita. Ecology. 71: 1204-1207.

Weller, D.E. 1991. The self-thinning rule: Dead or unsupported? – A reply to Lonsdale. Ecology. 72: 747-750.

Westoby, M. 1984. The self-thinning rule. Advances in Ecological Research. 14: 167- 225.

Williams, R.A. 1996. Stand density index for loblolly pine plantations in north Louisiana. Southern Journal of Applied Forestry. 20: 110-113.

Wittwer, R.F.; Lynch, T.B.; Huebschmann, M.M. 1998. Stand density index for shortleaf pine (*Pinus echinata* Mill.) natural stands. In: Waldrop, T.A. (ed.). Proceedings of the ninth biennial southern silvicultural research conference. Gen. Tech. Rep. SRS-GTR-20. Asheville, NC: U.S. Department of Agriculture, Forest Service, Southern Research Station: 590-596

Yang, Y.; Titus, S.J. 2002. Maximum size-density relationship for constraining individual tree mortality functions. Forest Ecology and Management. 168: 259-273.

Zeide, B. 1985. Tolerance and self-tolerance of trees. Forest Ecology and Management. 13: 149-166.

Zeide, B. 1995. A relationship between size of trees and their number. Forest Ecology and Management. 72: 265-272.