TECHNICAL NOTE

Comparison of the Arithmetic and Geometric Means in Estimating Crown Diameter and Crown Cross-Sectional Area

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ABSTRACT

The use of the geometric and arithmetic means for estimating tree crown diameter and crown cross-sectional area were examined for trees with crown width measurements taken at the widest point of the crown and perpendicular to the widest point of the crown. The average difference between the geometric and arithmetic mean crown diameters was less than 0.2 ft in absolute value. The mean difference between crown cross-sectional areas based on the geometric and arithmetic mean crown diameters was less than 6.0 ft^2 in absolute value. At the plot level, the average difference between cumulative crown cross-sectional areas based on the geometric and arithmetic mean crown diameters amounted to less than 2.5% of the total plot area. The practical significance of these differences will depend on the final application in which the mean crown diameters are used.

Keywords: forest inventory, crown width, eccentricity, AM–GM inequality

Tree crown widths, and associated crown cross-sectional areas, have value in many forestry applications including canopy cover estimation (Gill et al. 2000), competition indices development (Biging and Dobbertin 1995), foliage biomass estimation (Grote and Reiter 2004), irradiance reduction calculations in urban landscapes (McPherson and Rowntree 1988), tree stem and stand attribute models (Sprinz and Burkhart 1987), silvicultural prescriptions, and wildlife habitat assessments. In many forest surveys, crown width is measured either as a diameter from drip line to drip line (e.g., Nowak et al. 2005) or as a radius from the bole to drip line (e.g., Francis 1988). Because the crowns of forest-grown trees are often highly irregular in shape, multiple widths usually are measured and the average is used in subsequent applications.

Average crown width can be calculated in multiple ways, but the most commonly used methods are the arithmetic (e.g., Bechtold 2003, Zarnoch et al. 2004) and geometric means (e.g., Sumida and Komiyama 1997, Marshall et al. 2003). The arithmetic mean of a set of values $d_1, d_2, \ldots, d_n$ is

$$\bar{d}_a = \frac{1}{n} \sum_{i=1}^{n} d_i, \tag{1}$$

and the corresponding geometric mean is

$$\bar{d}_g = \left(\prod_{i=1}^{n} d_i \right)^{1/n}. \tag{2}$$

The classical arithmetic mean–geometric mean (AM–GM) inequality states that $\bar{d}_g \leq \bar{d}_a$ for all positive real numbers. (There are many proofs of this inequality; Alzer 1996 provides one example.) Therefore, results of applications in which an average crown width is used are dependent on the selected averaging method.

The objective of this study was to compare the arithmetic and geometric mean crown diameters and subsequent effects of their use in calculating crown cross-sectional areas of forest-grown trees. In addition, because canopy cover estimates are an important component of forest assessments (Jennings et al. 1999), even to the point of being included in the international definition of "forest" (Food and Agriculture Organization 2006), an additional objective was to compare the cumulative crown cross-sectional area based on the two means when summed to the plot level.

Methods

Between 1990 and 1999 the US Forest Health Monitoring (FHM) Program collected forest health data on a nationwide network of detection monitoring ground plots (Riitters and Tkacz 2004). Each detection monitoring plot was a cluster of four subplots each with a radius of 24.0 ft. On each subplot, two crown widths were measured with a tape measure to the nearest 0.1 ft for each live tree ≥5.0-in. dbh or diameter at root collar. The first width was taken drip line to drip line at the widest part of the crown (d_w). The second diameter was taken perpendicular to the widest axis (d_{90}), also drip line to drip line (US Forest Service 1999). For this study, only data from the 1999 assessments in Alabama, Georgia, North Carolina, South Carolina, Tennessee, and Virginia were used. The resulting data set included a total of 6,540 trees measured across 276 plots (Table 1). More than one-half (59.2%) of the trees measured were hardwoods. The most abundant species were loblolly pine (28.4%), sweetgum (6.1%), red maple (5.7%), and yellow-poplar (5.6%).

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Table 1. Descriptive statistics for trees measured by the US Forest Health Monitoring Program in the Southern United States in 1999.

Species Group	No. of trees[a]	dbh[b] Mean (in.)	dbh[b] Range (in.)	d_w[c] Mean (ft)	d_w[c] Range (ft)	d_{90}[d] Mean (ft)	d_{90}[d] Range (ft)	Eccentricity Mean	Eccentricity Range
All Trees	6540	8.9	5.0–41.2	19.3	3–77	15.4	2–72	0.20	0.00–0.77
All softwoods	2666	8.2	5.0–33.4	14.8	4–52	11.9	3–51	0.19	0.00–0.77
Eastern white pine	45	13.8	5.0–33.4	25.1	11–50	21.2	10–40	0.15	0.00–0.42
Virginia pine	284	8.5	5.0–19.8	16.7	5–35	13.9	4–30	0.17	0.00–0.61
Loblolly pine	1855	7.8	5.0–28.9	14.0	5–52	11.2	3–51	0.19	0.00–0.67
Eastern redcedar	50	7.8	5.0–13.9	16.2	4–28	12.8	3–23	0.21	0.00–0.64
Shortleaf pine	137	9.6	5.1–20.7	16.0	5–41	12.5	3–27	0.22	0.00–0.60
Longleaf pine	44	10.1	5.1–18.5	17.6	9–38	13.4	5–30	0.23	0.00–0.69
All hardwoods	3874	9.3	5.0–41.2	22.5	3–77	17.8	2–72	0.20	0.00–0.75
Sugar maple	71	8.6	5.0–17.1	25.8	12–45	21.4	10–42	0.16	0.00–0.53
American beech	51	11.9	5.1–30.3	34.0	20–65	28.1	11–56	0.17	0.00–0.52
Shagbark hickory	44	8.9	5.0–15.8	22.5	11–44	18.3	8–40	0.17	0.00–0.58
Swamp tupelo	52	9.5	5.1–20.3	18.4	10–33	15.3	7–31	0.17	0.00–0.47
Scarlet oak	108	11.2	5.2–26.9	25.0	5–75	20.7	5–58	0.17	0.00–0.61
Flowering dogwood	38	6.2	5.0–9.4	22.2	9–39	18.1	8–34	0.18	0.00–0.61
Yellow-poplar	367	10.1	5.0–30.1	21.3	8–53	17.4	4–47	0.18	0.00–0.60
Northern red oak	82	12.0	5.2–30.1	26.6	12–65	21.5	8–55	0.19	0.00–0.61
Winged elm	26	8.2	5.1–16.4	26.1	17–43	20.9	14–39	0.19	0.05–0.56
Pignut hickory	102	9.6	5.0–23.0	23.4	11–55	18.9	7–50	0.19	0.00–0.57
Sassafras	28	7.0	5.0–15.9	14.7	7–35	11.4	6–22	0.19	0.00–0.66
Red maple	375	8.4	5.0–24.3	22.8	8–51	18.1	4–45	0.20	0.00–0.73
Blackgum	126	8.1	5.0–17.0	19.6	8–38	15.5	5–32	0.20	0.00–0.63
Chestnut oak	311	10.8	5.0–32.8	24.1	8–62	19.1	6–60	0.20	0.00–0.69
Mockernut hickory	111	9.2	5.0–23.0	22.7	9–61	18.1	6–50	0.20	0.00–0.57
Southern red oak	82	10.1	5.1–20.2	24.4	10–50	19.3	9–45	0.20	0.00–0.55
White oak	303	9.7	5.0–29.8	24.1	7–63	19.0	4–58	0.20	0.00–0.74
Green ash	66	10.0	5.0–25.9	22.2	5–64	17.7	5–59	0.21	0.00–0.72
American elm	26	7.1	5.0–13.2	23.5	5–37	18.2	4–33	0.21	0.00–0.62
Post oak	44	9.1	5.3–15.3	21.6	9–45	16.5	8–35	0.21	0.00–0.56
Black locust	31	9.9	5.0–21.8	20.3	11–42	15.7	5–31	0.21	0.00–0.62
Sweetgum	401	8.6	5.0–26.6	17.6	3–52	13.7	2–38	0.21	0.00–0.70
Black oak	85	9.5	5.0–21.1	21.1	8–41	16.4	4–38	0.21	0.00–0.70
White ash	27	9.1	5.2–15.9	22.3	10–36	17.4	8–26	0.21	0.04–0.52
Slippery elm	26	8.7	5.1–14.2	26.1	14–56	20.4	10–39	0.22	0.00–0.48
Water oak	140	9.2	5.0–32.1	23.6	8–60	18.5	5–55	0.22	0.00–0.75
Willow oak	30	12.7	5.0–41.2	27.5	13–77	21.8	8–72	0.22	0.02–0.56
Black cherry	88	8.2	5.1–19.4	19.2	9–36	14.5	6–28	0.23	0.00–0.68
American hornbeam	29	7.1	5.0–11.0	27.9	18–45	21.2	12–37	0.24	0.00–0.57
Sourwood	129	6.9	5.0–14.5	19.4	6–43	14.5	4–31	0.24	0.00–0.68

[a] Number of trees within each taxonomic group does not sum to the total because not all individual species are listed.
[b] Diameter at breast height.
[c] Crown width, drip line to drip line, at the widest part of the crown.
[d] Crown width, drip line to drip line, perpendicular to d_w.

For each tree in the data set, two average crown diameters were calculated: the arithmetic average, $\bar{d}_a$, defined as

$$\bar{d}_a = \frac{d_{90} + d_w}{2}, \tag{3}$$

and the geometric average, $\bar{d}_g$, defined as

$$\bar{d}_g = \sqrt{d_{90} \cdot d_w}. \tag{4}$$

Crown cross-sectional area was calculated for each tree using $\bar{d}_a$ and $\bar{d}_g$ and the formula for a circle,

$$A_i = \pi \cdot r_i^2, \tag{5}$$

where $r_i = 0.5 \cdot \bar{d}_i$ and $i = a, g$ for arithmetic and geometric, respectively. When only two widths have been measured, using the geometric mean radius in the formula for circular area is equivalent to calculating the area of an ellipse. In addition, a measure of eccentricity was calculated for each tree as

$$\text{eccentricity} = 1 - \frac{d_{90}}{d_w} \tag{6}$$

so that as d_{90} and d_w become more similar, eccentricity approaches zero and the "true" crown cross-sectional area becomes more circular.

The differences $\delta_d = (\bar{d}_g - \bar{d}_a)$ and $\delta_A = (A_g - A_a)$ were made for each tree and the mean differences ($\bar{\delta}_d$ and $\bar{\delta}_A$) were calculated for all trees combined, hardwood and softwood species groups, and individual species with 25 or more observations across at least 10 plots (a total of 36 species). For some applications, total crown cover at the plot level may be more important than individual tree crown area, so cumulative crown areas based on A_a and A_g were calculated and compared for each plot with forested conditions on all four subplots (a total of 170 plots). Given the AM–GM inequality, hypothesis tests to determine if $\bar{\delta}_d$ and $\bar{\delta}_A$ were statistically different from zero were deemed unnecessary.

Table 2. Mean and range of the difference between the geometric ($\bar{d}_g$) and arithmetic ($\bar{d}_a$) average crown diameters and corresponding crown cross-sectional areas.

Species group	No. of trees[a]	$\delta_d = (\bar{d}_g - \bar{d}_a)$ Mean (ft)	Range (ft)	$\delta_A = (A_g - A_a)$ Mean (ft^2)	Range (ft^2)
All trees	6540	−0.19	−5.02, 0.00	−5.6	−201.1, 0.0
All softwoods	2666	−0.13	−2.18, 0.00	−2.9	−78.6, 0.0
Virginia pine	284	−0.10	−1.28, 0.00	−2.5	−28.3, 0.0
Loblolly pine	1855	−0.12	−2.18, 0.00	−2.7	−78.6, 0.0
Eastern white pine	45	−0.12	−0.67, 0.00	−4.5	−23.8, 0.0
Shortleaf pine	137	−0.15	−1.47, 0.00	−3.9	−70.9, 0.0
Eastern redcedar	50	−0.17	−1.13, 0.00	−3.6	−19.6, 0.0
Longleaf pine	44	−0.27	−1.88, 0.00	−6.5	−78.6, 0.0
All hardwoods	3874	−0.23	−5.02, 0.00	−7.6	−201.1, 0.0
Swamp tupelo	52	−0.11	−0.63, 0.00	−2.7	−15.9, 0.0
Yellow−poplar	367	−0.17	−1.68, 0.00	−5.3	−78.6, 0.0
Sugar maple	71	−0.17	−2.17, 0.00	−6.5	−103.9, 0.0
Scarlet oak	108	−0.18	−1.67, 0.00	−6.4	−95.1, 0.0
Blackgum	126	−0.18	−2.07, 0.00	−5.3	−56.8, 0.0
Flowering dogwood	38	−0.18	−1.95, 0.00	−5.7	−56.8, 0.0
Pignut hickory	102	−0.19	−2.09, 0.00	−6.3	−78.6, 0.0
Sweetgum	401	−0.20	−3.33, 0.00	−5.2	−103.9, 0.0
Shagbark hickory	44	−0.20	−1.63, 0.00	−7.1	−70.9, 0.0
Sassafras	28	−0.20	−3.01, 0.00	−5.7	−103.9, 0.0
Mockernut hickory	111	−0.21	−1.61, 0.00	−6.6	−63.6, 0.0
Green ash	66	−0.22	−2.01, 0.00	−6.5	−56.8, 0.0
Southern red oak	82	−0.22	−1.63, 0.00	−8.2	−95.1, 0.0
Northern red oak	82	−0.22	−1.72, 0.00	−8.7	−78.6, 0.0
Red maple	375	−0.23	−5.02, 0.00	−7.7	−201.1, 0.0
Water oak	140	−0.24	−2.74, 0.00	−7.9	−122.8, 0.0
Winged elm	26	−0.24	−1.83, −0.01	−8.8	−63.6, −0.2
Chestnut oak	311	−0.25	−2.84, 0.00	−8.3	−78.6, 0.0
Black locust	31	−0.25	−3.08, 0.00	−8.4	−132.8, 0.0
American beech	51	−0.25	−2.10, 0.00	−12.8	−122.8, 0.0
White ash	27	−0.26	−1.18, 0.00	−8.0	−38.5, −0.2
Black cherry	88	−0.26	−2.09, 0.00	−7.2	−63.6, 0.0
White oak	303	−0.27	−4.25, 0.00	−9.1	−176.8, 0.0
Slippery elm	26	−0.28	−1.83, 0.00	−11.3	−122.8, 0.0
Willow oak	30	−0.28	−2.42, 0.00	−10.3	−113.1, −0.2
Post oak	44	−0.28	−2.50, 0.00	−10.3	−122.8, 0.0
Black oak	85	−0.29	−3.60, 0.00	−8.6	−122.8, 0.0
Sourwood	129	−0.31	−2.03, 0.00	−8.4	−63.6, 0.0
American elm	26	−0.32	−2.74, 0.00	−11.0	−103.9, 0.0
American hornbeam	29	−0.37	−1.67, 0.00	−13.1	−50.3, 0.0

[a] Number of trees within each taxonomic group does not sum to the total because not all individual species are listed.

Results

For all trees in the data set, d_w ranged from 3.0 to 77.0 ft, d_{90} ranged from 2.0 to 72.0 ft, and eccentricity ranged from 0 to 0.77 (Table 1). Deviations between $\bar{d}_g$ and $\bar{d}_a$ ranged from 0 to 5.02 ft (Table 2). The $\bar{\delta}_d$ was greater for the hardwoods than for the softwoods and was −0.19 ft overall (Table 2). Likewise, $\bar{\delta}_A$ was greater for the hardwoods than for the softwoods and was −5.6 ft^2 overall (Table 2). The difference between A_g and A_a increased as eccentricity increased and for any given eccentricity, larger d_w resulted in greater δ_A; however, the greatest differences were not necessarily associated with the largest crowns (Figure 1). When summed to the plot level the difference between the cumulative areas based on A_g and A_a was −166.9 ft^2, which equates to 2.3% of the total plot area. Although the percentage is small, this magnitude of difference could be important if, e.g., percent canopy cover is used to classify forest from nonforest and canopy cover is near the threshold that separates the two categories.

At the species level, d_w ranged from a minimum of 3.0 ft for sweetgum to a maximum of 77.0 ft for willow oak and d_{90} ranged from a minimum of 2.0 ft for sweetgum to a maximum of 72.0 ft for willow oak (Table 1). Eccentricity ranged from 0 for all but three species to 0.75 for water oak (Table 1). The $\bar{\delta}_d$ were less than 0.4 ft in absolute value and ranged from −0.37 ft for American hornbeam to −0.10 ft for Virginia pine (Table 2). The $\bar{\delta}_A$ followed the same pattern ranging from −13.1 ft^2 for American hornbeam to −2.5 ft^2 for Virginia pine (Table 2).

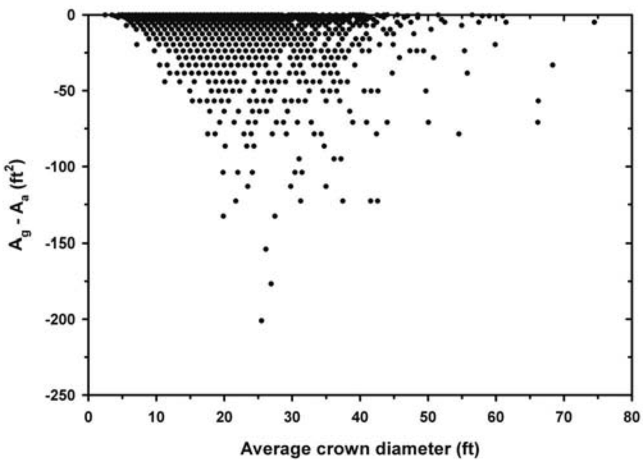

Figure 1. Scatterplot of the difference between the individual-tree crown cross-sectional areas based on the geometric mean crown diameter (A_g) and arithmetic mean crown diameter (A_a) versus the average of the geometric mean crown diameter and arithmetic mean crown diameter.

Conclusions

For the individual species examined and for all trees combined, $\bar{\delta}_d$ and $\bar{\delta}_A$ were minimal and may not be practically significant in most cases; however, testing the impact of the two averaging methods on applications beyond crown cross-sectional area calculations was beyond the scope of this study. Practitioners are encouraged to examine what differences may arise in their specific studies by using the two averaging methods. In the case of calculating crown cross-sectional areas, an accuracy assessment of using the geometric or arithmetic mean radius for calculating crown area can only be made against the true crown areas. Although the true crown areas were unknown here, a study by Biging and Wensel (1988) may provide some insight into what might be expected. Biging and Wensel (1988) examined the effect of eccentric tree boles on the estimation of basal area and observed that for a random sampling of trees the cross-sectional areas based on the geometric and arithmetic means of the major and minor axes of the bole overestimated the true basal area as measured by a compensating polar planimeter by 0.98 and 1.11%, respectively. For tree boles that were noticeably out-of-round (i.e., highly eccentric), they found that basal areas based on the geometric and arithmetic means overestimated the true basal area by 8.4 and 8.8%, respectively. Although not appreciably different from one another in either case, both the geometric and the arithmetic means considerably overestimated the true basal areas of eccentric boles. The tree crowns measured in the South showed a higher level of eccentricity on average than the out-of-round boles measured by Biging and Wensel (1988); hence, the estimated tree crown areas using either averaging method may overestimate true crown area by as much as 8%, if not more, on average.

Besides the methodology of measuring d_w and d_{90} used by FHM, a number of other methods for measuring crown dimensions have been used. For example, Jordan and Ducey (2007) measured four crown radii, the first in the direction away from the plot center and the remaining three at 90, 180, and 270° clockwise; Francis (1988) measured eight radii on the directions north, northeast, east, southeast, south, southwest, west, and northwest; and for the Urban Forest Effects model, Nowak et al. (2005) specify crown diameter measurements in two directions, north–south and east–west. Biging and Wensel (1988) observed that basal areas for eccentric trees based on a measurement of the longest axis or on an average that included the longest axis yielded less accurate estimates of the true basal area than estimates of the basal areas based on the length of the minor, or shortest, axis of the bole. Thus, to eliminate any potential bias in crown area estimations or other applications using an estimated crown width, further investigation should be made into the interaction between crown measurement protocols and the method for computing average crown width.

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CORRECTION

LEDUC, D., AND J. GOELZ. 2009. A Height–Diameter Curve for Longleaf Pine Plantations in the Gulf Coastal Plain *South. J. Appl. For.* 33(4):164–170.

It was discovered after publication that the δ and λ parameters were incorrectly transcribed from the analysis to the article even though they were used appropriately in all of the calculations. This results in changes to Table 5 and Equation 44, which are presented in their corrected forms below.

Table 5. Statistics that describe the estimated parameters for Equation 43

Parameter	Estimate	Approximate standard error
α	1.0491	0.00184
β	0.2172	0.00472
δ	4.0543	0.09710
λ	0.2541	0.00935

$$h = 4.5 + (90.3 - 4.5) * 1.0491\left(\frac{((11.0 - 0.5)/11.6)^{4.0543}}{0.2172 + ((11.0 - 0.5)/11.6)^{4.0543}}\right)^{0.2541}. \tag{44a}$$

This simplifies to

$$h = 4.5 + 85.8 * 1.0491\left(\frac{(0.9052)^{4.0543}}{0.2172 + (0.9052)^{4.0543}}\right)^{0.22541} \tag{44b}$$

and further to

$$h = 4.5 + 85.8 * 1.0491\left(\frac{0.6677}{0.2172 + 0.6677}\right)^{0.2541} \tag{44c}$$

then

$$h = 4.5 + 85.8 * 1.0491 * 0.9309, \tag{44d}$$