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Small Sample Properties of Asymptotically Efficient Estimators of the Parameters of a Bivariate Gaussian–Weibull Distribution

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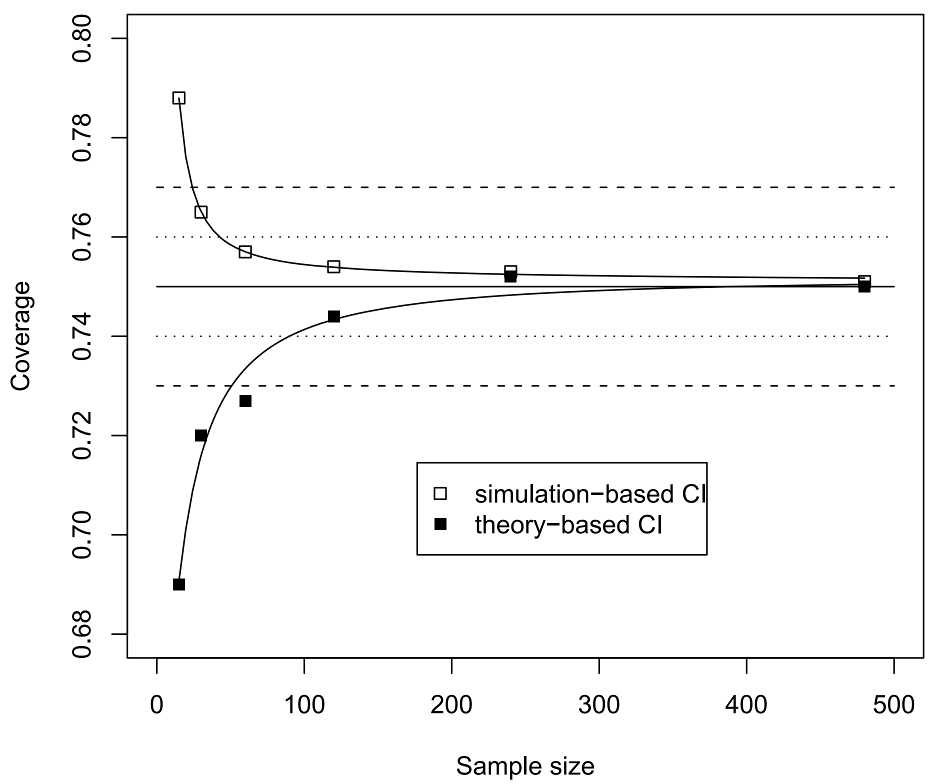

Abstract

Two important wood properties are stiffness (modulus of elasticity or MOE) and bending strength (modulus of rupture or MOR). In the past, MOE has often been modeled as a Gaussian and MOR as a lognormal or a two or three parameter Weibull. It is well known that MOE and MOR are positively correlated. To model the simultaneous behavior of MOE and MOR for the purposes of wood system reliability calculations, in a 2012 paper Verrill, Evans, Kretschmann, and Hatfield introduced a bivariate Gaussian–Weibull distribution and the associated pseudo-truncated Weibull. In that paper, they obtained asymptotically efficient estimators of the parameter vector of the bivariate Gaussian–Weibull. In this paper, we discuss computer simulations that investigated the small sample properties of these parameter estimates. We also discuss a Web-based computer program that obtains these estimates.

In the course of conducting the computer simulations we found that, as one would expect, assuming that the data truly have a bivariate Gaussian–Weibull distribution, bivariate Gaussian–Weibull estimators are superior to univariate (marginal) estimators. Also, under conditions likely to be encountered by wood scientists, univariate maximum likelihood Weibull estimators are generally superior to univariate ordinary least squares Weibull estimators. This latter result has implications for ASTM standard D 5457.

Keywords: Reliability, modulus of rupture, modulus of elasticity, normal distribution, Weibull distribution, likelihood methods, ASTM D 5457

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1 Introduction

Two important wood properties are stiffness (modulus of elasticity or MOE) and bending strength (modulus of rupture or MOR). In the past, MOE has often been modeled as a Gaussian and MOR as a lognormal or a two- or three-parameter Weibull. (See, for example, ASTM 2010a, Evans and Green 1988, and Green and Evans 1988.)

Design engineers must ensure that the loads to which wood systems are subjected rarely exceed the systems' strengths. To this end, ASTM D 2915 (ASTM 2010a), and ASTM D 245 or ASTM D 1990 (ASTM 2010b,c) describe the manner in which "allowable properties" are assigned to populations of structural lumber. In essence, an allowable strength property is calculated by estimating a fifth percentile of a population (actually a 95% content, lower, 75% tolerance bound) and then dividing that value by "duration of load" (aging) and safety factors. The intent is that the population can only be used in applications in which the load does not exceed the allowable property. Of course there are stochastic issues associated with variable loads, uncertainty in estimation, and the division of a percentile with no consideration of population variability. Thus, from a statistician's perspective, this is not an ideal approach to ensuring reliability of wood systems. However, it is the currently codified approach.

To apply this approach, one must obtain estimates of the fifth percentiles of MOR distributions. Currently, one method for obtaining estimates involves fitting a two-parameter Weibull distribution to a sample of MORs. To obtain this fit, either a maximum likelihood approach or a linear regression approach based on order statistics is permitted under ASTM D 5457 (ASTM 2010d).

Unfortunately, these methods are often applied to populations that are not really distributed as two-parameter Weibulls. For example, in the United States, construction grade 2 by 4's are often classified into visual categories—select structural, number 1, number 2—or into machine stress-rated (MSR) grades. In the case of MSR grades, MOE boundaries are selected, MOE is measured non-destructively, and boards are placed into categories based upon the MOE bins into which the boards fall. Because MOE and MOR are correlated, bins with higher MOE boundaries also tend to contain board populations with higher MOR values. The fifth percentiles of these MOR populations are sometimes estimated by fitting Weibull distributions to these populations. Statisticians recognize that this poses a problem. Even if the full population of lumber strengths were distributed as a Weibull, we would not expect that subpopulations formed by visual grades or MOE binning would continue to be distributed as Weibulls.

In fact, such a subpopulation is not distributed as a Weibull. Instead, if the full joint MOE–MOR population were distributed as a bivariate Gaussian–Weibull, the subpopulation would be distributed as a "pseudo-truncated Weibull" (PTW). Verrill, Evans, Kretschmann, and Hatfield (2012) obtained the distribution of a PTW and showed how to obtain estimates of its parameters by using asymptotically efficient methods to fit a bivariate Gaussian–Weibull to the full MOE–MOR distribution.

In this paper, we use computer simulations to investigate the small sample properties of these asymptotically efficient estimators. These simulations and their results are described in Sections 2 through 5. We then describe a Web-based computer program that calculates these estimates. This program is described in Section 6.

In the course of performing the computer simulations, we also found that bivariate Gaussian–Weibull estimators are superior to univariate (marginal) estimators, and that under conditions likely to be encountered by wood scientists, univariate maximum likelihood Weibull estimators are generally superior to univariate ordinary least squares Weibull estimators. These results are described in Sections 4 and 5.

As an aside, we remark that the bivariate Gaussian–Weibull distribution has uses other than as a generator of pseudo-truncated Weibulls. For example, engineers who are interested in simulating the performance of wood systems must begin with a model for the joint stiffness, strength distribution of the members of the system. Provided that we are considering the *full* population, a Gaussian–Weibull is one possible model for this joint distribution.

Bivariate Gaussian–Weibull distributions have not yet appeared in the literature. However, Gumbel (1960), Freund (1961), Marshall and Olkin (1967), Block and Basu (1974), Clayton (1978), Lee (1979), Hougaard (1986), Sarkar (1987), Lu and Bhattacharyya (1990), Patra and Dey (1999), Johnson *et al.* (1999), and others have previously investigated bivariate Weibulls.

We note that the bivariate Gaussian–Weibull distribution discussed in the current paper is not the only possible bivariate distribution with Gaussian and Weibull marginals. In essence we begin with a "Gaussian copula"—a bivariate uniform distribution generated by starting with a bivariate normal distribution and then applying normal cumulative distribution functions to its marginals. However, there is a large literature on alternative copulas (multivariate distributions with uniform marginals). See, for example, Nelsen (1999) and Jaworski (2010). (Also see Wang, Rennolls, and Tang (2008) for an application of copulas to joint models of tree heights and diameters.) These alternatives would lead to alternative bivariate Gaussian–Weibulls. Ultimately, the test of the usefulness of our proposed version of a Gaussian–Weibull for a particular application will depend on the match between the theoretical distribution and data. Still, we believe that the ability to fit the version discussed in the current paper represents a useful step in the construction and evaluation of bivariate Gaussian–Weibull distributions.

2 Simulations of Gaussian–Weibull Fits

In Verrill *et al.* (2012) we found that the joint probability density function of the proposed Gaussian–Weibull was

$$\begin{aligned} \text{gaussweib}(x, w; \mu, \sigma, \rho, \gamma, \beta) \quad &\equiv \quad \gamma^\beta \beta w^{\beta-1} \exp\left(-(\gamma w)^\beta\right) \\ &\times \frac{1}{\sqrt{2\pi}} \frac{1}{\sigma\sqrt{1-\rho^2}} \exp\left(-\left(\frac{x-\mu}{\sigma} - \rho y\right)^2 / (2(1-\rho^2))\right) \end{aligned} \tag{1}$$

where $x \in (-\infty, \infty)$ is the Gaussian value, $w > 0$ is the Weibull value, μ and σ are the mean and the standard deviation of the marginal Gaussian, ρ is the generating correlation, γ and β are the inverse scale and the shape of the marginal two-parameter Weibull (we assumed that $\beta > 1$ in our development),

$$y = \Phi^{-1}\left(1 - \exp\left(-(\gamma \times w)^{\beta}\right)\right) \tag{2}$$

and Φ denotes the N(0,1) cumulative distribution function. (In figures 1 – 9 of Verrill *et al.* (2012), we provide contour plots of the proposed bivariate Gaussian–Weibull distribution for coefficients of variation equal to 0.35, 0.25, and 0.15, and generating correlations equal to 0.5, 0.7, and 0.9.)

We also established that

$$\sqrt{n}\left(\begin{pmatrix}\hat{\mu}\\ \hat{\sigma}\\ \hat{\rho}\\ \hat{\gamma}\\ \hat{\beta}\end{pmatrix} - \begin{pmatrix}\mu\\ \sigma\\ \rho\\ \gamma\\ \beta\end{pmatrix}\right) \xrightarrow{D} \mathrm{N}(\mathbf{0}, I(\boldsymbol{\theta})^{-1}) \tag{3}$$

where $\boldsymbol{\theta} \equiv (\mu, \sigma, \rho, \gamma, \beta)^T$, $\hat{\mu}$ and $\hat{\sigma}$ are one-step Newton estimators based on the bivariate Gaussian–Weibull theory (that is, the gradient and Hessian used to calculate the Newton step correspond to the first and second partials of the full Gaussian–Weibull likelihood) that start at the standard univariate normal maximum likelihood estimators of the mean and standard deviation of a Gaussian, $\hat{\gamma}$ and $\hat{\beta}$ are one-step Newton estimators based on the bivariate Gaussian–Weibull theory that start at the standard univariate maximum likelihood estimators of 1/scale and shape for a Weibull, $\hat{\rho}$ is a one-step Newton estimator based on the bivariate Gaussian–Weibull theory that starts at the $\sqrt{n}$-consistent estimator of ρ introduced in appendix B of Verrill *et al.* (2012), and the elements of $I(\boldsymbol{\theta})$ are listed in Appendix A of the current paper. However, these are "asymptotic" or "large sample" results. To apply these results, we need to know what sample sizes are required to ensure that the large sample approximations are satisfactory. We are also interested in the biases, variabilities, and mean squared errors associated with these estimators.

To investigate these questions, we performed computer simulations. In particular, for coefficients of variation of 0.10, 0.20, 0.30, and 0.40 (for both the Gaussian and Weibull marginals), generating correlations of 0.5, 0.6, 0.7, 0.8, 0.9, and 0.95 (the "generating correlations" are approximately equal to the observed correlations between the Gaussian random variable and the Weibull random variable—see Table 1), and sample sizes of 15, 30, 60, 120, 240, and 480, we did the following:

1. Obtained the actual coverages of nominal 75%, 90%, 95%, and 99% confidence intervals on the five parameters of the distribution ($\mu, \sigma, \rho, \gamma$, and β).

2. Used these actual coverages at known sample sizes to estimate the sample sizes required to obtain actual coverages that fell in the narrow ranges [.74,.76], [.89,.91], [.94,.96], [.985,.995] and the wider ranges [.73,.77], [.88,.92], [.93,.97], [.98,.995].

3. Obtained estimates of the percent biases of the asymptotically efficient estimators.

4. Obtained estimates of the percent standard deviations of the estimators.

5. Calculated the ratios of the theoretical variances of the estimators of the parameters in the univariate and bivariate cases.

6. Obtained simulation estimates of the mean squared error ratios for the parameter estimators in the univariate and bivariate cases.

The simulations were based on 10,000 trials of each condition. In these simulations the generating μ was set at 100, the generating σ was set at 10, 20, 30, or 40 (for coefficients of variation equal to

0.10, 0.20, 0.30, or 0.40), the generating β was set at 12.154, 5.7974, 3.7138, or 2.6956 (for coefficients of variation equal to 0.10, 0.20, 0.30, or 0.40), and the generating γ was set at a value that would yield a Weibull median of 100 (given the β value). In the optimizations that were performed by the programs, no constraints were placed on the μ estimate, the ρ estimate was constrained to lie within the interval $[-1, 1]$, the σ and γ estimates were constrained to be non-negative, and the β estimate was constrained to lie within the interval $[1, 50.59]$ (that is, the coefficient of variation was constrained to lie between 1 and 0.025). Listings of the computer programs that were used to perform these simulations can be found at `http://www1.fpl.fs.fed.us/sim_gauss_weib.html`. (A single instance of a bivariate Gaussian–Weibull was generated as follows: Obtain independent N(0,1)'s, X_1, X_2, via a Gaussian random number generator. Set $X = \mu + \sigma X_1$ and $Y = \rho X_1 + \sqrt{1-\rho^2} X_2$. Then X is distributed as a $\mathrm{N}(\mu, \sigma^2)$, Y is distributed as a N(0,1), and their correlation is ρ. Now let $U = \Phi(Y)$. Then U is a Uniform(0,1) random variable that is correlated with X. Finally, let $W = (-\log(1-U))^{1\beta}/\gamma$. Then W is distributed as a Weibull with shape parameter β and scale parameter $1/\gamma$, and together X and W have our joint "bivariate Gaussian–Weibull" distribution.)

To perform the simulation work, we needed to be able to obtain maximum likelihood estimates of two-parameter Weibull parameters. To perform these optimizations, we needed initial estimates of the shape and scale parameters. We used the regression estimates specified in ASTM D 5457 (ASTM 2010d). This permitted us to compare the mean squared errors associated with the regression and maximum likelihood estimators, and to conclude that the regression estimators can be highly inefficient. This work is described in Section 5. Our (limited) results are in accord with the results from an extensive simulation study performed by Genschel and Meeker (2010). Other authors have found that for small samples ("small" from the perspective of wood research), regression methods (especially generalized least squares) can outperform maximum likelihood methods. See, for example, Engeman and Keefe (1982) and Al-Baidhani and Sinclair (1987).

3 n's Needed for Satisfactory Confidence Interval Coverages

The theory embodied in result (3) is asymptotic. That is, the approximation becomes better as sample sizes increase. Thus, for smaller samples, nominal confidence interval coverages based on (3) might not be good matches for actual confidence interval coverages.

To evaluate the sample sizes needed to yield actual confidence interval coverages that are good matches to nominal coverages, for coefficients of variation 0.10, 0.20, 0.30, and 0.40, and generating correlations 0.50, 0.60, 0.70, 0.80, 0.90, and 0.95, we performed simulations of samples of size 15, 30, 60, 120, 240, and 480. Then, for nominal coverages of 0.75, 0.90, 0.95, and 0.99, we used least squares to fit the linear model

$$\text{actual coverage} - \text{nominal coverage} = a_1/\sqrt{n} + a_2/n + a_3/n^{3/2} \tag{4}$$

to the data. Here n denotes the sample size. Two examples of the data and the associated fits to the data are given in Figures 1 and 2.

Given these fits, we could calculate the n needed to ensure that coverages lay in, for example, [.73,.77] (one of the "Wide" cases). In this case, if a curve approached the horizontal 0.77 line from above (as does the simulation-based curve in Fig. 1), then we would obtain the needed n by using a nonlinear equation solver to solve

$$0.77 = 0.75 + a_1/\sqrt{n} + a_2/n + a_3/n^{3/2}$$

for n where a_1, a_2, and a_3 were obtained from the least squares fit of model (4). That is, we would find the n at which the upper curved line in Figure 1 intersected the upper horizontal line. If a curve approached the horizontal 0.73 line from below (as does the theory-based curve in Fig. 1), then we would obtain the needed n by using a nonlinear equation solver to solve

$$0.73 = 0.75 + a_1/\sqrt{n} + a_2/n + a_3/n^{3/2}$$

for n. Note that if a curve already lay between 0.73 and 0.77 for $n = 15$, then we reported the needed n as 15. Thus, in our tables, 15 is the minimum value. Similar techniques were used to obtain the n needed to ensure that coverages lay in [.88,.92], [.93,.97], [.980,.995] ("Wide" cases) or [.74,.76], [.89,.91], [.94,.96], [.985,.995] ("Narrow" cases) for nominal 75%, 90%, 95%, 99% confidence intervals, respectively. Results from the complete simulations can be found at `http://www1.fpl.fs.fed.us/n_needed_gauss_weib.html`. Here, we report a summary of these simulations.

We considered three types of confidence intervals on the parameters.

1. Simulation-based (sim) confidence intervals:

$$\hat{\theta}_{\text{biv}} \pm z_{1-\alpha/2} \times \sigma_{\text{sim}} \tag{5}$$

2. Univariate (uni) asymptotic theory confidence intervals:

$$\hat{\theta}_{\text{uni}} \pm z_{1-\alpha/2} \times \sigma_{\text{uni}}/\sqrt{n} \tag{6}$$

3. Bivariate (biv) asymptotic theory confidence intervals:

$$\hat{\theta}_{\text{biv}} \pm z_{1-\alpha/2} \times \sigma_{\text{biv}}/\sqrt{n} \tag{7}$$

Here, θ denotes one of the five parameters (μ, σ, ρ, γ, β). $z_{1-\alpha/2}$ denotes the appropriate N(0,1) quantile. (For example, for a 95% confidence interval, $\alpha = 0.05$ and $z_{1-\alpha/2} = 1.96$.) $\hat{\theta}_{\text{uni}}$ denotes the maximum likelihood estimate of the parameter based on standard univariate theory. ($\hat{\mu}$ and $\hat{\sigma}$ from univariate maximum likelihood theory for a Gaussian, $\hat{\gamma}$ and $\hat{\beta}$ from univariate maximum likelihood theory for a 2-parameter Weibull.) $\hat{\theta}_{\text{biv}}$ denotes the asymptotically efficient estimate of the parameter based on the bivariate Gaussian–Weibull theory developed in Verrill *et al.* (2012).

σ_{uni} denotes the square root of the appropriate element of the appropriate asymptotic covariance matrix (obtained from the inverse of the information matrix) in the univariate case. σ_{biv} denotes the square root of the appropriate element of the asymptotic covariance matrix (obtained from the inverse of the information matrix) in the bivariate case. σ_{sim} is obtained from the simulation. It is given by

$$\sigma_{\text{sim}} = \sqrt{\sum_{i=1}^{10000} \left(\hat{\theta}_i - \bar{\hat{\theta}}\right)^2 /9999} \tag{8}$$

where $\hat{\theta}_i$ is the asymptotically efficient estimate of the parameter in the ith simulation trial and $\bar{\hat{\theta}}$ is the average of the 10000 $\hat{\theta}_i$'s.

For a given coefficient of variation, correlation, and sample size the actual coverage associated with the sim (for example) type of confidence interval was the fraction of trials in which $\hat{\theta}_{\text{biv}} \pm z_{1-\alpha/2} \times \sigma_{\text{sim}}$ included the true θ value.

For μ, we found that in all cases, a sample size of 15 was sufficient to ensure that the actual coverages of all three types of confidence interval lay within narrow bounds around the nominal coverages ([.74,.76], [.89,.91], [.94,.96], and [.985,.995] for nominal 75%, 90%, 95%, and 99% coverages, respectively).

For γ, we found that in most but not all cases sample sizes of 15 were sufficient. The n's needed did not appear to depend on the generating ρ, but they did depend on the coefficient of variation and the nominal confidence level. Details are provided in Table 2.

For σ and ρ, we found that needed n's did not depend on the coefficient of variation, but they did depend on the generating ρ and the nominal confidence level. Details are provided in Table 3.

For β, we found that needed n's did not depend on either the coefficient of variation or the generating ρ. They did depend on the nominal confidence level. The sim confidence intervals yielded adequate coverages for lower sample sizes than did the uni and biv intervals. Details are provided in Table 4.

In designing an experiment in which one wants to obtain confidence intervals on bivariate Gaussian–Weibull parameters, one should consult Tables 2–4 to choose an adequate sample size. However, our Web program provides some protection against inadequate sample sizes at the analysis stage. It provides nominal 75%, 90%, 95%, and 99% simulation and theoretical confidence intervals on the parameters. However, it also provides on-the-fly simulation estimates of the actual coverages of these confidence intervals. (Details on the manner in which on-the-fly simulation estimates of coverages are calculated are provided in point 7 of Appendix B.) If these simulation estimates of coverages significantly diverge from the nominal coverages, then this fact should be reported and the simulation estimates of coverages should be used rather than the nominal coverages.

It is *very important* to draw a distinction between two types of "needed sample sizes." We have been talking about the n's needed to ensure that we can trust the nominal confidence levels. That is, we want the sample size that ensures that a confidence interval constructed to cover the true value of a parameter at least 95% (for example) of the time really does cover the parameter at least 95% of the time. This is distinct from a separate sample size issue. The separate issue is whether the sample size is large enough to ensure that a confidence interval is narrow enough or that our ability to detect a difference (statistical power) is large enough. It is quite possible that the n needed to ensure that actual confidence levels are close to nominal confidence levels could be as low as 15 while the n needed to ensure that confidence interval widths are sufficiently small could be much higher than 15. These are two separate issues. We do not consider the second issue in this paper.

4 Biases, Variances, and Mean Squared Errors of Parameter Estimates

Likelihood theory tells us that our estimators are *asymptotically* efficient. However, this is large sample theory and will hold to a greater or lesser extent for small samples. In the course of our coverage simulations, we also investigated the biases, variances, and mean squared errors of the estimators for coefficients of variation 0.10, 0.20, 0.30, and 0.40, generating correlations 0.50, 0.60, 0.70, 0.80, 0.90, and 0.95, and sample sizes 15, 30, 60, 120, 240, and 480.

The percent biases in the uni and biv estimators are provided in Tables 5–8. Clearly, the absolute percent biases decrease as sample sizes increase. (Also see Figs. 3 and 4.) The generating correlation has little to no effect for the μ, σ, and γ estimators. It has a more significant effect on the bias of the ρ estimator (absolute percent bias decreases as generating correlation increases). See Figure 3. It has a subtle effect on the bias of the β estimators—the bias in the biv estimator declines below that of the uni estimator as the generating correlation increases. The only noticeable effect of an increase in the coefficient of variation is an increase in the (small) biases in the estimators of γ as the coefficient of variation increases. See Figure 4.

The percent standard deviations (sample standard deviation times 100 divided by generating parameter) associated with the uni and biv estimators are provided in Tables 9–12. The generating correlation has no effect on these values for the μ and γ estimators. It has a very large effect on the percent standard deviation for the ρ estimator—the percent standard deviation decreases significantly as the generating correlation increases. See Figure 5. It has a more subtle effect on the percent standard deviations of the σ and β estimators—the percent standard deviations in the biv estimators of these two parameters decline below those of the uni estimators as the generating correlation increases. The coefficient of variation has no effect on percent standard deviation for σ, ρ, and β. However, as the coefficent of variation increases, the percent standard deviations of the μ and γ estimators increase. See Figure 6.

We were interested in any efficiency increases that we could obtain by fitting the bivariate Gaussian–Weibull rather than by fitting the marginal Gaussian and Weibull distributions separately. For large samples, the relative efficiency ratio is given by the asymptotic variabilities that we obtain from the inverses of the information matrices. For small samples, we can look at the ratios of mean squared errors obtained from the simulations. These ratios did not depend (except in a negligible fashion) on the coefficients of variation. We summarize the results in Table 13. Note that for smaller sample sizes, there is bias in the estimates so the mean squared error ratios differ from the relative asymptotic efficiencies. However, as the sample sizes increase, the simulation estimates of mean squared error ratios approach the expected asymptotic ratios. Also note that for high generating correlations, there can be important efficiency increases in the σ and β estimations when we take a bivariate Gaussian–Weibull approach rather than fitting the Gaussian and Weibull data separately. See Figure 7.

5 Regression and Maximum Likelihood (ML) Estimators of the Parameters of a Two-Parameter Weibull

As noted above, to obtain asymptotically efficient bivariate Gaussian–Weibull estimators for γ and β, we begin with univariate ML estimates, and to obtain these, we begin with regression estimates. The regression estimates are based on the approximations

$$\log(w_{in}) \approx -\log(\gamma) + \frac{1}{\beta} \times \log(-\log(1-(i-0.3)/(n+0.4))) \tag{9}$$

and

$$\log(-\log(1-(i-0.3)/(n+0.4))) \approx \beta\log(\gamma) + \beta \times \log(w_{in}) \tag{10}$$

where w_{in} is the ith order statistic in a sample of n from a Weibull population with scale parameter $1/\gamma$ and shape parameter β. (These approximations are in turn based on the fact that if U is a Uniform(0,1) random variable and F_W is the distribution function of a Weibull(γ,β) then $F_W^{-1}(U)$ is distributed as a Weibull(γ,β). See, for example, the discussion of Weibull probability plots in D'Agostino and Stephens (1986).)

Approximation (9) suggests that we can regress

$$y_i = \log(w_{in})$$

on

$$x_i = \log(-\log(1-(i-0.3)/(n+0.4)))$$

to obtain

$$\hat{\beta} = 1/\hat{b}$$

and

$$\hat{\gamma} = \exp(-\hat{a})$$

where $\hat{a}, \hat{b}$ are the intercept and the slope from the regression. We refer to this as regression approach 1. Regression approach 1 is permitted as an alternative to a maximum likelihood approach to estimating γ and β in ASTM standard D 5457 (ASTM 2010d).

Approximation (10) suggests that we can regress

$$y_i = \log(-\log(1 - (i - 0.3)/(n + 0.4)))$$

on

$$x_i = \log(w_{in})$$

to obtain

$$\hat{\beta} = \hat{b}$$

and

$$\hat{\gamma} = \exp\left(\hat{a}/\hat{b}\right)$$

where $\hat{a}, \hat{b}$ are the intercept and the slope from the regression. This is regression approach 2.

Our simulations permitted us to compare the mean squared errors of the regression 1, regression 2, and univariate maximum likelihood (ML) estimators of γ and β. The ratios of the regression 1 to ML and regression 2 to ML estimators are reported in Table 14 and plotted in Figure 8 (there will be no correlation effect and we found no coefficient of variation effect). These ratios indicate that the regression estimates can be competitive with the maximum likelihood estimates for samples of size 15. However, for samples of size 30 and larger, the maximum likelihood estimators are superior to the regression estimators, and *much* superior for β. We believe that ASTM D 5457 (ASTM 2010d) should be modified to reflect these facts.

We note that other authors have previously investigated the efficiency of regression estimators of Weibull parameters. Our limited results are in accord with the results from an extensive simulation study performed by Genschel and Meeker (2010). Other authors have found that for small samples ("small" from the perspective of wood research), generalized least squares techniques can outperform maximum likelihood techniques. See, for example, Engeman and Keefe (1982) and Al-Baidhani and Sinclair (1987). Any modification of ASTM D 5457 (ASTM 2010d) would also have to take into account these results. (It is possible that other standards (e.g., IEC 2008), should also be modified. However, censoring and very small samples cloud the issue, and we can make no general recommendations for other standards based on our limited studies. However, see Genschel and Meeker (2010) for results that are more broadly relevant.)

For sample sizes of 15 or larger, we found that regression approach 1 is as good as or better than regression approach 2 for estimating γ. For sample sizes of 30 or larger, regression approach 1 is as good as or better than regression approach 2 for estimating β. See Table 14 and Figure 8. Lawrence and Shier (1981) compared the two regression approaches for samples of size 20, 30, 40, 50, and 100. They used the Hazen plotting position, $(i - .5)/n$, rather than the Benard position, $(i - .3)/(n + .4)$, and they performed 100 trials rather than 10,000. Their results conflict with ours (they find an advantage for regression approach 2 more often than we do). We suspect that this is due to the limited number of trials that they conducted. We reran a portion of their simulation with 10,000 trials per condition. In Table 15, we compare our results with those that they reported in table III of their paper.

6 Web Program to Estimate the Parameters of a Bivariate Gaussian–Weibull

Based on the theory in Verrill *et al.* (2012), we have developed a computer program that obtains asymptotically efficient estimates of the parameters of a bivariate Gaussian–Weibull. The program also returns nominal 75%, 90%, 95%, and 99% sim and biv (Equations (5) and (7)) confidence intervals on the parameters. Finally, it performs simulations to obtain estimates of the actual coverages of these intervals. Algorithmic details of the program are provided in Appendix B. The Web program can be run at `http://www1.fpl.fs.fed.us/fit_gauss_weib.html`. The code for a standalone FORTRAN program that performs these same functions can be found at `http://www1.fpl.fs.fed.us/fit_gauss_weib_code.html`.

In Figures 9–13 we provide screen shots of the Web program. There are are five fields that need to be filled:

1. A user must specify a data file name. The data file must be a txt file that contains two columns. (By default, Notepad and Emacs create txt files. Wordpad and Word can be directed to create txt files.) The first column must contain the values from the Gaussian distribution. The second column must contain the corresponding values from the Weibull distribution. The user must have previously sent the data file by FTP to our Web server. Directions for doing this are provided at the "provide the data file" link near the top of the Web page.

2. A user can specify a results file name. Directions for retrieving the results file appear above the results field on the page. In fact, however, a user need not specify a results file as the results are displayed in tabular form after the execute button is clicked. These results can be printed or saved to the user's machine.

3. The user must specify the sample size (n in this paper). Currently, the program cannot handle sample sizes larger than 2000 observations. If this presents a problem for you, please contact Steve Verrill at sverrill@fs.fed.us.

4. The user must specify the number of trials in the simulation. Currently, this cannot exceed 10000. If this presents a problem for you, please contact Steve Verrill at sverrill@fs.fed.us.

5. The user must provide an integer starting value for the random number generator. This istart value cannot exceed $2^{31} - 1 = 2147483647$.

After filling these five fields, the user clicks the execute button and the program runs. Results are then displayed in tabular form. The program produces results for four confidence levels—75%, 90%, 95%, and 99%. For each confidence level, it produces two tables. The first table (see, for example, Fig. 12) contains the asymptotically efficient estimates of the five parameters together with simulation-based and asymptotic theory-based confidence intervals on those parameters. The second table (see, for example, Fig. 13) contains simulation estimates of the actual coverages (rather than the nominal 75%, 90%, 95%, or 99%) of the two types of confidence intervals. It also contains confidence intervals (based on the arcsin square root transformation) on these actual coverages. If simulation estimates of actual coverages differ significantly from nominal coverages, then this fact should be reported and the simulation estimates of coverages should be used rather than the nominal coverages.

Important The response *will not* be immediate. Because simulations are being run, there will be a delay before the results appear. An approximate formula for the number of seconds needed to perform the simulations is $(3.3 + 0.36 \times n) \times N/10000$ where n is the sample size and N is the number of trials. The time needed to run 10000 trials of samples of size 15 is approximately 9 seconds. The time needed to run 10000 trials of samples of size 480 is approximately 177 seconds.

If you encounter problems while running this program, please contact Steve Verrill at sverrill@fs.fed.us or 608-231-9375. As of June 2012, the program is a beta program. That is, we have tried to be very careful in its development. However, it might still contain bugs. If you believe that you have encountered a bug, please contact us.

7 Summary

In the context of wood strength modeling, Verrill *et al.* (2012) introduced a bivariate Gaussian–Weibull distribution and the associated pseudo-truncated Weibull distribution. In that paper, we also developed asymptotically efficient estimators of the parameters of the bivariate Gaussian–Weibull. In this paper, we have discussed a Web-based computer program that implements the asymptotically efficient estimation technique. We have also discussed computer simulations that investigate the small sample properties of this technique.

In the course of conducting these computer simulations we also found that, as one would expect, bivariate Gaussian–Weibull estimators are superior to univariate (marginal) estimators, and that, under conditions likely to be encountered by wood scientists, univariate maximum likelihood Weibull estimators are generally superior to univariate ordinary least squares Weibull estimators. This latter result suggests that ASTM standard D 5457 should be modified to reflect this superiority of maximum likelihood Weibull estimators.

In a future paper, we will investigate the question of whether allowable property estimates based on a Weibull assumption can be poor if the strength population is actually a pseudo-truncated Weibull population.

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9 Appendix A–Elements of the Information Matrix

Denote the information by

$$I(\boldsymbol{\theta}) \equiv \begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} & a_{15} \\ a_{12} & a_{22} & a_{23} & a_{24} & a_{25} \\ a_{13} & a_{23} & a_{33} & a_{34} & a_{35} \\ a_{14} & a_{24} & a_{34} & a_{44} & a_{45} \\ a_{15} & a_{25} & a_{35} & a_{45} & a_{55} \end{pmatrix}$$

Then, from appendices D and E2 of Verrill *et al.* (2012) we have

$$a_{11} = -E\left(\frac{\partial^2 \log(f(x,w))}{\partial\mu^2}\right) = \frac{1}{\sigma^2(1-\rho^2)} \tag{11}$$

$$a_{22} = -E\left(\frac{\partial^2 \log(f(x,w))}{\partial\sigma^2}\right) = \frac{2-\rho^2}{\sigma^2(1-\rho^2)} \tag{12}$$

$$a_{33} = -E\left(\frac{\partial^2 \log(f(x,w))}{\partial\rho^2}\right) = \frac{(1+\rho^2)}{(1-\rho^2)^2} \tag{13}$$

$$a_{44} = -E\left(\frac{\partial^2 \log(f(x,w))}{\partial\gamma^2}\right) = \frac{\rho^2}{1-\rho^2}E\left(\left(\frac{\partial y}{\partial\gamma}\right)^2\right) + \frac{\beta^2}{\gamma^2} \tag{14}$$

where y is given by (2) and

$$\frac{\partial y}{\partial\gamma} = \sqrt{2\pi} \times \beta\gamma^{\beta-1} \times w^\beta \times \exp\left(-(\gamma w)^\beta\right) \times \exp\left(y^2/2\right)$$

$$\begin{aligned} a_{55} = -E\left(\frac{\partial^2 \log(f(x,w))}{\partial\beta^2}\right) &= \frac{\rho^2}{1-\rho^2}E\left(\left(\frac{\partial y}{\partial\beta}\right)^2\right) + \frac{1}{\beta^2} \\ &+E\left((\log(w))^2\right) + \frac{2}{\beta}E\left(\log(w)\right) \\ &+2\log(\gamma)E\left(\log(w)\right) + \frac{2\log(\gamma)}{\beta} \\ &+\left(\log(\gamma)\right)^2 \end{aligned} \tag{15}$$

where

$$\frac{\partial y}{\partial\beta} = \sqrt{2\pi} \times (\gamma w)^\beta \log(\gamma w) \times \exp\left(-(\gamma w)^\beta\right) \times \exp\left(y^2/2\right)$$

$$a_{12} = -E\left(\frac{\partial^2 \log(f(x,w))}{\partial\mu\partial\sigma}\right) = 0 \tag{16}$$

$$a_{13} = -E\left(\frac{\partial^2 \log(f(x,w))}{\partial\mu\partial\rho}\right) = 0 \tag{17}$$

$$a_{14} = -E\left(\frac{\partial^2 \log(f(x,w))}{\partial\mu\partial\gamma}\right) = \frac{\rho}{1-\rho^2}E\left(\frac{\partial y}{\partial\gamma}\right)\left(\frac{1}{\sigma}\right) \tag{18}$$

$$a_{15} = -E\left(\frac{\partial^2 \log(f(x,w))}{\partial\mu\partial\beta}\right) = \frac{\rho}{1-\rho^2}E\left(\frac{\partial y}{\partial\beta}\right)\left(\frac{1}{\sigma}\right) \quad (19)$$

$$a_{23} = -E\left(\frac{\partial^2 \log(f(x,w))}{\partial\sigma\partial\rho}\right) = \frac{-\rho}{\sigma(1-\rho^2)} \quad (20)$$

$$a_{24} = -E\left(\frac{\partial^2 \log(f(x,w))}{\partial\sigma\partial\gamma}\right) = \frac{\rho^2}{\sigma(1-\rho^2)}E\left(y\frac{\partial y}{\partial\gamma}\right) \quad (21)$$

$$a_{25} = -E\left(\frac{\partial^2 \log(f(x,w))}{\partial\sigma\partial\beta}\right) = \frac{\rho^2}{\sigma(1-\rho^2)}E\left(y\frac{\partial y}{\partial\beta}\right) \quad (22)$$

$$a_{34} = -E\left(\frac{\partial^2 \log(f(x,w))}{\partial\rho\partial\gamma}\right) = \frac{\rho}{1-\rho^2} \times E\left(y\frac{\partial y}{\partial\gamma}\right) \quad (23)$$

$$a_{35} = -E\left(\frac{\partial^2 \log(f(x,w))}{\partial\rho\partial\beta}\right) = \frac{\rho}{1-\rho^2} \times E\left(y\frac{\partial y}{\partial\beta}\right) \quad (24)$$

$$a_{45} = -E\left(\frac{\partial^2 \log(f(x,w))}{\partial\gamma\partial\beta}\right) = \frac{\rho^2}{1-\rho^2}E\left(\frac{\partial y}{\partial\gamma}\frac{\partial y}{\partial\beta}\right) + \frac{\beta}{\gamma} \times E(\log(w)) + \frac{1}{\gamma} + \frac{\beta\log(\gamma)}{\gamma} \quad (25)$$

10 Appendix B—The Algorithm

The program is straightforward. It performs the following tasks:

1. It obtains initital estimates of μ and σ. These are simply the standard univariate maximum likelihood estimates—$\bar{x}$ and $\sqrt{\sum_{i=1}^{n}(x_i-\bar{x})^2/n}$.

2. It obtains initial estimates of γ and β. First, it obtains regression estimates. We have

$$F_W(w) = 1 - \exp\left(-(\gamma \times w)^\beta\right)$$

so

$$F_W^{-1}(y) = (-\log(1-y))^{1/\beta}/\gamma$$

If U denotes a Uniform(0,1) random variable, then we know that $F_W^{-1}(U)$ is distributed as W. Thus, if U_{in} denotes the ith order statistic from a sample of n Uniform(0,1)'s, $F_W^{-1}(U_{in})$ is the ith order statistic from a sample of n Weibulls. So, if we approximate U_{in} by $(i-.3)/(n+.4)$, we obtain

$$W_{in} \approx F_W^{-1}((i-.3)/(n+.4))$$

where W_{in} is the ith order statistic from the Weibull distribution. Thus,

$$W_{in} \approx (-\log(1-(i-.3)/(n+.4)))^{1/\beta}/\gamma$$

and

$$\log(W_{in}) \approx (1/\beta) \times \log(-\log(1-(i-.3)/(n+.4))) - \log(\gamma)$$

So, if we regress the $\log(W_{in})$ on the $\log(-\log(1-(i-.3)/(n+.4)))$ we obtain

$$\beta \approx 1/\hat{b} \quad (26)$$

and

$$\gamma \approx \exp(-\hat{a}) \tag{27}$$

where $\hat{b}$ and $\hat{a}$ are the slope and intercept from the regression.

We can then use the β and γ from Equations (26) and (27) as starting values in a nonlinear optimization of the two parameter Weibull likelihood function. We use the public domain nonlinear optimizer UNCMIN to perform this optimization. The results from this optimization are univariate maximum likelihood estimates of β and γ. We use these as the starting values in the bivariate optimization.

3. We obtain an initial estimate of ρ. We do this by calculating the sample correlation between the univariate Gaussian values, x_i, and the corresponding transformed Weibull values:

$$y_i = \Phi^{-1}\left(1 - \exp\left(-(\hat{\gamma} \times w_i)^{\hat{\beta}}\right)\right)$$

where $\hat{\beta}$ and $\hat{\gamma}$ are the univariate maximum likelihood estimators of β and γ.

4. Given these $\sqrt{n}$-consistent initial estimates of μ, σ, ρ, γ, and β, UNCMIN then employs the Newton method modified by a backtracking line search technique to find a maximum of the of the bivariate likelihood function. It can be shown (see Appendix C) that this approach ensures that we are left with (at least) a $\sqrt{n}$-consistent estimate of the parameter vector. We then perform a full Newton step to obtain our final estimate of the parameter vector.

5. We calculate the information matrix from the results in Appendix A. To do so, we need to perform a number of numerical integrations — $E\left(\left(\frac{\partial y}{\partial \gamma}\right)^2\right)$, $E\left(\left(\frac{\partial y}{\partial \beta}\right)^2\right)$, $E\left(\frac{\partial y}{\partial \gamma}\frac{\partial y}{\partial \beta}\right)$, $E\left(y\frac{\partial y}{\partial \gamma}\right)$, $E\left(y\frac{\partial y}{\partial \beta}\right)$, $E\left(\frac{\partial y}{\partial \gamma}\right)$, and $E\left(\frac{\partial y}{\partial \beta}\right)$. We use the QUADPACK routine dqags to perform these. $E\left((\log(w))^2\right)$ and $E\left(\log(w)\right)$ are related to the Euler–Mascheroni constant (see Verrill *et al.* 2012) and can be calculated from it.

6. We invert the information matrix using the LINPACK routines dpofa and dpodi.

7. We perform a simulation that has two purposes. First, it permits us to calculate the σ_{sim} used to calculate the simulation-based confidence intervals (5). Second, it permits us to estimate the coverage of both the simulation-based and theory-based (7) confidence intervals.

Let n denote the size of a sample provided by a user of the Web program. From the sample, the program first obtains asymptotically efficient estimates $\hat{\mu}$, $\hat{\sigma}$, $\hat{\rho}$, $\hat{\gamma}$, and $\hat{\beta}$ as described above. From Equations (11)–(25), it also obtains an estimate of the information matrix and (via dpofa and dpodi) its inverse. This yields $\hat{\sigma}_{\text{biv}}$ for all five parameters. Then it generates N samples of size n from a bivariate Gaussian–Weibull distribution with parameters $\hat{\mu}$, $\hat{\sigma}$, $\hat{\rho}$, $\hat{\gamma}$, and $\hat{\beta}$. (Here, N is the number of trials specified by the user.) For the ith generated sample, it obtains estimates (as above) of the five parameters, $\hat{\hat{\mu}}_i$, $\hat{\hat{\sigma}}_i$, $\hat{\hat{\rho}}_i$, $\hat{\hat{\gamma}}_i$, and $\hat{\hat{\beta}}_i$. For each of the five parameters, the program calculates

$$\hat{\sigma}_{\text{sim}} = \sqrt{\sum_{i=1}^{N}\left(\hat{\hat{\theta}}_i - \bar{\hat{\hat{\theta}}}\right)^2/(N-1)}$$

For all five parameters, the program reports the theory-based confidence intervals

$$\hat{\theta} \pm z_{1-\alpha/2} \times \hat{\sigma}_{\text{biv}}/\sqrt{n}$$

and the simulation-based confidence intervals

$$\hat{\theta} \pm z_{1-\alpha/2} \times \hat{\sigma}_{\text{sim}}$$

It obtains estimates of the coverages of these intervals by going back through the N groups of $\hat{\hat{\mu}}_i$, $\hat{\hat{\sigma}}_i$, $\hat{\hat{\rho}}_i$, $\hat{\hat{\gamma}}_i$, and $\hat{\hat{\beta}}_i$ and calculating the fraction of the time in which (for the theory-based intervals)

$$\hat{\theta} \in \hat{\hat{\theta}}_i \pm z_{1-\alpha/2} \times \hat{\sigma}_{\text{biv}}/\sqrt{n} \tag{28}$$

or (for the simulation-based confidence intervals)

$$\hat{\theta} \in \hat{\hat{\theta}}_i \pm z_{1-\alpha/2} \times \hat{\sigma}_{\text{sim}}$$

Note that, ideally, the $\hat{\sigma}_{\text{biv}}$ in Equation (28) should be based on the ith simulation sample of size n rather than the original sample of data. However, the calculation of the information matrix involves numerical integrations and, if done for a number of trials, could lead to an additional slowdown.

11 Appendix C—UNCMIN and $\sqrt{n}$-consistent Estimators

In Appendix B we noted that the optimization program UNCMIN employs the Newton method modified by a backtracking line search technique to find a maximum of the bivariate likelihood function (actually, it finds the minimum of the negative log likelihood). Thus, at each iteration, rather than taking the full Newton step

$$-\mathbf{H}^{-1}\mathbf{g}$$

where $\mathbf{g}$ is the gradient of the negative log likelihood and $\mathbf{H}$ is the corresponding Hessian, the modified algorithm takes a step of the form

$$-\delta \times \mathbf{H}^{-1}\mathbf{g}$$

where $\delta \in (0, 1]$. Our claim is that such a "partial Newton step" leaves us with an estimate of the parameter vector that is still $\sqrt{n}$-consistent. We will only sketch a proof of the claim here.

Let $\boldsymbol{\theta}_{n,c}$ denote the vector of $\sqrt{n}$-consistent initial estimates of the parameters, $\boldsymbol{\theta}_{n,\delta,\text{Newt}}$ denote the result of a partial Newton step from $\boldsymbol{\theta}_{n,c}$, and $\boldsymbol{\theta}_0$ denote the true vector of parameters. Then, by the definition of $\sqrt{n}$-consistency, we have

$$\sqrt{n}(\boldsymbol{\theta}_{n,c} - \boldsymbol{\theta}_0) = \mathbf{O}_p(1) \tag{29}$$

Thus, if we can show

$$\sqrt{n}(\boldsymbol{\theta}_{n,\delta,\text{Newt}} - \boldsymbol{\theta}_{n,c}) = \mathbf{O}_p(1) \tag{30}$$

we will have

$$\sqrt{n}(\boldsymbol{\theta}_{n,\delta,\text{Newt}} - \boldsymbol{\theta}_0) = \mathbf{O}_p(1) \tag{31}$$

which is what we are claiming.

We have

$$\begin{aligned}
\sqrt{n}(\boldsymbol{\theta}_{n,\delta,\text{Newt}} - \boldsymbol{\theta}_{n,c}) &= \sqrt{n}(\boldsymbol{\theta}_{n,c} - \delta\mathbf{H}_{n,c}^{-1}\mathbf{g}_{n,c} - \boldsymbol{\theta}_{n,c}) \\
&= \sqrt{n}\left(-\delta\mathbf{H}_{n,c}^{-1}\mathbf{g}_{n,c}\right) \\
&= \delta\left(-\mathbf{H}_{n,c}/n\right)^{-1}\sqrt{n}\mathbf{g}_{n,c}/n
\end{aligned} \tag{32}$$

Under the conditions needed to establish our Theorem 1 in Verrill *et al.* (2012), we can use Taylor expansions and the law of large numbers to show that

$$(-\mathbf{H}_{n,c}/n)^{-1} \xrightarrow{p} I(\boldsymbol{\theta})^{-1} \tag{33}$$

Similarly, we can show that

$$\sqrt{n}\left(\mathbf{g}_{n,c}/n - \mathbf{g}_{\boldsymbol{\theta}_0}/n\right) = O_p(1) \tag{34}$$

and

$$\sqrt{n}\mathbf{g}_{\boldsymbol{\theta}_0}/n \xrightarrow{D} N(\mathbf{0}, I(\boldsymbol{\theta})) \tag{35}$$

Results (32)–(35) establish result (30), which is what we needed to obtain result (31).

	Generating Correlation			
CV	.5	.7	.9	1.0
.05	.489	.685	.881	.979
.10	.494	.690	.888	.986
.15	.498	.695	.893	.992
.20	.498	.697	.897	.996
.25	.497	.699	.899	.999
.30	.498	.699	.899	.999
.35	.499	.699	.899	.999
.40	.498	.698	.897	.997

Table 1. Sample correlations between Gaussian and Weibull. Based on 1,000,000 trials. CV denotes coefficient of variation. (For the Gaussian, CV $= \sigma/\mu$. For the Weibull, CV depends solely on the shape parameter, β. Higher CV's correspond to lower β's.)

COV	Nominal confidence level	Interval type	Bounds Wide	Bounds Narrow
.10	75	sim	15	16
		uni	15	15
		biv	15	15
	90	sim	15	15
		uni	15	15
		biv	15	15
	95	sim	15	15
		uni	15	15
		biv	15	15
	99	sim	15	15
		uni	15	16
		biv	15	16
.20	75	sim	15	16
		uni	15	15
		biv	15	15
	90	sim	15	15
		uni	15	15
		biv	15	15
	95	sim	15	15
		uni	15	16
		biv	15	16
	99	sim	15	16
		uni	15	20
		biv	15	20

Table 2. Sample sizes needed for "correct" γ confidence intervals. Wide—n needed for the actual coverage to lie in [.73,.77], [.88,.92], [.93,.97], [.980,.995] for nominal 75%, 90%, 95%, and 99% confidence intervals, respectively. Narrow—n needed for the actual coverage to lie in [.74,.76], [.89,.91], [.94,.96], [.985,.995] for nominal 75%, 90%, 95%, and 99% confidence intervals, respectively. 15 indicates that the actual coverage already lies between the limits for sample sizes of 15.

COV	Nominal confidence level	Interval type	Bounds	
			Wide	Narrow
.30	75	sim	15	20
		uni	15	15
		biv	15	15
	90	sim	15	15
		uni	15	15
		biv	15	15
	95	sim	15	15
		uni	15	16
		biv	15	16
	99	sim	15	19
		uni	15	26
		biv	16	26
.40	75	sim	15	29
		uni	15	15
		biv	15	15
	90	sim	15	15
		uni	15	15
		biv	15	15
	95	sim	15	15
		uni	15	17
		biv	15	17
	99	sim	15	26
		uni	17	36
		biv	17	34

Table 2 continued. Sample sizes needed for "correct" γ confidence intervals. Wide—n needed for the actual coverage to lie in [.73,.77], [.88,.92], [.93,.97], [.980,.995] for nominal 75%, 90%, 95%, and 99% confidence intervals, respectively. Narrow—n needed for the actual coverage to lie in [.74,.76], [.89,.91], [.94,.96], [.985,.995] for nominal 75%, 90%, 95%, and 99% confidence intervals, respectively. 15 indicates that the actual coverage already lies between the limits for sample sizes of 15.

Nominal confidence level	Generating correlation	Interval type	Bounds σ Wide	Bounds σ Narrow	Bounds ρ Wide	Bounds ρ Narrow
75	.50	sim	21	38	16	23
		uni	17	28	–	–
		biv	17	31	24	45
	.60	sim	22	39	24	47
		uni	19	37	–	–
		biv	19	36	18	41
	.70	sim	23	43	37	76
		uni	21	35	–	–
		biv	22	39	15	17
	.80	sim	21	39	52	100
		uni	18	32	–	–
		biv	21	37	15	15
	.90	sim	21	41	69	127
		uni	18	34	–	–
		biv	29	53	15	16
	.95	sim	19	37	70	135
		uni	20	38	–	–
		biv	40	73	15	35
90	.50	sim	15	24	15	25
		uni	15	20	–	–
		biv	15	21	15	23
	.60	sim	15	23	15	38
		uni	15	18	–	–
		biv	15	20	15	22
	.70	sim	15	24	17	48
		uni	15	21	–	–
		biv	15	25	15	17
	.80	sim	15	22	25	63
		uni	15	17	–	–
		biv	15	22	15	16
	.90	sim	15	27	37	81
		uni	15	16	–	–
		biv	19	38	15	17
	.95	sim	16	26	41	92
		uni	15	23	–	–
		biv	29	53	15	27

Table 3. Sample sizes needed for "correct" σ and ρ confidence intervals. Wide—n needed for the actual coverage to lie in [.73,.77], [.88,.92], [.93,.97], [.980,.995] for nominal 75%, 90%, 95%, and 99% confidence intervals, respectively. Narrow—n needed for the actual coverage to lie in [.74,.76], [.89,.91], [.94,.96], [.985,.995] for nominal 75%, 90%, 95%, and 99% confidence intervals, respectively. 15 indicates that the actual coverage already lies between the limits for sample sizes of 15.

Nominal confidence level	Generating correlation	Interval type	Bounds			
			σ		ρ	
			Wide	Narrow	Wide	Narrow
95	.50	sim	15	15	15	15
		uni	15	15	–	–
		biv	15	15	15	22
	.60	sim	15	15	15	15
		uni	15	15	–	–
		biv	15	15	15	25
	.70	sim	15	15	15	15
		uni	15	15	–	–
		biv	15	17	18	30
	.80	sim	15	15	15	15
		uni	15	15	–	–
		biv	15	16	19	36
	.90	sim	15	17	15	15
		uni	15	15	–	–
		biv	15	24	23	45
	.95	sim	15	17	15	15
		uni	15	15	–	–
		biv	20	42	26	49
99	.50	sim	15	15	15	21
		uni	15	15	–	–
		biv	15	15	22	36
	.60	sim	15	15	18	36
		uni	15	15	–	–
		biv	15	15	31	65
	.70	sim	15	15	23	51
		uni	15	15	–	–
		biv	15	15	38	78
	.80	sim	15	15	31	68
		uni	15	15	–	–
		biv	15	15	53	108
	.90	sim	15	15	37	77
		uni	15	15	–	–
		biv	15	15	65	122
	.95	sim	15	15	40	84
		uni	15	15	–	–
		biv	15	22	66	128

Table 3 continued. Sample sizes needed for "correct" σ and ρ confidence intervals. Wide—n needed for the actual coverage to lie in [.73,.77], [.88,.92], [.93,.97], [.980,.995] for nominal 75%, 90%, 95%, and 99% confidence intervals, respectively. Narrow—n needed for the actual coverage to lie in [.74,.76], [.89,.91], [.94,.96], [.985,.995] for nominal 75%, 90%, 95%, and 99% confidence intervals, respectively. 15 indicates that the actual coverage already lies between the limits for sample sizes of 15.

Nominal confidence level	Interval type	Bounds	
		Wide	Narrow
75	sim	31	64
	uni	47	104
	biv	51	110
90	sim	15	17
	uni	57	118
	biv	61	124
95	sim	20	43
	uni	57	113
	biv	61	122
99	sim	50	108
	uni	82	165
	biv	84	165

Table 4. Sample sizes needed for "correct" β confidence intervals. Wide—n needed for the actual coverage to lie in [.73,.77], [.88,.92], [.93,.97], [.980,.995] for nominal 75%, 90%, 95%, and 99% confidence intervals, respectively. Narrow—n needed for the actual coverage to lie in [.74,.76], [.89,.91], [.94,.96], [.985,.995] for nominal 75%, 90%, 95%, and 99% confidence intervals, respectively. 15 indicates that the actual coverage already lies between the limits for sample sizes of 15.

Generating correlation	Sample size	Parameter									
		μ		σ		ρ		γ		β	
		uni	biv	uni	biv	uni	biv	uni	biv	uni	biv
.50	15	0.0	0.0	−4.9	−5.0	—	−2.9	0.2	0.2	10.3	10.3
	30	0.0	0.0	−2.6	−2.6	—	−1.6	0.1	0.1	5.1	5.1
	60	0.0	0.0	−1.3	−1.4	—	−0.9	0.0	0.0	2.5	2.5
	120	0.0	0.0	−0.6	−0.6	—	−0.4	0.0	0.0	1.2	1.3
	240	0.0	0.0	−0.3	−0.3	—	−0.2	0.0	0.0	0.6	0.6
	480	0.0	0.0	−0.2	−0.2	—	−0.2	0.0	0.0	0.3	0.3
.60	15	0.0	0.0	−5.2	−5.3	—	−2.7	0.2	0.2	10.6	10.7
	30	0.0	0.0	−2.5	−2.6	—	−1.5	0.1	0.1	5.1	5.1
	60	0.0	0.0	−1.3	−1.3	—	−0.7	0.1	0.1	2.5	2.5
	120	0.0	0.0	−0.7	−0.7	—	−0.3	0.0	0.0	1.2	1.2
	240	0.0	0.0	−0.3	−0.3	—	0.0	0.0	0.0	0.6	0.6
	480	0.0	0.0	−0.2	−0.2	—	−0.1	0.0	0.0	0.3	0.3
.70	15	0.0	0.0	−5.0	−5.1	—	−2.0	0.2	0.2	10.9	10.9
	30	−0.1	0.0	−2.6	−2.7	—	−1.0	0.1	0.1	4.7	4.7
	60	0.0	0.0	−1.2	−1.2	—	−0.3	0.1	0.1	2.2	2.1
	120	0.0	0.0	−0.7	−0.8	—	−0.3	0.0	0.0	1.2	1.2
	240	0.0	0.0	−0.3	−0.4	—	−0.1	0.0	0.0	0.5	0.5
	480	0.0	0.0	−0.2	−0.2	—	−0.1	0.0	0.0	0.4	0.4
.80	15	0.0	0.0	−5.1	−5.2	—	−1.5	0.3	0.3	10.6	10.5
	30	0.0	0.0	−2.5	−2.5	—	−0.4	0.1	0.1	4.7	4.6
	60	0.0	0.0	−1.2	−1.3	—	−0.2	0.1	0.1	2.2	2.2
	120	0.0	0.0	−0.6	−0.6	—	−0.1	0.0	0.0	1.2	1.2
	240	0.0	0.0	−0.3	−0.3	—	−0.1	0.0	0.0	0.6	0.6
	480	0.0	0.0	−0.1	−0.1	—	0.0	0.0	0.0	0.2	0.2
.90	15	0.0	0.0	−5.1	−5.2	—	−0.6	0.2	0.2	10.6	10.3
	30	0.0	0.0	−2.6	−2.8	—	−0.3	0.1	0.1	5.0	5.0
	60	0.0	0.0	−1.3	−1.4	—	−0.1	0.1	0.1	2.4	2.3
	120	0.0	0.0	−0.6	−0.6	—	−0.1	0.0	0.0	1.1	1.1
	240	0.0	0.0	−0.3	−0.3	—	0.0	0.0	0.0	0.6	0.5
	480	0.0	0.0	−0.1	−0.2	—	0.0	0.0	0.0	0.3	0.3
.95	15	0.0	0.1	−4.9	−5.1	—	−0.2	0.2	0.2	10.5	10.0
	30	0.0	0.0	−2.4	−2.6	—	−0.1	0.1	0.1	4.9	4.7
	60	0.0	0.0	−1.1	−1.2	—	−0.1	0.1	0.1	2.3	2.2
	120	0.0	0.0	−0.6	−0.6	—	0.0	0.0	0.0	1.2	1.1
	240	0.0	0.0	−0.3	−0.4	—	0.0	0.0	0.0	0.6	0.6
	480	0.0	0.0	−0.1	−0.1	—	0.0	0.0	0.0	0.2	0.2

Table 5. Simulation estimates of percent biases. uni—univariate estimates. biv—bivariate estimates. Coefficient of variation equal to 0.10.

Generating correlation	Sample size	Parameter									
		μ		σ		ρ		γ		β	
		uni	biv	uni	biv	uni	biv	uni	biv	uni	biv
.50	15	0.0	0.0	−5.1	−5.1	—	−2.9	0.6	0.6	10.5	10.5
	30	0.0	0.0	−2.7	−2.7	—	−1.1	0.3	0.3	4.7	4.7
	60	0.0	0.0	−1.2	−1.2	—	−0.4	0.1	0.1	2.2	2.2
	120	0.0	0.0	−0.6	−0.7	—	−0.3	0.1	0.1	1.2	1.2
	240	0.0	0.0	−0.3	−0.4	—	−0.2	0.0	0.0	0.5	0.5
	480	0.0	0.0	−0.2	−0.2	—	−0.1	0.0	0.0	0.3	0.3
.60	15	0.0	0.0	−5.0	−5.2	—	−2.5	0.6	0.6	10.3	10.4
	30	0.0	0.0	−2.5	−2.6	—	−1.2	0.3	0.3	4.9	4.9
	60	0.0	0.0	−1.2	−1.2	—	−0.6	0.1	0.1	2.5	2.5
	120	0.0	0.0	−0.6	−0.7	—	−0.3	0.1	0.1	1.2	1.2
	240	0.0	0.0	−0.2	−0.2	—	−0.1	0.0	0.0	0.5	0.5
	480	0.0	0.0	−0.1	−0.1	—	0.0	0.0	0.0	0.3	0.3
.70	15	0.0	0.1	−5.2	−5.4	—	−2.1	0.5	0.5	10.4	10.4
	30	0.0	0.0	−2.8	−2.8	—	−0.9	0.3	0.3	4.5	4.5
	60	0.0	0.0	−1.3	−1.3	—	−0.4	0.1	0.1	2.5	2.5
	120	0.0	0.0	−0.6	−0.6	—	−0.2	0.1	0.1	1.1	1.1
	240	0.0	0.0	−0.3	−0.3	—	−0.1	0.0	0.0	0.6	0.6
	480	0.0	0.0	−0.1	−0.2	—	0.0	0.0	0.0	0.3	0.3
.80	15	−0.1	0.0	−4.9	−5.1	—	−1.3	0.6	0.6	10.0	10.0
	30	0.0	0.0	−2.6	−2.7	—	−0.7	0.3	0.3	4.9	4.9
	60	0.0	0.0	−1.2	−1.3	—	−0.2	0.1	0.1	2.4	2.3
	120	0.0	0.0	−0.7	−0.7	—	−0.1	0.1	0.1	1.1	1.1
	240	0.0	0.0	−0.3	−0.3	—	−0.1	0.0	0.0	0.6	0.6
	480	0.0	0.0	−0.1	−0.2	—	−0.1	0.0	0.0	0.3	0.3
.90	15	0.0	0.0	−5.1	−5.4	—	−0.7	0.6	0.6	10.5	10.3
	30	0.0	0.0	−2.6	−2.7	—	−0.3	0.3	0.3	4.9	4.9
	60	0.0	0.0	−1.2	−1.3	—	−0.1	0.1	0.1	2.4	2.4
	120	0.0	0.0	−0.6	−0.6	—	−0.1	0.1	0.1	1.2	1.2
	240	0.0	0.0	−0.3	−0.3	—	−0.1	0.0	0.0	0.7	0.6
	480	0.0	0.0	−0.1	−0.1	—	0.0	0.0	0.0	0.3	0.3
.95	15	0.0	0.0	−5.0	−5.2	—	−0.3	0.6	0.6	10.3	9.9
	30	0.0	0.0	−2.5	−2.6	—	−0.1	0.3	0.3	4.8	4.6
	60	0.0	0.0	−1.2	−1.2	—	0.0	0.1	0.1	2.3	2.1
	120	0.0	0.0	−0.6	−0.6	—	0.0	0.1	0.0	1.1	1.1
	240	0.0	0.0	−0.4	−0.4	—	0.0	0.0	0.0	0.6	0.6
	480	0.0	0.0	−0.2	−0.2	—	0.0	0.0	0.0	0.3	0.3

Table 6. Simulation estimates of percent biases. uni—univariate estimates. biv—bivariate estimates. Coefficient of variation equal to 0.20.

Generating correlation	Sample size	Parameter									
		μ		σ		ρ		γ		β	
		uni	biv	uni	biv	uni	biv	uni	biv	uni	biv
.50	15	0.0	0.1	−5.1	−5.2	—	−2.7	1.0	1.0	10.6	10.6
	30	0.0	0.0	−2.3	−2.4	—	−0.9	0.5	0.5	4.8	4.8
	60	0.0	0.0	−1.3	−1.3	—	−0.5	0.2	0.2	2.3	2.3
	120	0.0	0.0	−0.6	−0.6	—	−0.3	0.1	0.1	1.2	1.2
	240	0.0	0.0	−0.3	−0.3	—	−0.1	0.0	0.0	0.7	0.7
	480	0.0	0.0	−0.2	−0.2	—	−0.1	0.0	0.0	0.3	0.3
.60	15	0.0	0.0	−5.1	−5.3	—	−2.6	1.0	1.0	10.6	10.6
	30	0.0	0.0	−2.3	−2.4	—	−1.2	0.5	0.5	4.9	4.9
	60	0.0	0.0	−1.3	−1.4	—	−0.5	0.2	0.2	2.4	2.4
	120	0.0	0.0	−0.6	−0.7	—	−0.4	0.1	0.1	1.2	1.2
	240	0.0	0.0	−0.3	−0.4	—	0.0	0.0	0.0	0.6	0.6
	480	0.0	0.0	−0.1	−0.1	—	−0.1	0.0	0.0	0.3	0.3
.70	15	0.0	0.1	−5.4	−5.5	—	−2.3	1.1	1.1	10.4	10.3
	30	0.1	0.1	−2.5	−2.6	—	−0.8	0.4	0.4	5.0	5.0
	60	0.1	0.1	−1.3	−1.4	—	−0.5	0.2	0.2	2.4	2.4
	120	0.0	0.0	−0.6	−0.6	—	−0.2	0.1	0.1	1.2	1.2
	240	0.0	0.0	−0.3	−0.4	—	−0.1	0.1	0.1	0.6	0.6
	480	0.0	0.0	−0.1	−0.1	—	−0.1	0.0	0.0	0.2	0.2
.80	15	0.0	0.1	−5.0	−5.2	—	−1.1	1.0	1.0	10.0	9.9
	30	0.1	0.1	−2.5	−2.6	—	−0.6	0.4	0.4	4.8	4.8
	60	0.1	0.1	−1.5	−1.5	—	−0.5	0.2	0.2	2.6	2.5
	120	0.0	0.0	−0.6	−0.6	—	−0.2	0.1	0.1	1.2	1.2
	240	0.0	0.0	−0.3	−0.4	—	−0.1	0.1	0.1	0.6	0.6
	480	0.0	0.0	−0.1	−0.1	—	0.0	0.0	0.0	0.3	0.3
.90	15	0.0	0.0	−4.9	−5.2	—	−0.6	1.0	1.0	10.3	10.2
	30	−0.2	−0.1	−2.6	−2.8	—	−0.3	0.6	0.6	4.8	4.7
	60	0.0	0.0	−1.1	−1.2	—	−0.1	0.2	0.2	2.1	2.1
	120	0.0	0.0	−0.6	−0.6	—	0.0	0.1	0.1	1.1	1.1
	240	0.0	0.0	−0.3	−0.3	—	0.0	0.1	0.1	0.5	0.5
	480	0.0	0.0	−0.2	−0.2	—	0.0	0.0	0.0	0.3	0.3
.95	15	−0.2	−0.1	−4.9	−5.2	—	−0.3	1.1	1.1	10.1	9.8
	30	0.0	0.0	−2.5	−2.6	—	−0.1	0.5	0.5	4.9	4.6
	60	0.0	0.0	−1.3	−1.4	—	−0.1	0.3	0.3	2.4	2.2
	120	0.0	0.0	−0.6	−0.7	—	0.0	0.1	0.1	1.2	1.1
	240	0.0	0.0	−0.3	−0.4	—	0.0	0.1	0.1	0.6	0.6
	480	0.0	0.0	−0.2	−0.2	—	0.0	0.0	0.0	0.3	0.3

Table 7. Simulation estimates of percent biases. uni—univariate estimates. biv—bivariate estimates. Coefficient of variation equal to 0.30.

Generating correlation	Sample size	Parameter									
		μ		σ		ρ		γ		β	
		uni	biv	uni	biv	uni	biv	uni	biv	uni	biv
.50	15	0.0	0.1	−5.3	−5.4	—	−2.7	1.5	1.5	10.3	10.2
	30	0.1	0.1	−2.5	−2.6	—	−1.3	0.7	0.7	4.9	4.9
	60	0.0	0.0	−1.3	−1.4	—	−0.8	0.4	0.4	2.4	2.4
	120	0.0	0.0	−0.6	−0.6	—	−0.2	0.2	0.2	1.2	1.2
	240	0.0	0.0	−0.3	−0.3	—	−0.1	0.1	0.1	0.7	0.7
	480	0.0	0.0	−0.1	−0.1	—	−0.1	0.0	0.0	0.3	0.3
.60	15	0.1	0.1	−5.3	−5.4	—	−2.7	1.4	1.4	10.6	10.6
	30	−0.1	0.0	−2.7	−2.7	—	−0.9	0.8	0.8	4.7	4.7
	60	0.0	0.1	−1.3	−1.4	—	−0.7	0.4	0.4	2.4	2.4
	120	0.0	0.0	−0.6	−0.6	—	−0.1	0.1	0.1	1.2	1.2
	240	0.0	0.0	−0.3	−0.3	—	−0.1	0.1	0.1	0.6	0.6
	480	0.0	0.0	−0.2	−0.2	—	−0.1	0.0	0.0	0.3	0.3
.70	15	−0.1	0.0	−5.1	−5.3	—	−2.2	1.6	1.6	10.5	10.4
	30	−0.1	−0.1	−2.5	−2.5	—	−0.9	0.7	0.7	5.1	5.1
	60	−0.1	−0.1	−1.3	−1.3	—	−0.4	0.4	0.4	2.4	2.4
	120	0.0	0.0	−0.6	−0.7	—	−0.2	0.2	0.2	1.2	1.2
	240	0.0	0.0	−0.4	−0.3	—	−0.1	0.1	0.1	0.6	0.6
	480	0.0	0.0	−0.2	−0.2	—	−0.1	0.0	0.0	0.4	0.4
.80	15	0.0	0.0	−5.0	−5.1	—	−1.2	1.5	1.5	10.2	10.1
	30	0.0	0.0	−2.2	−2.3	—	−0.5	0.8	0.8	4.7	4.6
	60	0.0	0.0	−1.3	−1.4	—	−0.3	0.4	0.4	2.4	2.4
	120	0.0	0.0	−0.7	−0.7	—	−0.1	0.2	0.2	1.2	1.1
	240	0.0	0.0	−0.3	−0.3	—	0.0	0.1	0.1	0.5	0.5
	480	0.0	0.0	−0.2	−0.2	—	0.0	0.0	0.0	0.3	0.3
.90	15	−0.2	−0.1	−5.1	−5.3	—	−0.6	1.7	1.7	10.0	9.8
	30	0.0	0.1	−2.5	−2.7	—	−0.3	0.7	0.7	5.1	5.1
	60	−0.1	0.0	−1.4	−1.4	—	−0.1	0.4	0.4	2.3	2.2
	120	0.0	0.0	−0.6	−0.6	—	−0.1	0.2	0.2	1.1	1.1
	240	0.0	0.0	−0.4	−0.3	—	0.0	0.1	0.1	0.6	0.5
	480	0.0	0.0	−0.2	−0.2	—	0.0	0.0	0.0	0.3	0.3
.95	15	0.0	0.0	−5.2	−5.4	—	−0.3	1.6	1.5	10.3	9.9
	30	0.0	0.1	−2.5	−2.6	—	−0.1	0.7	0.7	5.0	4.7
	60	0.0	0.0	−1.4	−1.3	—	−0.1	0.4	0.4	2.4	2.2
	120	0.0	0.0	−0.6	−0.7	—	0.0	0.2	0.2	1.2	1.2
	240	0.1	0.1	−0.2	−0.3	—	0.0	0.0	0.0	0.6	0.6
	480	0.0	0.0	−0.2	−0.2	—	0.0	0.1	0.1	0.3	0.2

Table 8. Simulation estimates of percent biases. uni—univariate estimates. biv—bivariate estimates. Coefficient of variation equal to 0.40.

Generating correlation	Sample size	Parameter									
		μ		σ		ρ		γ		β	
		uni	biv	uni	biv	uni	biv	uni	biv	uni	biv
.50	15	2.6	2.6	18.2	18.2	—	42.2	2.3	2.3	24.6	24.6
	30	1.8	1.8	12.9	12.9	—	28.7	1.6	1.6	15.9	15.9
	60	1.3	1.3	9.1	9.1	—	19.6	1.1	1.1	10.7	10.7
	120	0.9	0.9	6.4	6.4	—	13.8	0.8	0.8	7.4	7.4
	240	0.6	0.6	4.6	4.6	—	9.5	0.6	0.6	5.1	5.1
	480	0.5	0.5	3.2	3.2	—	6.8	0.4	0.4	3.6	3.5
.60	15	2.6	2.6	18.3	18.2	—	30.3	2.3	2.3	25.1	25.1
	30	1.8	1.8	12.8	12.8	—	20.5	1.6	1.6	15.9	15.8
	60	1.3	1.3	9.1	9.0	—	14.0	1.1	1.1	10.7	10.7
	120	0.9	0.9	6.5	6.4	—	9.8	0.8	0.8	7.3	7.3
	240	0.6	0.6	4.6	4.6	—	6.8	0.6	0.6	5.1	5.1
	480	0.5	0.5	3.2	3.2	—	4.8	0.4	0.4	3.6	3.6
.70	15	2.6	2.6	18.1	18.0	—	21.1	2.3	2.3	24.9	24.8
	30	1.8	1.8	13.0	12.9	—	14.2	1.6	1.6	15.9	15.9
	60	1.3	1.3	9.1	9.1	—	9.6	1.1	1.1	10.7	10.6
	120	0.9	0.9	6.5	6.4	—	6.6	0.8	0.8	7.3	7.3
	240	0.6	0.6	4.5	4.5	—	4.7	0.6	0.6	5.1	5.0
	480	0.5	0.5	3.2	3.2	—	3.3	0.4	0.4	3.6	3.6
.80	15	2.6	2.6	18.0	17.8	—	13.7	2.3	2.3	25.3	25.1
	30	1.8	1.8	12.9	12.7	—	8.7	1.6	1.6	15.9	15.8
	60	1.3	1.3	9.1	8.9	—	6.0	1.1	1.1	10.5	10.4
	120	0.9	0.9	6.4	6.3	—	4.1	0.8	0.8	7.4	7.2
	240	0.6	0.6	4.6	4.5	—	2.9	0.6	0.6	5.1	5.0
	480	0.5	0.5	3.2	3.1	—	2.0	0.4	0.4	3.6	3.5
.90	15	2.6	2.6	17.9	17.7	—	6.5	2.3	2.3	25.2	24.7
	30	1.8	1.8	12.8	12.3	—	4.2	1.6	1.6	16.0	15.6
	60	1.3	1.3	9.2	8.8	—	2.8	1.1	1.1	10.7	10.4
	120	0.9	0.9	6.4	6.1	—	1.9	0.8	0.8	7.3	7.0
	240	0.6	0.6	4.5	4.3	—	1.3	0.6	0.6	5.0	4.8
	480	0.5	0.5	3.2	3.1	—	0.9	0.4	0.4	3.6	3.5
.95	15	2.6	2.6	18.1	17.2	—	3.2	2.3	2.3	25.2	24.2
	30	1.8	1.8	12.8	11.9	—	2.0	1.6	1.6	15.9	14.9
	60	1.3	1.3	9.2	8.4	—	1.3	1.1	1.1	10.6	9.8
	120	0.9	0.9	6.5	5.9	—	0.9	0.8	0.8	7.3	6.8
	240	0.6	0.6	4.6	4.1	—	0.6	0.6	0.6	5.1	4.7
	480	0.5	0.5	3.2	2.9	—	0.4	0.4	0.4	3.5	3.2

Table 9. Simulation estimates of sample standard deviation of parameter estimate times 100 divided by parameter value. uni—univariate estimates. biv—bivariate estimates. The coefficients of variation of the generating normal and Weibull distributions were equal to 0.10.

Generating correlation	Sample size	Parameter									
		μ		σ		ρ		γ		β	
		uni	biv	uni	biv	uni	biv	uni	biv	uni	biv
.50	15	5.1	5.1	18.2	18.2	—	42.0	4.8	4.8	24.9	24.9
	30	3.6	3.6	12.8	12.8	—	28.5	3.3	3.3	15.7	15.7
	60	2.6	2.6	9.0	9.0	—	19.5	2.3	2.3	10.4	10.4
	120	1.8	1.8	6.4	6.4	—	13.8	1.7	1.7	7.3	7.2
	240	1.3	1.3	4.6	4.5	—	9.6	1.2	1.2	5.1	5.0
	480	0.9	0.9	3.2	3.2	—	6.8	0.8	0.8	3.6	3.5
.60	15	5.2	5.2	18.1	18.0	—	30.5	4.9	4.9	25.2	25.2
	30	3.7	3.7	12.9	12.8	—	20.4	3.3	3.3	15.9	15.9
	60	2.6	2.6	9.2	9.2	—	14.2	2.4	2.4	10.8	10.7
	120	1.8	1.8	6.5	6.4	—	9.8	1.7	1.7	7.3	7.3
	240	1.3	1.3	4.6	4.6	—	6.8	1.2	1.2	5.0	5.0
	480	0.9	0.9	3.2	3.2	—	4.8	0.8	0.8	3.6	3.6
.70	15	5.1	5.1	18.2	18.1	—	21.3	4.8	4.8	25.5	25.5
	30	3.6	3.6	12.9	12.9	—	13.9	3.3	3.3	15.6	15.5
	60	2.6	2.6	9.1	9.0	—	9.6	2.3	2.3	10.6	10.5
	120	1.8	1.8	6.5	6.4	—	6.8	1.6	1.6	7.3	7.2
	240	1.3	1.3	4.6	4.5	—	4.6	1.2	1.2	5.1	5.0
	480	0.9	0.9	3.2	3.2	—	3.3	0.8	0.8	3.6	3.5
.80	15	5.1	5.1	18.2	18.0	—	13.5	4.8	4.8	24.3	24.2
	30	3.6	3.6	12.7	12.5	—	8.9	3.4	3.4	15.8	15.6
	60	2.6	2.6	9.1	8.9	—	5.9	2.4	2.4	10.6	10.4
	120	1.8	1.8	6.4	6.3	—	4.1	1.6	1.6	7.2	7.1
	240	1.3	1.3	4.6	4.5	—	2.9	1.2	1.2	5.0	5.0
	480	0.9	0.9	3.2	3.1	—	2.0	0.8	0.8	3.6	3.5
.90	15	5.2	5.2	18.0	17.6	—	6.6	4.8	4.8	24.8	24.5
	30	3.6	3.6	13.0	12.5	—	4.2	3.4	3.3	15.9	15.5
	60	2.6	2.6	9.1	8.7	—	2.8	2.4	2.4	10.7	10.4
	120	1.8	1.8	6.5	6.2	—	1.9	1.7	1.7	7.3	7.0
	240	1.3	1.3	4.6	4.4	—	1.3	1.2	1.2	5.2	4.9
	480	0.9	0.9	3.2	3.0	—	0.9	0.8	0.8	3.6	3.4
.95	15	5.2	5.2	18.4	17.3	—	3.3	4.9	4.9	25.4	24.4
	30	3.7	3.7	13.0	12.0	—	2.0	3.4	3.4	16.1	15.2
	60	2.6	2.6	9.2	8.4	—	1.3	2.4	2.4	10.7	10.0
	120	1.8	1.8	6.4	5.8	—	0.9	1.7	1.7	7.2	6.7
	240	1.3	1.3	4.5	4.1	—	0.6	1.2	1.2	5.1	4.7
	480	0.9	0.9	3.2	2.9	—	0.4	0.8	0.8	3.6	3.3

Table 10. Simulation estimates of sample standard deviation of parameter estimate times 100 divided by parameter value. uni—univariate estimates. biv—bivariate estimates. The coefficients of variation of the generating normal and Weibull distributions were equal to 0.20.

Generating correlation	Sample size	Parameter									
		μ		σ		ρ		γ		β	
		uni	biv	uni	biv	uni	biv	uni	biv	uni	biv
.50	15	7.8	7.8	18.1	18.0	—	42.5	7.6	7.6	25.0	25.0
	30	5.6	5.6	12.8	12.7	—	28.1	5.3	5.3	15.7	15.7
	60	3.9	3.9	9.1	9.1	—	19.5	3.7	3.7	10.6	10.6
	120	2.7	2.7	6.5	6.5	—	13.9	2.6	2.6	7.3	7.3
	240	1.9	1.9	4.6	4.6	—	9.8	1.8	1.8	5.1	5.1
	480	1.4	1.4	3.2	3.2	—	6.7	1.3	1.3	3.6	3.6
.60	15	7.6	7.6	18.1	18.0	—	30.2	7.6	7.6	24.9	24.9
	30	5.4	5.4	12.8	12.8	—	20.4	5.2	5.2	15.9	15.8
	60	3.9	3.9	9.0	9.0	—	14.1	3.7	3.7	10.7	10.6
	120	2.7	2.7	6.4	6.3	—	9.7	2.6	2.6	7.3	7.2
	240	1.9	1.9	4.6	4.6	—	6.9	1.8	1.8	5.1	5.0
	480	1.4	1.4	3.2	3.2	—	4.8	1.3	1.3	3.6	3.6
.70	15	7.7	7.7	18.0	17.9	—	21.5	7.6	7.6	24.6	24.6
	30	5.5	5.5	12.8	12.7	—	13.8	5.2	5.2	15.7	15.6
	60	3.9	3.9	9.1	9.0	—	9.5	3.7	3.7	10.5	10.4
	120	2.7	2.7	6.4	6.3	—	6.7	2.6	2.6	7.3	7.2
	240	1.9	1.9	4.6	4.6	—	4.7	1.8	1.8	5.2	5.1
	480	1.4	1.4	3.2	3.2	—	3.3	1.3	1.3	3.6	3.6
.80	15	7.7	7.7	18.2	17.9	—	13.5	7.5	7.5	24.8	24.6
	30	5.5	5.5	12.8	12.6	—	8.8	5.3	5.3	15.9	15.8
	60	3.9	3.9	9.1	8.9	—	6.1	3.7	3.7	10.8	10.6
	120	2.8	2.8	6.5	6.3	—	4.2	2.6	2.6	7.3	7.2
	240	1.9	1.9	4.6	4.5	—	2.9	1.9	1.9	5.1	5.0
	480	1.4	1.4	3.2	3.2	—	2.0	1.3	1.3	3.6	3.5
.90	15	7.7	7.7	18.3	17.7	—	6.6	7.6	7.5	25.7	25.3
	30	5.5	5.5	12.7	12.2	—	4.1	5.3	5.3	15.9	15.4
	60	3.9	3.9	9.2	8.8	—	2.8	3.7	3.7	10.6	10.3
	120	2.8	2.8	6.4	6.2	—	1.9	2.6	2.6	7.3	7.0
	240	1.9	1.9	4.6	4.3	—	1.3	1.8	1.8	5.1	4.9
	480	1.4	1.4	3.2	3.0	—	0.9	1.3	1.3	3.6	3.4
.95	15	7.8	7.8	18.0	17.1	—	3.3	7.6	7.6	24.5	23.7
	30	5.4	5.4	12.9	11.9	—	2.0	5.2	5.2	16.0	15.0
	60	3.9	3.9	9.0	8.3	—	1.3	3.7	3.7	10.5	9.8
	120	2.7	2.7	6.4	5.8	—	0.9	2.6	2.6	7.3	6.7
	240	1.9	1.9	4.6	4.2	—	0.6	1.8	1.8	5.1	4.7
	480	1.4	1.4	3.2	2.9	—	0.4	1.3	1.3	3.6	3.3

Table 11. Simulation estimates of sample standard deviation of parameter estimate times 100 divided by parameter value. uni—univariate estimates. biv—bivariate estimates. The coefficients of variation of the generating normal and Weibull distributions were equal to 0.30.

Generating correlation	Sample size	Parameter									
		μ		σ		ρ		γ		β	
		uni	biv	uni	biv	uni	biv	uni	biv	uni	biv
.50	15	10.4	10.4	18.1	18.1	—	42.0	10.6	10.6	24.9	24.9
	30	7.3	7.3	12.8	12.8	—	28.4	7.2	7.2	15.9	15.9
	60	5.1	5.1	9.1	9.1	—	19.8	5.1	5.1	10.6	10.6
	120	3.6	3.6	6.4	6.4	—	13.7	3.6	3.6	7.3	7.3
	240	2.5	2.5	4.6	4.5	—	9.7	2.5	2.5	5.1	5.1
	480	1.8	1.8	3.3	3.3	—	6.7	1.8	1.8	3.6	3.6
.60	15	10.3	10.3	18.2	18.1	—	30.8	10.3	10.3	25.0	25.0
	30	7.3	7.3	12.9	12.8	—	20.5	7.3	7.3	16.0	16.0
	60	5.1	5.1	9.2	9.1	—	14.2	5.1	5.1	10.6	10.5
	120	3.6	3.6	6.4	6.4	—	9.8	3.6	3.6	7.3	7.3
	240	2.6	2.6	4.5	4.5	—	6.9	2.5	2.5	5.0	5.0
	480	1.8	1.8	3.2	3.2	—	4.9	1.8	1.8	3.5	3.5
.70	15	10.3	10.3	18.3	18.1	—	21.5	10.4	10.4	25.2	25.1
	30	7.2	7.2	12.7	12.6	—	14.1	7.3	7.3	15.9	15.8
	60	5.1	5.1	9.0	8.9	—	9.6	5.1	5.1	10.5	10.4
	120	3.6	3.6	6.5	6.4	—	6.6	3.6	3.6	7.2	7.2
	240	2.6	2.6	4.6	4.5	—	4.7	2.5	2.5	5.1	5.1
	480	1.9	1.9	3.3	3.2	—	3.3	1.8	1.8	3.6	3.6
.80	15	10.3	10.3	18.2	18.1	—	13.6	10.5	10.5	24.9	24.8
	30	7.3	7.3	12.8	12.6	—	8.7	7.3	7.3	15.9	15.7
	60	5.1	5.1	9.0	8.9	—	5.9	5.1	5.1	10.6	10.5
	120	3.7	3.7	6.5	6.4	—	4.1	3.6	3.6	7.4	7.3
	240	2.6	2.6	4.6	4.5	—	2.9	2.6	2.6	5.1	5.0
	480	1.8	1.8	3.2	3.2	—	2.0	1.8	1.8	3.6	3.5
.90	15	10.2	10.2	18.1	17.5	—	6.6	10.5	10.5	24.4	23.9
	30	7.3	7.3	12.9	12.3	—	4.2	7.3	7.3	16.0	15.6
	60	5.1	5.1	9.1	8.7	—	2.8	5.1	5.1	10.6	10.2
	120	3.6	3.6	6.5	6.2	—	1.9	3.6	3.6	7.3	7.0
	240	2.6	2.5	4.5	4.3	—	1.3	2.5	2.5	5.0	4.8
	480	1.8	1.8	3.2	3.1	—	0.9	1.8	1.8	3.6	3.4
.95	15	10.4	10.3	18.1	17.3	—	3.3	10.5	10.5	25.1	24.3
	30	7.3	7.2	12.9	12.0	—	2.0	7.3	7.3	15.9	15.0
	60	5.2	5.2	9.2	8.4	—	1.3	5.1	5.1	10.6	10.0
	120	3.7	3.7	6.5	5.9	—	0.9	3.6	3.6	7.3	6.8
	240	2.6	2.6	4.5	4.1	—	0.6	2.5	2.5	5.1	4.7
	480	1.8	1.8	3.2	2.9	—	0.4	1.8	1.8	3.6	3.3

Table 12. Simulation estimates of sample standard deviation of parameter estimate times 100 divided by parameter value. uni—univariate estimates. biv—bivariate estimates. The coefficients of variation of the generating normal and Weibull distributions were equal to 0.40.

Generating correlation	Sample size	Parameter							
		μ		σ		γ		β	
		sim	th	sim	th	sim	th	sim	th
.50	15	1.000	1.001	1.001	1.005	1.000	1.000	1.001	1.007
	30	1.000		1.004		1.000		1.003	
	60	1.000		1.003		1.000		1.004	
	120	1.001		1.003		1.000		1.004	
	240	1.001		1.003		1.000		1.007	
	480	1.000		1.005		1.000		1.008	
.60	15	1.000	1.001	1.004	1.010	1.000	1.000	1.000	1.012
	30	1.000		1.008		1.000		1.007	
	60	1.001		1.009		1.000		1.010	
	120	1.000		1.010		1.000		1.012	
	240	1.001		1.007		1.000		1.010	
	480	1.001		1.010		1.000		1.013	
.70	15	1.001	1.001	1.008	1.021	1.000	1.000	1.005	1.020
	30	1.000		1.010		1.000		1.012	
	60	1.001		1.018		1.000		1.018	
	120	1.001		1.021		1.000		1.018	
	240	1.001		1.022		1.000		1.020	
	480	1.002		1.020		1.000		1.018	
.80	15	1.001	1.002	1.023	1.044	1.000	1.000	1.011	1.037
	30	1.001		1.030		1.000		1.019	
	60	1.001		1.036		1.000		1.028	
	120	1.002		1.042		1.000		1.036	
	240	1.002		1.045		1.000		1.037	
	480	1.002		1.043		1.000		1.034	
.90	15	1.002	1.003	1.046	1.109	1.001	1.001	1.035	1.087
	30	1.002		1.074		1.000		1.056	
	60	1.003		1.089		1.000		1.067	
	120	1.003		1.097		1.001		1.081	
	240	1.004		1.106		1.001		1.088	
	480	1.002		1.109		1.000		1.083	
.95	15	1.002	1.005	1.100	1.228	1.002	1.001	1.079	1.188
	30	1.004		1.150		1.002		1.125	
	60	1.004		1.181		1.002		1.153	
	120	1.005		1.202		1.002		1.168	
	240	1.005		1.213		1.001		1.176	
	480	1.005		1.236		1.002		1.185	

Table 13. Ratio of Univariate MSE to Bivariate MSE. sim—from the simulation (averages over four coefficients of variation). th—from the ratio of the appropriate elements of the inverses of the information matrices.

Sample size	Parameter			
	γ		β	
	reg1/mle	reg2/mle	reg1/mle	reg2/mle
15	1.006	0.995	0.991	0.890
30	1.018	1.031	1.233	1.228
60	1.028	1.052	1.432	1.506
120	1.034	1.062	1.568	1.689
240	1.039	1.063	1.667	1.793
480	1.046	1.065	1.720	1.827

Table 14. Ratio of regression MSE to mle MSE (average over four coefficients of variation). reg1—see Equation (9). reg2—see Equation (10).

Sample size	Whose simulation	β			
		1.0	3.0	5.0	10.0
20	Theirs	1.16	1.31	1.27	1.24
	Ours	1.078	1.077	1.065	1.058
30	Theirs	1.21	1.27	1.08	1.18
	Ours	0.996	1.015	1.006	1.000
40	Theirs	1.01	1.03	1.09	1.11
	Ours	0.975	0.969	0.980	0.979
50	Theirs	1.09	1.08	0.966	1.20
	Ours	0.966	0.964	0.956	0.953
100	Theirs	0.989	1.08	0.901	0.994
	Ours	0.934	0.930	0.921	0.931

Table 15. Ratio of regression 1 MSE to regression 2 MSE for the shape parameter. reg1—see Equation (9). reg2—see Equation (10). However, for the simulation reported in this table, in both Equations (9) and (10), $(i - .3)/(n + .4)$ was replaced by $(i - .5)/n$ to be in accord with Lawrence and Shier's (1981) simulation. "Theirs" refers to table III in Lawrence and Shier. "Ours" refers to a simulation that we conducted using 10,000 trials per condition.

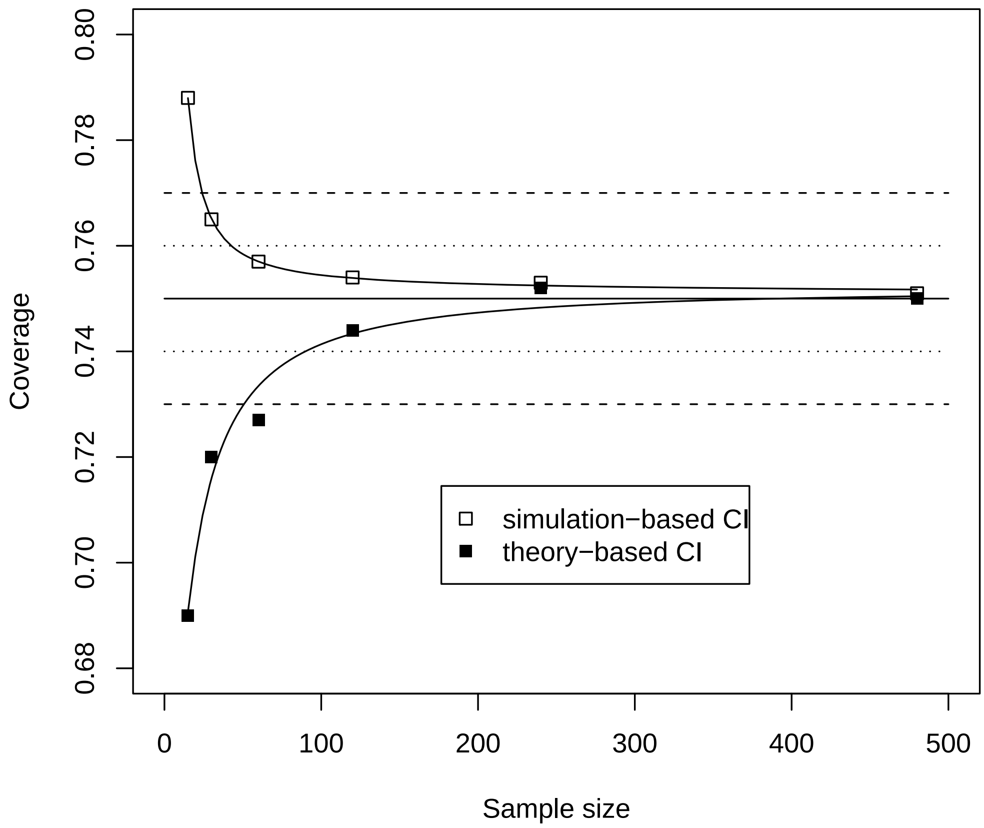


Figure 1: The open squares correspond to the simulation estimates of the coverages of simulation-based 75% confidence intervals. The solid squares are the simulation estimates of the coverages of Gaussian–Weibull theory-based 75% confidence intervals. The curved lines correspond to regression fits of the model $\textit{actual coverage} - 0.75 = a_1/\sqrt{n} + a_2/n + a_3/n^{3/2}$ to the data. The horizontal lines are at coverages 0.73, 0.74, 0.75, 0.76, and 0.77. For this simulation, the coefficient of variation of both marginals was 0.20 and the generating correlation was 0.70. The parameter was β.

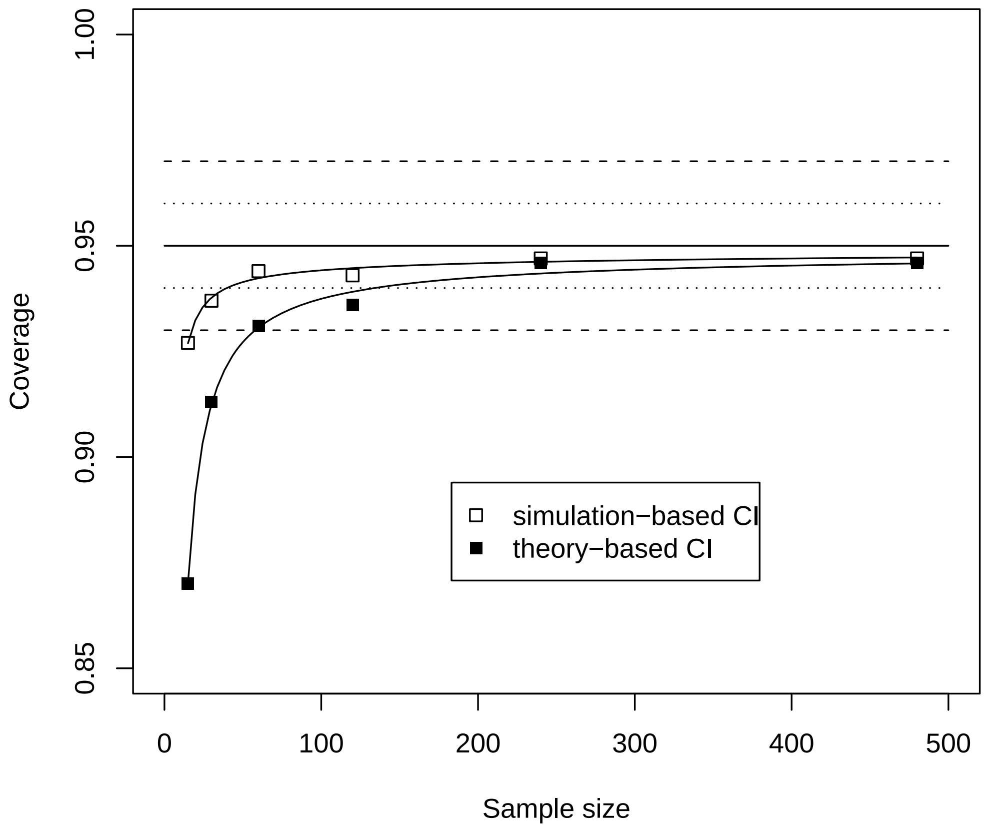


Figure 2: The open squares correspond to the simulation estimates of the coverages of simulation-based 95% confidence intervals. The solid squares are the simulation estimates of the coverages of Gaussian–Weibull theory-based 95% confidence intervals. The curved lines correspond to regression fits of the model $\textit{actual coverage} - 0.95 = a_1/\sqrt{n} + a_2/n + a_3/n^{3/2}$ to the data. The horizontal lines are at coverages 0.93, 0.94, 0.95, 0.96, and 0.97. For this simulation, the coefficient of variation of both marginals was 0.20 and the generating correlation was 0.70. The parameter was β.

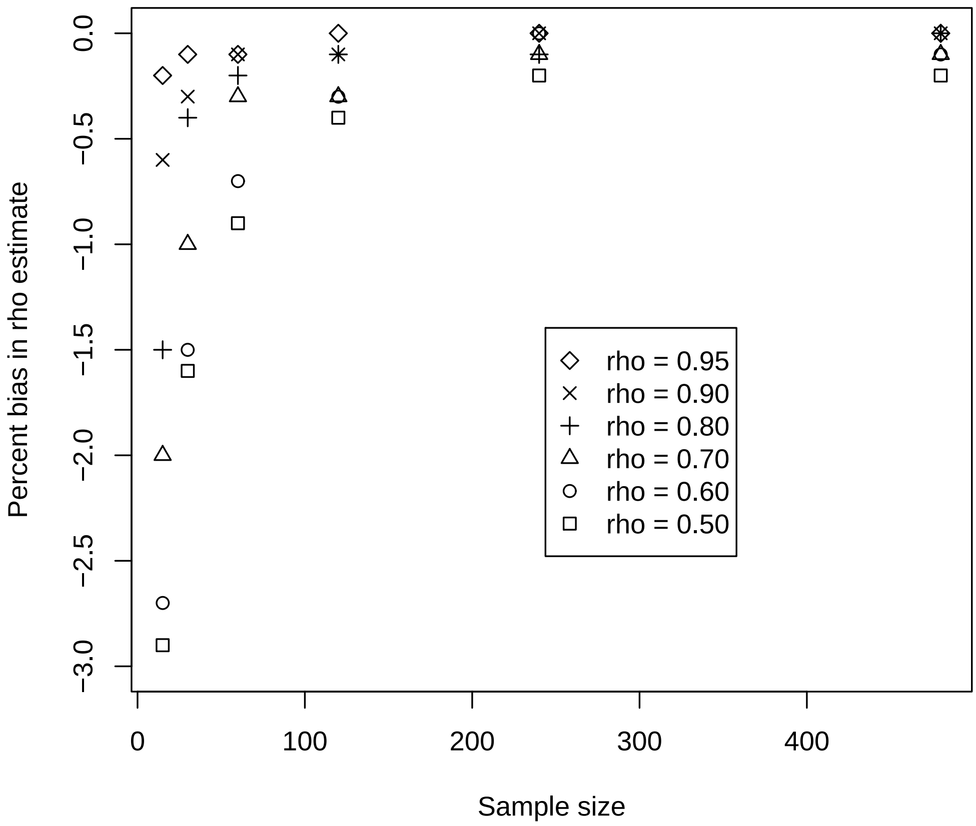


Figure 3: This figure corresponds to the data in Table 5 for a coefficient of variation of 0.10. The plots for the other three coefficients of variation are very similar.

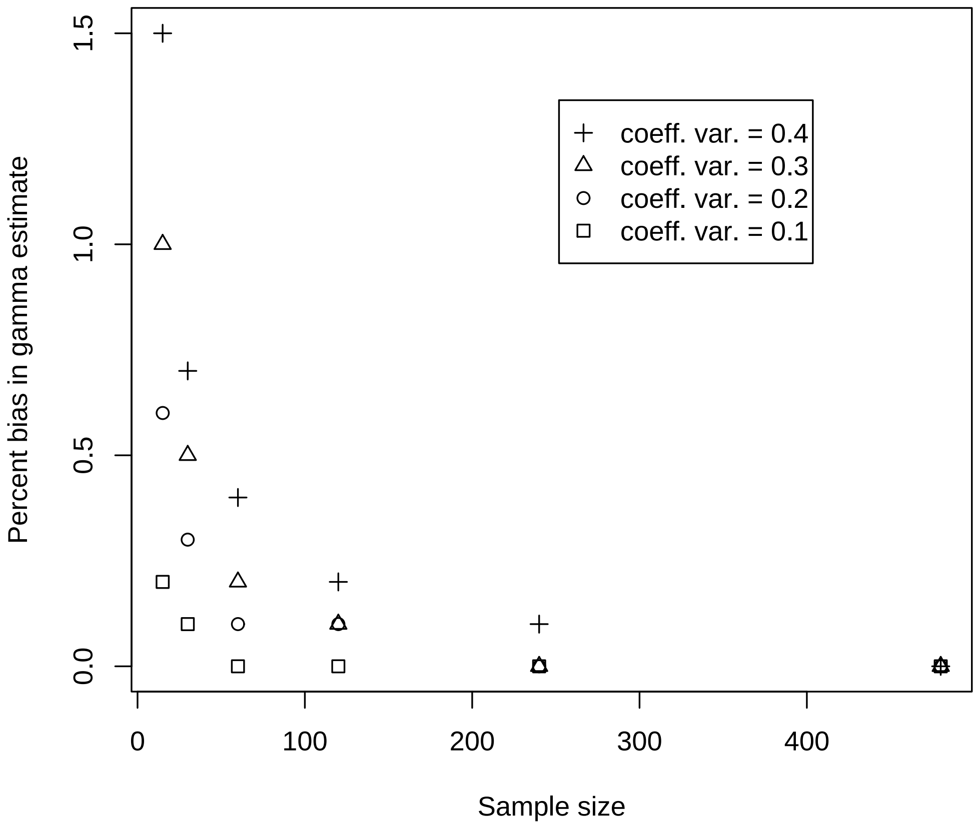


Figure 4: This figure corresponds to the data in Tables 5–8 for a generating correlation of 0.50. The plots for the other five generating correlations are very similar.

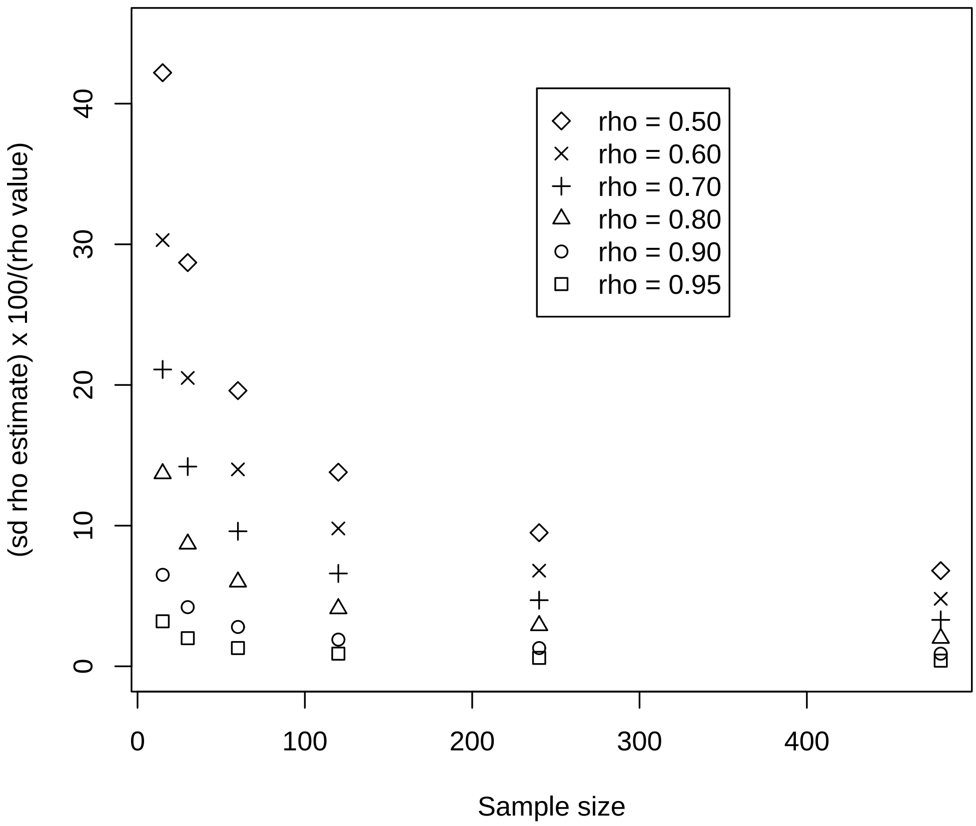


Figure 5: This figure corresponds to the data in Table 9 for a coefficient of variation of 0.10. The plots for the other three coefficients of variation are very similar.

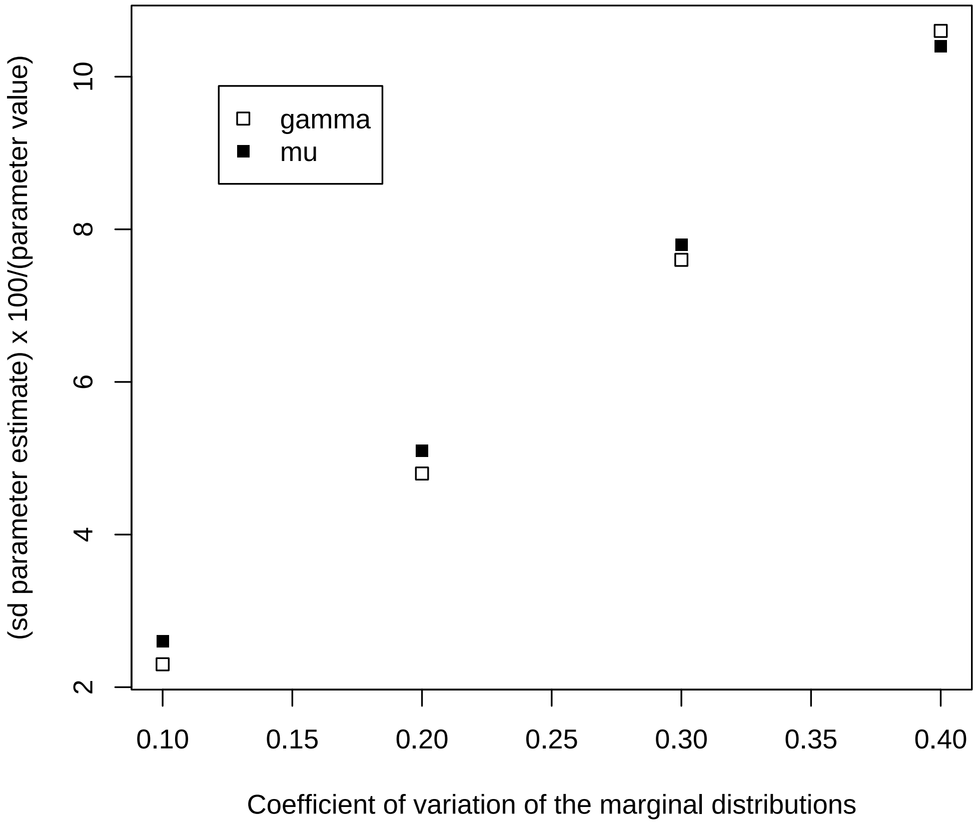


Figure 6: This figure corresponds to the data in Tables 9–12 for a generating correlation of 0.50 and a sample size of 15. The plots for the other five generating correlations are very similar. The plots for the other five sample sizes are similar. However, the slope decreases with increasing sample size.

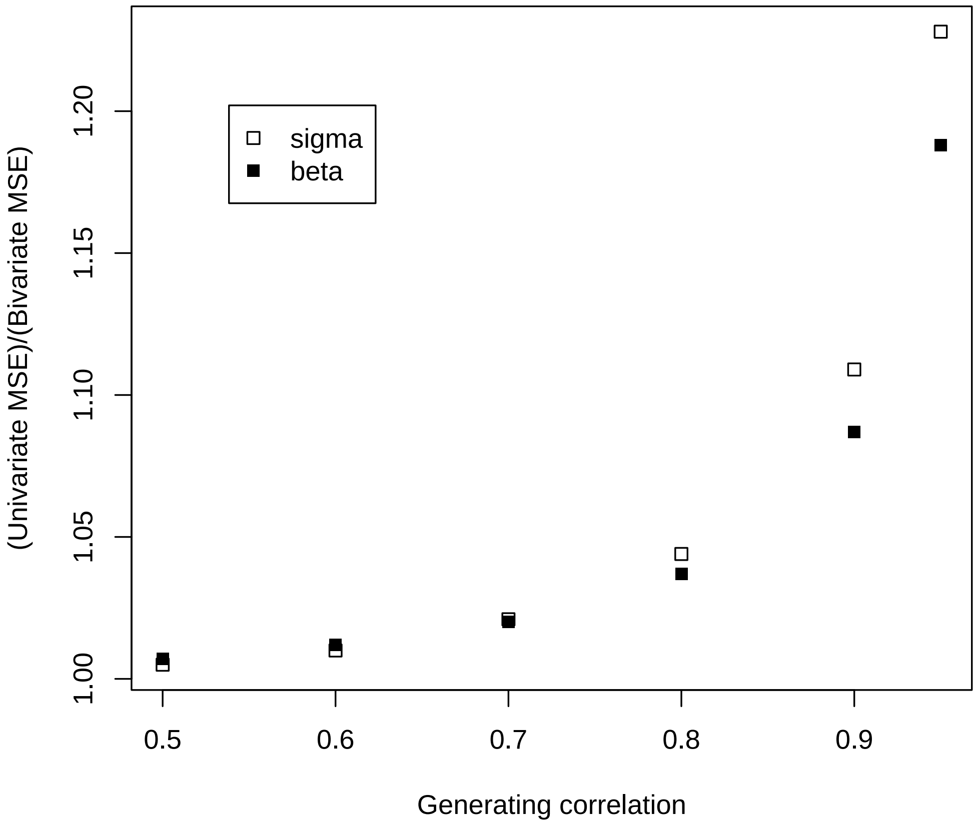


Figure 7: This figure corresponds to the data in the th columns for the σ and β parameters in Table 13.

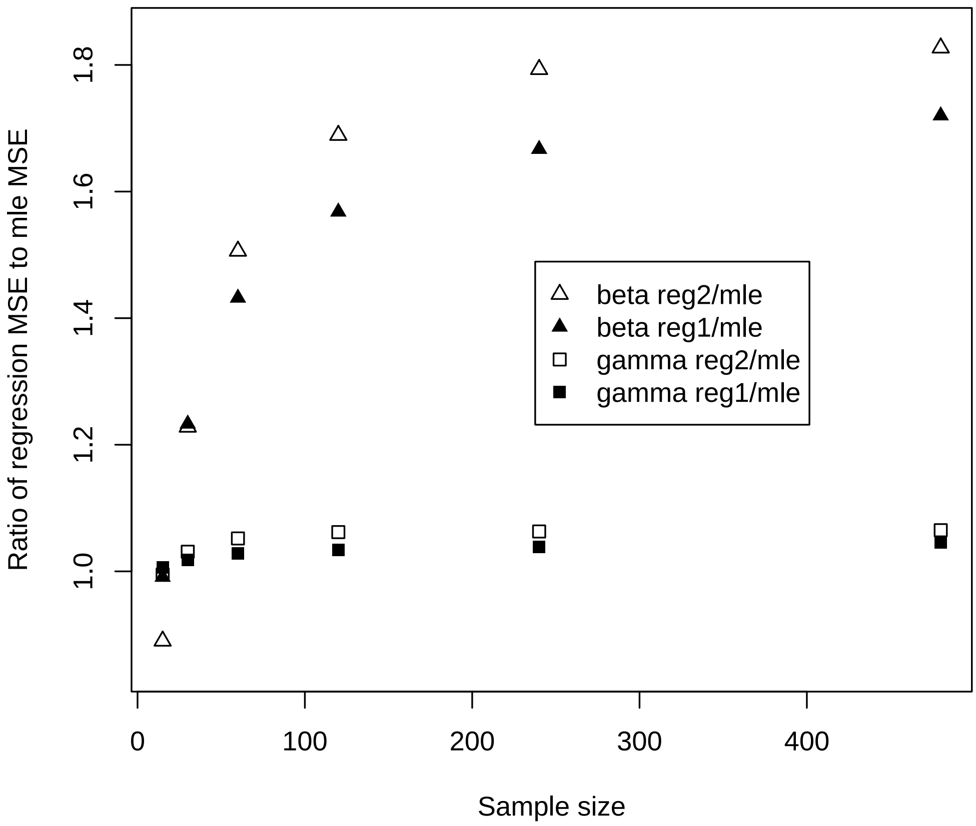


Figure 8: This figure corresponds to the data in Table 14. Simulations were performed for sample sizes 15, 30, 60, 120, 240, and 480.

Figure 9: Top of the input page of the Web program. Includes the data file input box.

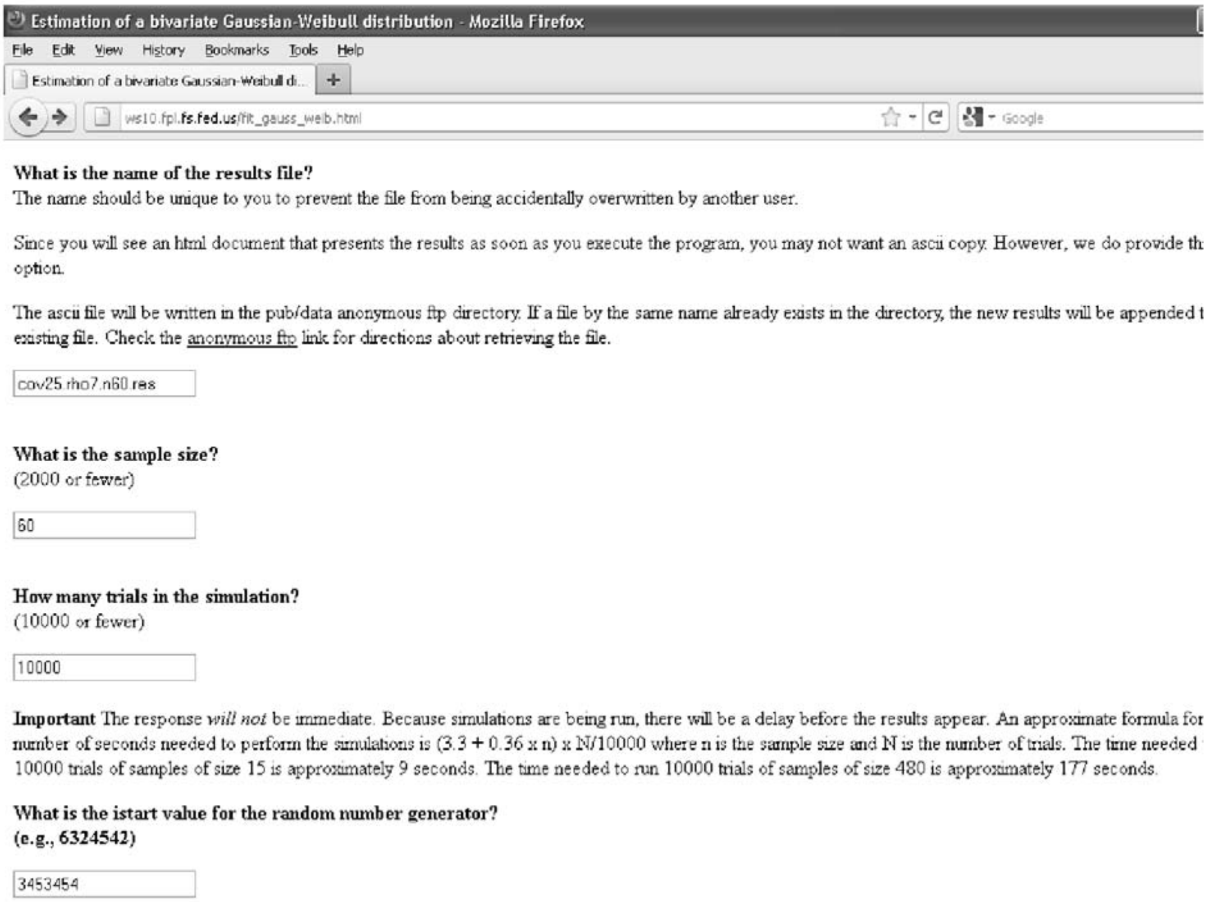


Figure 10: Middle of the input page of the Web program. Includes the results file, sample size, number of trials, and random number starting value input boxes.

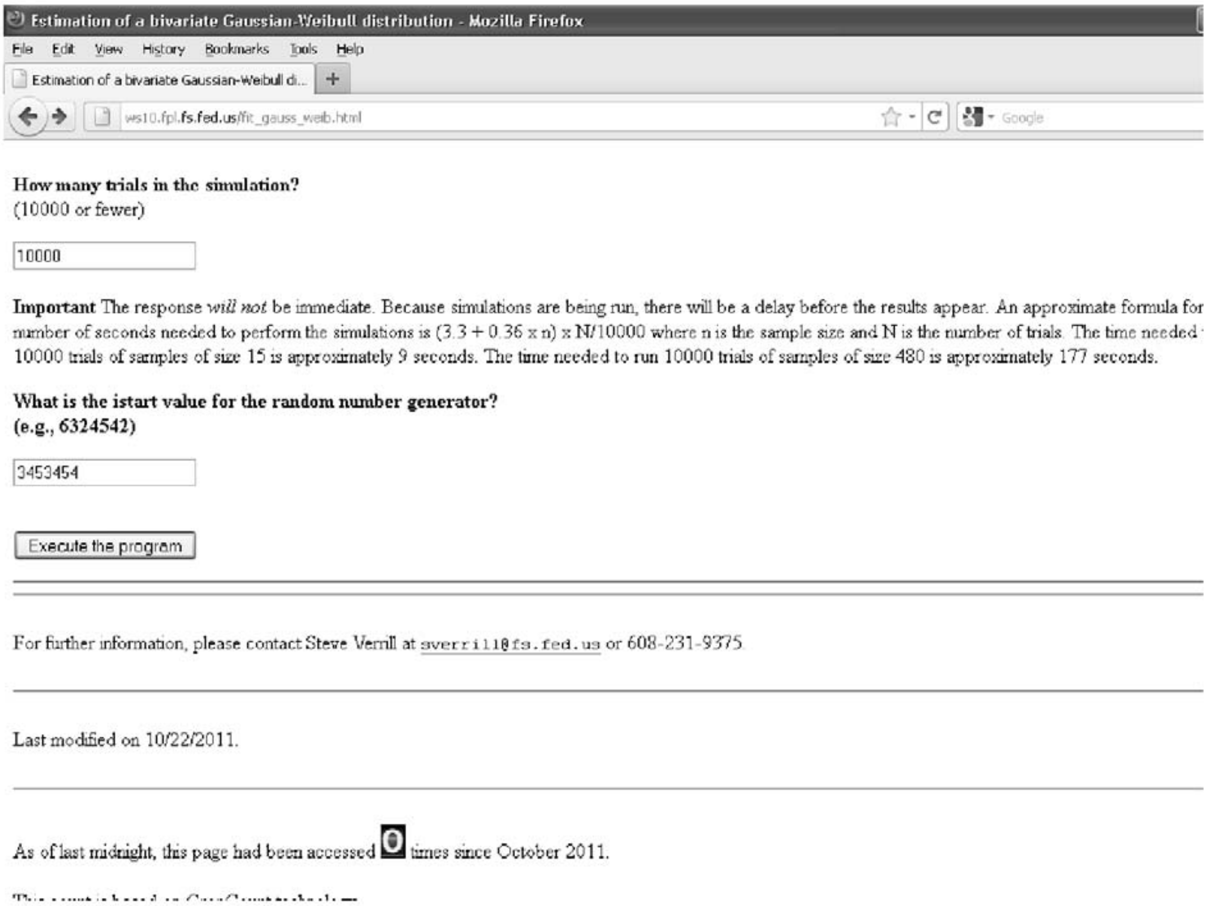
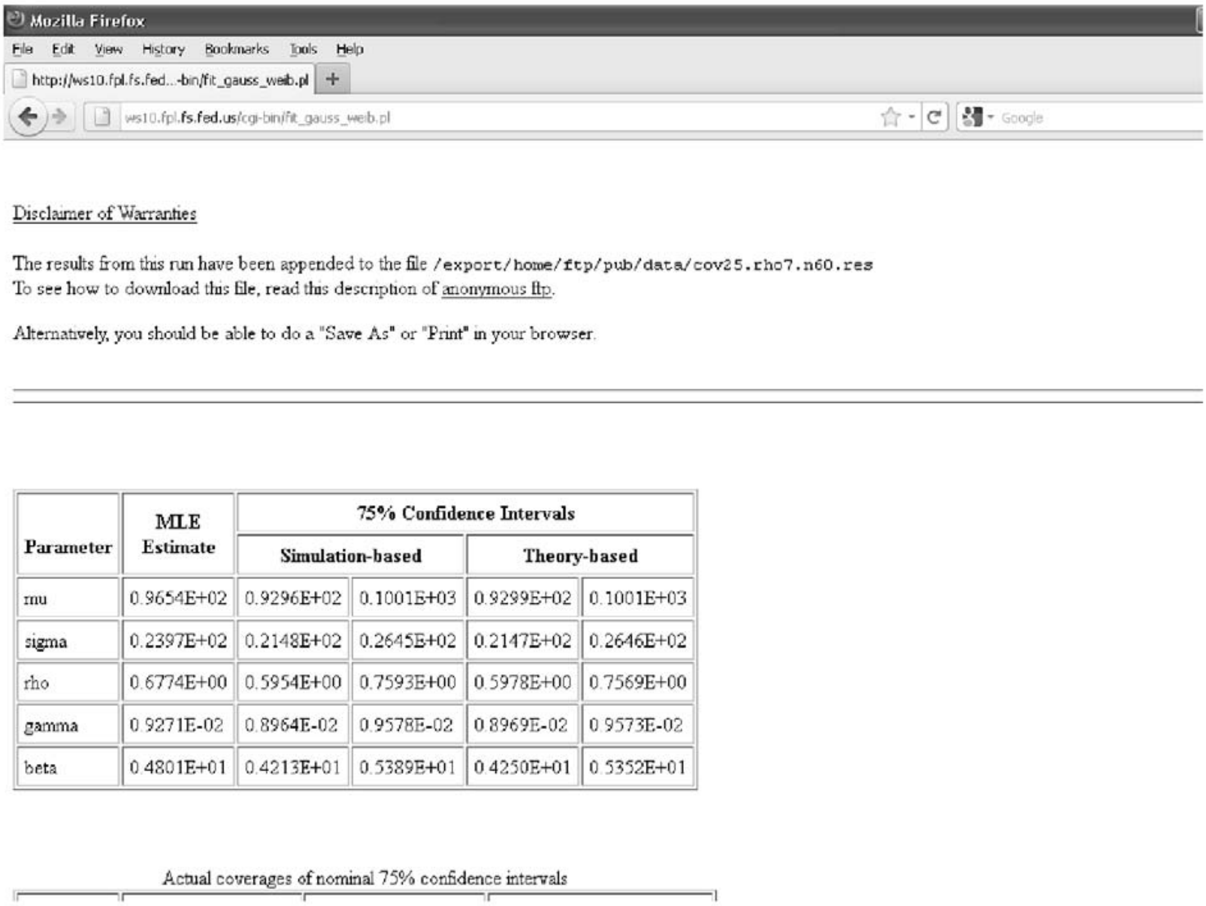


Figure 11: Bottom of the input page of the Web program. Includes the number of trials and random number starting value input boxes, and the execute button.



Disclaimer of Warranties

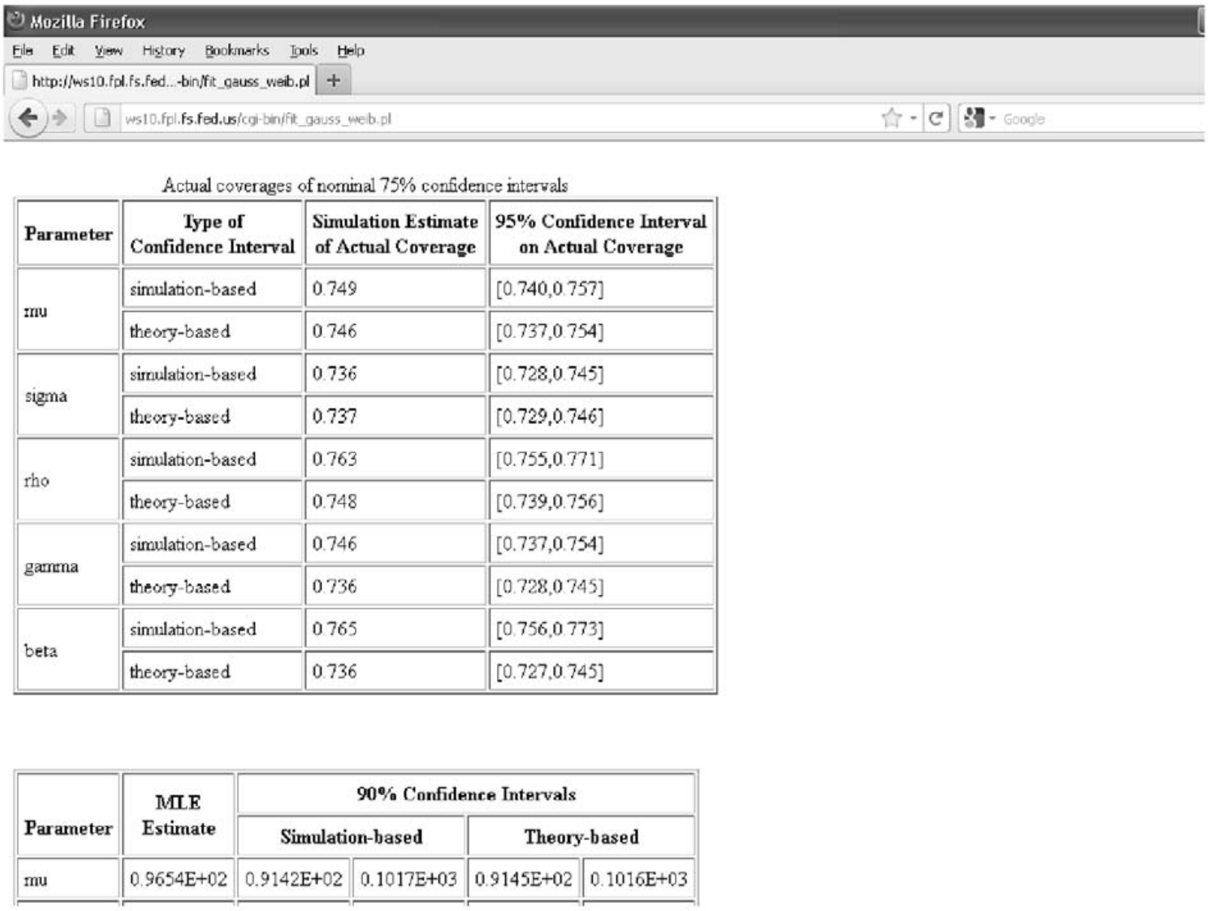
The results from this run have been appended to the file `/export/home/ftp/pub/data/cov25.rho7.n60.res`
To see how to download this file, read this description of anonymous ftp.

Alternatively, you should be able to do a "Save As" or "Print" in your browser.

Parameter	MLE Estimate	75% Confidence Intervals			
		Simulation-based		Theory-based	
mu	0.9654E+02	0.9296E+02	0.1001E+03	0.9299E+02	0.1001E+03
sigma	0.2397E+02	0.2148E+02	0.2645E+02	0.2147E+02	0.2646E+02
rho	0.6774E+00	0.5954E+00	0.7593E+00	0.5978E+00	0.7569E+00
gamma	0.9271E-02	0.8964E-02	0.9578E-02	0.8969E-02	0.9573E-02
beta	0.4801E+01	0.4213E+01	0.5389E+01	0.4250E+01	0.5352E+01

Actual coverages of nominal 75% confidence intervals

Figure 12: First results table. Includes the asymptotically efficient estimates of the five parameters of the bivariate Gaussian–Weibull together with asymptotic and simulation 75% confidence intervals on those parametes.



Actual coverages of nominal 75% confidence intervals

Parameter	Type of Confidence Interval	Simulation Estimate of Actual Coverage	95% Confidence Interval on Actual Coverage
mu	simulation-based	0.749	[0.740,0.757]
	theory-based	0.746	[0.737,0.754]
sigma	simulation-based	0.736	[0.728,0.745]
	theory-based	0.737	[0.729,0.746]
rho	simulation-based	0.763	[0.755,0.771]
	theory-based	0.748	[0.739,0.756]
gamma	simulation-based	0.746	[0.737,0.754]
	theory-based	0.736	[0.728,0.745]
beta	simulation-based	0.765	[0.756,0.773]
	theory-based	0.736	[0.727,0.745]

Parameter	MLE Estimate	90% Confidence Intervals			
		Simulation-based		Theory-based	
mu	0.9654E+02	0.9142E+02	0.1017E+03	0.9145E+02	0.1016E+03

Figure 13: Second table produced by the program. Simulation estimates of the actual coverages of nominal 75% confidence intervals on the parameters.