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Complex Compatible Taper and Volume Estimation Systems for Red and Loblolly Pine

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ABSTRACT. Five equation systems are described which can be used to estimate upper stem diameter, total individual tree cubic-foot volume, and merchantable cubic-foot volumes to any merchantability limit (expressed in terms of diameter or height), both inside and outside bark. The equations provide consistent results since they are mathematically related and are fit using stem analysis data from plantation-grown red and loblolly pine. Comparisons are made to determine which equation system provides the best overall fit to a set of validation data for each species. Results indicate that a system based on a segmented taper equation outperformed all other systems for both species. FOREST SCI. 32:423–443.

ADDITIONAL KEY WORDS. *Pinus resinosa, Pinus taeda.*

FOREST INVENTORY can be much more efficient if a system of equations is used to predict total and merchantable volumes to any merchantability limit. Such equations may also be used to predict diameter at any height and height at any diameter based only on commonly taken tree measurements (dbh and total height). Volume prediction to any merchantability limit has been accomplished in many ways but two are most common. One is to develop volume ratio equations that predict merchantable volume as a percentage of total tree volume (Honer 1964, Burkhart 1977). The other is to define an equation describing the stem taper. Integration of the taper equation from the ground to any height will give a merchantable volume estimate to that height (Kozak and others 1969). Predictions of diameter at any height and, in many cases, height at any diameter can be obtained by using a stem taper equation. Furthermore, it is often possible to derive volume ratio equations from a taper equation (Reed and Green 1984).

Ideally, such equation systems should be compatible. Demaerschalk (1972) defines compatible to mean that volumes estimated by integration of the taper curve are identical to the volumes obtained from the total volume or appropriate volume ratio equations. Volume estimation systems derived from integration of taper equations are compatible (i.e., mathematically related) when the coefficients of the derived volume equations can be written in terms of the taper equation coefficients. Besides the compatible equation systems derived from taper equations, Demaerschalk (1972) and Goulding and Murray (1976) have derived compatible total volume and taper equations by deriving the expression of taper from

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an existing total volume equation. Compatible taper equations have also been derived from existing volume ratio equations (Clutter 1980, Reed and Green 1984).

The accuracy and precision of the estimates of volume derived from a taper equation are dependent upon how well the taper equation fits the tree profile. It seems logical that a better fitting taper equation will result in more accurate and precise estimates of volume upon integration. Recent studies have shown that complex taper equations, such as segmented taper equations, provide a better fit of the stem profile than simple single taper equations, especially in the high volume butt region (Cao and others 1980; Martin 1981, 1984; Amidon 1984). Segmented taper equations describe each of several sections of a tree bole with separate equations. It is generally assumed that a tree stem can be divided into three geometric shapes, i.e., the top approaches a cone, the central section a frustum of a paraboloid, and the butt a frustum of a neiloid (Husch and others 1982). The method commonly used to describe these shapes is to fit each with a polynomial equation, usually quadratic, and then mathematically provide for a continuous curve at the two join points of the segments (Max and Burkhart 1976, Cao and others 1980). An attempt to describe the geometric shapes in terms of mathematical functions related to the shapes is the numerical iterative procedure used by the Weyerhaeuser Company (Frazer 1979). However, there has been no evaluation of how compatible volume estimation systems derived from complex taper equations perform in predicting both taper and volume.

The purpose of this study is threefold. The first of these is bringing together several compatible taper and volume equation systems from the literature that are based on highly recommended simple and complex taper equations. In conjunction with these previously defined systems we will derive a compatible volume estimation system from the segmented taper equation of Cao and others (1980). The second objective is to define a compatible taper and volume equation system based on a segmented taper equation using the mathematical equations related to the accepted geometric shapes of a tree stem. And finally, we will calibrate the taper and volume equation systems identified above for two conifers, red pine (*Pinus resinosa*) and loblolly pine (*Pinus taeda*). An attempt will be made to reduce total system squared error by utilizing a simultaneous fitting procedure described by Reed (1982) and used by Burkhart and Sprinz (1984) and Reed and Green (1984). The systems will be evaluated and recommendation will be made of the volume estimation system that provides the most accurate and precise estimates of stem taper and volume for each species.

NOTATION

The following notation will be used throughout the rest of this paper. Other notations specific to a particular equation will be listed with the equation.

a_i, b_i, c = regression coefficients estimated from sample data, where $i = 1, 2, 3, \ldots$
D = diameter at breast height (4.5 feet above ground),
d = top diameter at height h,
H = total tree height,
h = height above the ground to top diameter d,
K = 0.005454,
TV = total cubic-foot volume above the ground,
MV = cubic-foot volume from the ground to some top diameter or height limit (i.e., merchantable volume),
ib = inside bark,

ob = outside bark,
p = $H - h$,
z = $(H - h)/H$, relative tree height from the tip to top diameter d,
R = volume ratio, MV/TV; ratio which when multipled by total tree volume gives merchantable volume,
Rh = volume ratio for MV prediction to an upper height limit (h), and
Rd = volume ratio for MV prediction to an upper diameter limit (d).

DESCRIPTION OF VOLUME ESTIMATION SYSTEMS

Each of the systems has been derived from a taper equation. All of the taper equations are expressions of d^2 or d in terms of h, H, and D. Total volume and volume ratio equations are derived by integration of the taper equation. All equations assume that D and H are the only variables found by actual measurement. We define a volume estimation system as consisting of four equations: a taper equation, a total volume equation, a volume ratio equation for volume prediction to an upper height limit, and a volume ratio equation for volume prediction to an upper diameter limit.

The typical procedures for deriving volume equations from a taper equation are described here. Total volume is found by integrating an expression for basal area from zero (ground level) to H

$$\mathrm{TV} = \int_0^H K\{d(h)\}^2 \, \delta h. \tag{1}$$

Volume ratio equations are found by first deriving an expression for MV and then dividing it by the TV equation. The MV equation to a height limit is found by the same integration as TV except the upper limit of integration is h instead of H. The MV equation to a diameter limit is found by first algebraically redefining the taper equation so that h is in terms of d, D, and H. This expression for h is then substituted into the MV equation to a height limit for all partial heights (h), resulting in MV being expressed in terms of d, D, and H. After the volume ratio equations have been derived, MV is found by multiplying estimated TV by the estimated volume ratio:

$$\mathrm{MV} = \mathrm{TV}{*}R, \tag{2}$$

where $R = Rh$ or Rd. Two of the equation systems that will be described do not have an Rd component because expressions for h cannot be derived from the taper equation.

Equation Systems from Previous Work

Three of the equation systems are from other sources. The first one is based on a simple taper equation given by Demaerschalk (1972) that is compatible to the form factor total volume equation, TV = cD^2H. The taper equation is

$$d^2 = b_1 D^2 z^{b_2}. \tag{3}$$

This equation system will be referred to as the form factor equation system.

The second equation system is based on Max and Burkhart's (1976) segmented taper equation

$$d^2 = D^2[b_1(h/H - 1) + b_2\{(h/H)^2 - 1\} + b_3(a_1 - h/H)^2 I_1 + b_4(a_2 - h/H)^2 I_2], \tag{4}$$

where

$$I_i = \begin{cases} 1 \text{ if } h/H \le a_i \\ 0 \text{ if } h/H > a_i; \; i = 1, 2. \end{cases}$$

Martin (1981) described height and volume equations that are compatible with this taper equation; Green and Reed (1985) described volume ratio equations that are compatible with this taper equation.

The third equation system comes from the single taper equation defined by Bruce and others (1968)

$$
\begin{aligned}
d^2 = D^2[&b_1 x^{1.5}(10^{-1}) + b_2(x^{1.5} - x^3)D(10^{-2}) + b_3(x^{1.5} - x^3)H(10^{-3}) \\
&+ b_4(x^{1.5} - x^{32})HD(10^{-5}) + b_5(x^{1.5} - x^{32})H^{1/2}(10^{-3}) \\
&+ b_6(x^{1.5} - x^{40})H^2(10^{-6})],
\end{aligned}
\tag{5}
$$

where

$$x = \frac{(H - h)}{(H - 4.5)}.$$

Martin (1981) has previously defined both a TV and a MV equation to a height limit from this taper equation. In order to make it comparable to the other equation systems we defined an Rh equation by dividing the MV equation by the TV equation. This is one of the equation systems in which a volume ratio equation to a diameter limit cannot be derived algebraically. All three of these equation systems are described in the Appendix.

A volume estimation system is derived from the segmented taper equation given by Cao and others (1980)

$$(D^2KH/\mathrm{TV}) - 2z = b_1(3z^2 - 2z) + b_2(z - a_1)^2 I_1 + b_3(z - a_2)^2 I_2, \tag{6}$$

where

$$I_i = \begin{cases} 1 \text{ if } z \geq a_i \\ 0 \text{ if } z < a_i;\ i = 1, 2. \end{cases}$$

If we consider $\mathrm{TV} = cD^2H$ (the form factor TV equation), then

$$D^2 = \mathrm{TV}/cH, \tag{7}$$

and the taper equation can be rewritten as

$$
\begin{aligned}
d^2 = D^2(c/K)[&2z + b_1(3z^2 - 2z) + b_2(z - a_1)^2 I_1 \\
&+ b_3(z - a_2)^2 I_2],
\end{aligned}
\tag{8}
$$

where I_1 and I_2 are as previously defined. A TV equation can be derived by summing the following three integrals

$$\mathrm{TV} = \int_0^{h_1} K\{d(h)\}^2\,\delta h + \int_{h_1}^{h_2} K\{d(h)\}^2\,\delta h + \int_{h_2}^{H} K\{d(h)\}^2\,\delta h, \tag{9}$$

where h_1 and h_2 are variables representing the heights at the two join points of the model (i.e., a_1 and a_2, respectively). When these integrations are carried out the following TV equation results

$$\mathrm{TV} = c[1 + (b_2/3)(1 - a_1)^3 + (b_3/3)(1 - a_2)^3]D^2H. \tag{10}$$

This equation is in the form of a form factor total volume equation. An equation for merchantable volume to a height limit can be found in a similar way as the TV equation by integrating to height h instead of H. The resulting equation is

$$
\begin{aligned}
\mathrm{MV}_h = c[&1 + (b_1 - 1)z^2 - b_1 z^3 - (b_2/3)\{(z - a_1)^3 I_1 - (1 - a_1)^3\} \\
&- (b_3/3)\{(z - a_2)^3 I_2 - (1 - a_2)^3\}]D^2H.
\end{aligned}
\tag{11}
$$

A volume ratio equation to a height limit (Rh) is found by dividing the above equation by the TV equation

$$Rh = (1/\gamma)[1 + (b_1 - 1)z^2 - b_1z^3 - (b_2/3)\{(z - a_1)^3I_1 - (1 - a_1)^3\} - (b_3/3)\{(z - a_2)^3I_2 - (1 - a_2)^3\}], \tag{12}$$

where

$$\gamma = 1 + (b_2/3)(1 - a_1)^3 + (b_3/3)(1 - a_2)^3.$$

To obtain a volume ratio equation to a diameter limit (Rd) we must first algebraically redefine the taper equation in terms of d, D, and H:

$$h = H[1 - \{(-B \pm (B^2 - 4AC)^{1/2})/2A\}], \tag{13}$$

where

$$A = (c/K)(3b_1 + b_2J_1 + b_3J_2),$$
$$B = (2c/K)(1 - b_1 - a_1b_2J_1 - a_2b_3J_2),$$
$$C = (c/K)(a_1^2b_2J_1 + a_2^2b_3J_2) - (d^2/D^2),$$
$$J_i = \begin{cases} 1 \text{ if } d \geq M_i \\ 0 \text{ if } d < M_i; \; i = 1, 2, \end{cases}$$
$$M_i = \text{estimated diameter at } h_i$$
$$= D\{(c/K)[2a_i + b_1(3a_i^2 - 2a_i) + b_3(a_i - a_2)^2]\}^{1/2}.$$

Substitution of equation (13) into equation (12) results in the following Rd equation:

$$Rd = (1/\gamma)[1 + (b_1 - 1)w^2 - b_1w^3 - (b_2/3)\{(w - a_1)^3I_1 - (1 - a_1)^3\} - (b_3/3)\{(w - a_2)^3I_2 - (1 - a_2)^3\}], \tag{14}$$

where

$$w = [-B \pm (B^2 - 4AC)^{1/2}]/2A,$$

and A, B, C, and γ are as previously defined.

Equation System Based on Geometric Shapes

The usually accepted geometric shapes of a tree stem are neiloidic, parabolic, and conic for the lower, mid, and upper bole sections, respectively. As stated by Grosenbaugh (1954) and again by Forslund (1982), these shapes can be represented by a simple power function

$$y = mx^p, \tag{15}$$

where m is an appropriate constant and the form is neiloidic if $p = 3/2$, parabolic if $p = 1/2$, and conic if $p = 1$. It is apparent that a taper equation using these shapes will have x as a function of height and y as a function of diameter. If

$$y = d, \tag{16}$$

$$x = (H - h)/(H - 4.5), \tag{17}$$

$$m = D, \tag{18}$$

then the taper equation defined by Ormerod (1973) is in the general form for these shapes (Reed and Byrne 1985)

$$d = D\left(\frac{H - h}{H - 4.5}\right)^b. \tag{19}$$

The coefficient b can be defined as the power of the equations used for the shapes. A convenient way to join the three equations for the shapes into a smooth curve is to define b as a segmented equation similar to several of the previous taper equations. In order to correspond with the accepted ideas of tree form, this segmented equation should have $b = 3/2$ at the bottom part of the tree, $b = 1/2$ at the center, and $b = 1$ at the top of the tree. After considering several segmented forms, we decided to use a two-segmented equation to define the coefficient b, where b is dependent on the relative height from the ground (h/H) and one fitted coefficient (a_1). The form of the taper equation is

$$d = Dx^b,$$

where

$$b = 3/2 - (h/H)/a_1 - [1 - \{(h/H)/a_1\}]I_1 + (1/2)[\{(h/H) - a_1\}/(1 - a_1)]I_1, \quad (20)$$

and

$$I_1 = 1 \text{ if } h/H \geq a_1, \text{ otherwise } I_1 = 0.$$

This equation has the constraints that $b = 3/2$ at the base of the tree, and then decreases linearly (as h/H increases) to $b = 1/2$ at the join point a_1 ($a_1 = h_1/H$). Above a_1, b increases linearly from $b = 1/2$ to $b = 1$ at the top of the tree. Defining b in this way does not insure that the base is a frustum of a neiloid, the center is a frustum of a paraboloid, or that the top is a cone but allows for varying forms, fairly similar to the shapes, with increasing height. One discrepancy in this model is the transition between the neiloid and the paraboloid. A linearly smooth transition from $b = 3/2$ to $b = 1/2$ cannot be carried out without passing through $b = 1$ (a cone). The regression fit of the one join point should optimize the shapes that best fit the tree form since no way could be found to define b to conform to the accepted shapes of the tree. A similar system using two join points was developed but found to be inferior to the single join point geometric equation early in the testing process.

Unfortunately, the geometric taper equation cannot be integrated to an exact form. For estimates of total volume and volume ratio to a height limit, numerical integration must be used. Since the equation cannot be put in the form $h = f(d, D, H)$, estimates of volume ratio to a diameter limit are not possible.

DATA

Stem analysis data from red and loblolly pine trees were used in this study. The red pine data came mostly from the Upper Peninsula of Michigan, with a few trees from northern Wisconsin and the northern Lower Peninsula of Michigan. This old-field plantation red pine data came from three sources: (1) Champion Timberlands, Inc., of Norway, Michigan, (2) a red pine growth study carried out at the Ford Forestry Center, Michigan Technological University, and (3) a red pine thinning study conducted by Michigan Technological University. The loblolly data came from old-field plantations scattered throughout the Virginia Piedmont and Coastal Plain and the Coastal Plain of Delaware, Maryland, and North Carolina. These data were made available by Virginia Polytechnic Institute and State University, Blacksburg, Virginia. The same general stem analysis procedure was used for all studies.

Single-stemmed trees were felled and cut into sections. Trees were sectioned at dbh and then every 4 or 6 feet after that. For the red pine, cuts were made until a full section could not be obtained but for loblolly pine, the last cut was made at approximately a 2-inch upper diameter. At each cut, diameters (both inside and outside bark) were measured to the nearest 0.1 inch and the height from the ground to the cut was measured to the nearest 0.1 foot. Total tree height from the ground to the tip was also measured to the nearest 0.1 foot. Section volumes were determined using Smalian's formula (Avery and Burkhart 1983). Volumes of the tree top and stump were found by treating these parts as a cone and a cylinder, respectively. Total tree volume was found by summing the section, tree top, and stump volumes. All outside and inside bark volumes are in cubic feet.

Data from 249 red pine and 378 loblolly pine trees were used in the study. Each species data set was split into developmental and test data sets. Seventy percent of the trees (178 for red pine and 265 for loblolly pine) were randomly selected and the height/diameter observations associated with these trees (2,176 for red pine and 2,482 for loblolly pine) made up developmental data sets used in fitting the equations. The other thirty percent (71 for red pine and 113 for loblolly pine) were considered as being representative of the population and used in testing the validity of the fitted volume estimation systems. The number of height/diameter observations for the validation data sets are 907 for red pine and 1,039 for loblolly pine. The data for the four data sets are summarized by height and diameter classes in Table 1.

MODEL FITTING

Two methods were used in fitting the volume estimation systems to the developmental data. The first method is to fit the taper equation to the data using least squares techniques and then algebraically solve for the coefficients of the other equations based on the fitted taper equation coefficients. The second is to simultaneously fit all equations in each system using a numerical minimization procedure described by Reed and Green (1984). These methods are used for both outside and inside bark data. Table 2 provides a summary of the equation systems and fitting procedures used in this study.

The taper equations were fit to the data using an International Mathematical and Statistical Library (IMSL) minimization routine, ZXMIN (IMSL 1982). This routine was used to minimize sum of squared error (i.e., sum of squared observed minus predicted values, SSE). All of the taper equations were made to be expressions of d when fitting so that the SSE would be consistent. The sum of squared error for each model is given in Table 3. It is apparent that the segmented taper equations, Max and Burkhart's and Cao's, along with the complex single taper equation by Bruce, provided much better fits to the data than any of the other models. In all cases the SSE for the two segmented equations were nearly identical. Though the two equations are of a slightly different form, they both use quadratic equations to describe each of the three segments of a tree stem. This probably explains why the SSE are so similar. The geometric model gave the worst fit in most cases; it outperformed the form factor system on loblolly pine, outside bark, and was very similar to the form factor system for red pine, inside bark. The form factor and geometric models, though obviously poorer fitting models, were kept for further study for comparison against the more complex but better fitting models.

In the simultaneous fitting procedure, all four equations in each volume estimation system (taper, TV, Rh, Rd) are fit to the data at the same time. This procedure, using ZXMIN, minimizes the total system squared error (TSSE) for each model. TSSE is defined as the summation of the squared observed minus predicted values for each of the equations in a system (Reed and Green 1984).

$$\mathrm{TSSE} = \sum_{i=1}^{N} \frac{(Y_i - \hat{Y}_i)^2}{\hat{\sigma}_y^2} + \sum_{i=1}^{n} \frac{(V_i - \hat{V}_i)^2}{\hat{\sigma}_V^2} + \sum_{i=1}^{N} \frac{(Rh_i - \hat{Rh}_i)^2}{\hat{\sigma}_{Rh}^2} + \sum_{i=1}^{N} \frac{(Rd_i - \hat{Rd}_i)^2}{\hat{\sigma}_{Rd}^2}$$

TABLE 1. Red pine and loblolly pine. Data distribution of sample trees into diameter and height classes.

Dbh class (inches)	Total height class (feet) 20–29[b]	30–39	40–49	50–59	60–69	70–79	80–89	Total
RED PINE								
3[a]								0 (0)
4	4	I						5 (0)
5	8 (1)	2 (2)	2 (1)					12 (6)
6	4 (1)	4 (2)	3 (2)	2 (1)				13 (6)
7	3	5 (4)	12 (3)	10 (3)	1 (1)			31 (11)
8		2 (I)	10 (2)	18 (7)	3 (4)			33 (14)
9			4	23 (5)	5 (3)			32 (8)
10		I	1	13 (5)	16 (9)	(1)		31 (15)
11			I	4 (1)	8 (4)	1		14 (5)
12				2	2 (4)	2		6 (4)
13				I		(2)		1 (2)
Total	19 (2)	15 (11)	33 (8)	73 (22)	35 (25)	3 (3)	0 (0)	178 (71)
LOBLOLLY PINE								
3	3							3 (I)
4	8 (6)	15 (6)	I (1)					24 (13)
5	14 (4)	22 (8)	18 (12)	3 (1)				57 (25)
6	6 (I)	22 (9)	35 (17)	13 (8)	I			77 (35)
7	I	4 (3)	23 (13)	21 (4)	2 (2)			51 (22)
8			7 (4)	17 (4)	4 (I)		1	29 (9)
9			4	7 (2)	3 (1)	(2)		14 (5)
10				2 (1)	2 (I)	2		6 (2)
11				1	2			3 (0)
12					(1)		I	I (0)
13								0 (0)
Total	32 (12)	63 (26)	88 (47)	64 (20)	14 (6)	2 (2)	2 (0)	265 (113)

TABLE 2. Summary of models.

Model	Model description	Model	Model description
A	Form Factor—Taper curve fit	F	Form Factor—Simultaneous fit
B	Bruce—Taper curve fit	G	Bruce—Simultaneous fit
C	Max and Burkhart—Taper curve fit	H	Max and Burkhart—Simultaneous fit
D	Cao—Taper curve fit	I	Cao—Simultaneous fit
E	Geometric—Taper curve fit		

where

Y_i, $\hat{Y}_i$ = observed and predicted diameters for the taper function, respectively,

V_i, $\hat{V}_i$ = observed and predicted total cubic-foot volume,

Rh_i, $\hat{Rh}_i$ = observed and predicted volume ratios for MV prediction to an upper height limit,

Rd_i, $\hat{Rd}_i$ = observed and predicted volume ratios for MV prediction to an upper diameter limit,

$\hat{\sigma}_Y{}^2$ = mean square error from the least squares fit of the taper equation,

$\hat{\sigma}_V{}^2$ = mean square error from the least squares fit of the total volume equation,

$\hat{\sigma}_{Rh}{}^2$ = mean square error from the least squares fit of the volume ratio to a height limit equation,

$\hat{\sigma}_{Rd}{}^2$ = mean square error from the least squares fit of the volume ratio to a diameter limit equation,

N = number of height/diameter observations for fitting the equation, and

n = number of trees for fitting the equation.

The reuslts are shown in Table 3. Bruce's system cannot be involved in comparison here because only three component equations (no *Rd*) are available. The geometric system cannot be simultaneously fit because volumes can only be obtained by numerical integration. As with the taper curve fitting method, the two segmented systems outperform the simpler form factor system. The estimates of the regression coefficients for all of the systems and fitting procedures are shown in Table 3.

TESTING ACCURACY AND PRECISION

To further compare the volume estimation systems and fitting procedures, the fitted equations from each system-fitting approach are used to predict taper, total volume, and volume ratios from each of the height/diameter observations in the validation data. For the geometric taper equation, where no total volume or volume-ratio equations are algebraically possible, numerical integration (using IMSL routine DCADRE (IMSL 1982)) is used to predict total volume and volume ratio to a height limit. Neither the geometric system or Bruce's system can be used to predict volume ratio to a diameter limit.

Four criteria are used in comparing the models. Each criterion is based on the

←

[a] Diameter classes: 3 = 2.6–3.5 inches.

[b] Height classes: 20–29 = 19.6–29.5 feet.

Note: Parentheses indicate the number in the validation data set; no parentheses indicate the number in the developmental data set.

TABLE 3. Regression coefficients and fit statistics for each model for red pine and loblolly pine, outside and inside bark.

Species and model	Coefficient estimates						Fit statistics	
	b_1	b_2	b_3	$b_4(c)$	$b_5(a_1)$	$b_6(a_2)$	SSE[a]	TSSE[b]
RED PINE								
Outside bark								
A	1.267704	1.423807					486.762	
B	9.999849	−8.508098	20.507620	−0.111857	7.373357	−16.488880	268.089	
C	−5.106921	2.509328	65.683792	−2.766808	0.089224	0.743607	259.765	
D	0.949934	74.693730	−3.142097	0.004803	0.910842	0.256365	259.765	
E					0.431250		662.937	
F	1.298522	1.412098						8,728.62
G	10.385146	−6.972308	17.628113	−6.147729	2.334114	−1.978681		[c]5,063.16
H	−5.107827	2.508776	65.683795	−2.765967	0.089612	0.737490		6,826.54
I	0.985003	79.839830	−3.016924	0.005487	0.919393	0.210650		7,021.54
Inside bark								
A	1.116233	1.405224					446.634	
B	8.936711	−7.272773	17.981730	0.588732	5.264584	−11.957750	311.735	
C	−4.669834	2.313933	50.143560	−2.700603	0.085609	0.724035	268.918	
D	0.973539	63.079675	−3.408825	0.004322	0.914280	0.275954	268.918	
E					0.446867		473.527	
F	1.121683	1.365076						7,605.19
G	9.199652	−6.695521	16.910534	−3.609651	1.550591	−1.471831		[c]4,854.95
H	−4.670087	2.310000	50.143000	−2.699804	0.090363	0.719321		6,952.29
1	0.973538	63.079675	−3.408859	0.004443	0.914493	0.275412		6,793.89

TABLE 3. *Continued.*

Species and model	Coefficient estimates						Fit statistics	
	b_1	b_2	b_3	$b_4(c)$	$b_5(a_1)$	$b_6(a_2)$	SSE[a]	TSSE[b]
LOBLOLLY PINE								
Outside bark								
A	1.315396	1.665845					716.752	
B	9.728571	−5.553458	9.345601	−5.492790	11.226790	−19.155330	319.391	
C	−4.681835	2.281302	96.661845	−1.793894	0.087034	0.846905	300.808	
D	0.868883	147.922142	−1.924000	0.003862	0.915926	0.179294	300.893	
E					0.475850		674.935	
F	1.470445	1.806498						13,822.74
G	10.335325	−4.278748	5.537427	−8.430992	3.982315	−4.631713		[c]6,115.60
H	−4.096076	1.993723	107.053600	−1.516756	0.090950	0.799030		8,350.09
I	0.951004	145.450708	−2.003189	0.004428	0.915990	0.137474		8,509.54
Inside bark								
A	0.941520	1.505741					526.285	
B	7.060716	−6.843039	13.751920	−3.207006	8.421680	−14.612010	297.277	
C	−3.222793	1.553754	95.929052	−1.521432	0.073542	0.762259	294.117	
D	0.906743	149.933632	−2.663458	0.003181	0.923505	0.234427	294.163	
E					0.999950		910.819	
F	1.032872	1.581755						11,879.84
G	7.536934	−4.952386	9.328257	−5.205323	2.861503	−3.506438		[c]5,891.77
H	−3.209642	1.557651	96.012643	−1.517059	0.080098	0.746226		8,168.97
I	0.927994	146.655491	−2.543955	0.003367	0.918194	0.206147		8,100.21

[a] Sum of Squared Error (i.e., sum of squared observed minus predicted values).
[b] Total System Squared Error (i.e., the summation of the squared observed minus predicted values for each of the equations in a system).
[c] Bruce TSSE is based on simultaneous fit of only 3 equations (*Rd* equation does not exist).
Note: Simultaneous fitting not feasible for geometric model because volume and volume-ratio equations cannot be algebraically obtained from the taper equation.

residuals (or differences) between the observed quantities and predicted quantities

$$D_i = X_i - \hat{X}_i,$$

where

D_i = residual or difference,
X_i = observed value, and
$\hat{X}_i$ = predicted value.

The four performance criteria are defined below.

(1) Average Residual or Bias ($\bar{D}$):

$$\bar{D} = \frac{\sum_{i=1}^{N} D_i}{N}$$

(2) Standard Deviation of the Residuals or Precision(s):

$$s = \sqrt{\frac{\sum_{i=1}^{N} D_i^2 - \frac{\left(\sum_{i=1}^{N} D_i\right)^2}{N}}{N - 1}}$$

(3) Average Absolute Residual ($|\overline{D}|$):

$$|\overline{D}| = \frac{\sum_{i=1}^{N} |D_i|}{N}$$

(4) Percent Variation Explained (PVE):

$$\text{PVE} = \frac{\sum_{i=1}^{N} (X_i - \bar{X})^2 - \sum_{i=1}^{N} D_i^2}{\sum_{i=1}^{N} (X_i - \bar{X})^2}$$

where

$\bar{X}$ = average observed value, and
N = number of values to be compated in the validation data.

Using the calculated test statistics (criteria), taper, total volume, and volume ratio to a height limit equations, for all systems and fitting approaches, are all excellent predictors, except for the geometric model, which is slightly surpassed in prediction. But the volume ratio to a diameter limit equation produces less accurate and less precise predictions than the volume ratio to a height limit equation for all systems and both species. This is consistent with similar results reported by Van Deusen and others (1982), Reed and Green (1984), and Reed and Byrne (1985). The volume ratios to a height limit are overpredicted (negative average residuals) for all models on red pine but not with loblolly pine. No other such patterns are apparent with the other three equations.

It is also of value to note how the models fit the different species. In general,

TABLE 4. Values of the test statistics for the validation data for the best models—red pine.

Equation		Criteria	Max and Burkhart TC[a]	Max and Burkhart Sim[b]	Cao TC	Cao Sim
Taper (inches)	Outside bark	$\bar{D}$[c]	0.06	0.00	0.06	0.03
		s[d]	0.35	0.36	0.35	0.36
		$\lvert\bar{D}\rvert$[e]	0.25	0.26	0.25	0.25
		PVE[f]	98.9	98.8	98.9	98.8
	Inside bark	$\bar{D}$	0.10	0.03	0.10	0.03
		s	0.33	0.35	0.33	0.34
		$\lvert\bar{D}\rvert$	0.24	0.25	0.24	0.24
		PVE	98.8	98.7	98.8	98.9
Total volume (ft^3)	Outside bark	$\bar{D}$	0.267	0.006	0.267	0.127
		s	0.710	0.727	0.710	0.715
		$\lvert\bar{D}\rvert$	0.573	0.523	0.573	0.542
		PVE	99.1	99.2	99.1	99.2
	Inside bark	$\bar{D}$	0.414	0.131	0.413	0.112
		s	0.714	0.698	0.714	0.698
		$\lvert\bar{D}\rvert$	0.591	0.512	0.591	0.512
		PVE	98.7	99.1	98.7	99.1
Volume ratio (height)	Outside bark	$\bar{D}$	−0.0089	−0.0097	−0.0089	−0.0097
		s	0.0169	0.0171	0.0169	0.0172
		$\lvert\bar{D}\rvert$	0.0127	0.0132	0.0127	0.0132
		PVE	99.6	99.6	99.6	99.6
	Inside bark	$\bar{D}$	−0.0093	−0.0107	−0.0093	−0.0067
		s	0.0172	0.0175	0.0172	0.0171
		$\lvert\bar{D}\rvert$	0.0130	0.0139	0.0130	0.0129
		PVE	99.6	99.6	99.6	99.6
Volume ratio (diam.)	Outside bark	$\bar{D}$	−0.0030	−0.0140	−0.0030	−0.0095
		s	0.0428	0.0447	0.0428	0.0436
		$\lvert\bar{D}\rvert$	0.0262	0.0288	0.0262	0.0279
		PVE	98.2	97.8	98.2	98.0
	Inside bark	$\bar{D}$	0.0028	−0.0104	0.0028	−0.0102
		s	0.0459	0.0460	0.0459	0.0459
		$\lvert\bar{D}\rvert$	0.0277	0.0030	0.0277	0.0297
		PVE	97.9	97.8	97.9	97.8

[a] Taper curve fit.
[b] Simultaneous fit.
[c] Average residual.
[d] Standard deviation of residuals.
[e] Average absolute residual.
[f] Percent variation explained.

all of the models fit the red pine data closer than the loblolly pine data, especially when the statistics for the volume ratio to a diameter limit equation are considered. The taper equation also shows this trend but not as markedly as the volume ratio to a diameter limit equation. Statistics from the total volume and volume ratio to a height limit equations are very similar for both species.

For red pine, the fitting approaches tended to be roughly the same in their predictive ability. But for loblolly pine the simultaneous fitting produced a definite improvement in how the model fit the data. This is especially apparent with the volume ratio to a diameter limit equation.

To aid in the comparison of the systems, a mid-ranking procedure was used to rank the nine system-fitting procedure combinations. For each component equation (taper, total volume, volume ratios to height and diameter limits) and each test statistic, a rank from 1 (the best) to 9 (the worst) was assigned to each system-

TABLE 5. Values of the test statistics for the validation data for the best models—loblolly pine.

Equation		Criteria	Max and Burkhart TC[a]	Max and Burkhart Sim[b]	Cao TC	Cao Sim
Taper (inches)	Outside bark	$\bar{D}$[c]	0.02	−0.03	0.02	−0.02
		s[d]	0.35	0.38	0.35	0.38
		$\lvert\bar{D}\rvert$[e]	0.26	0.27	0.26	0.28
		PVE[f]	96.9	96.4	96.9	96.3
	Inside bark	$\bar{D}$	0.01	−0.04	0.01	−0.03
		s	0.33	0.34	0.33	0.35
		$\lvert\bar{D}\rvert$	0.24	0.26	0.24	0.26
		PVE	96.2	95.8	96.2	95.8
Total volume (ft^3)	Outside bark	$\bar{D}$	0.081	−0.020	0.084	−0.015
		s	0.497	0.533	0.496	0.531
		$\lvert\bar{D}\rvert$	0.356	0.353	0.356	0.352
		PVE	97.9	97.6	97.9	97.7
	Inside bark	$\bar{D}$	0.095	0.001	0.092	0.008
		s	0.473	0.474	0.472	0.473
		$\lvert\bar{D}\rvert$	0.303	0.300	0.303	0.299
		PVE	97.1	97.2	97.1	97.2
Volume ratio (height)	Outside bark	$\bar{D}$	0.0088	0.0020	0.0093	−0.0016
		s	0.0248	0.0242	0.0249	0.0241
		$\lvert\bar{D}\rvert$	0.0186	0.0173	0.0188	0.0173
		PVE	99.2	99.4	99.2	99.4
	Inside bark	$\bar{D}$	0.0090	0.0026	0.0088	0.0024
		s	0.0235	0.0229	0.0235	0.0227
		$\lvert\bar{D}\rvert$	0.0179	0.0165	0.0178	0.0164
		PVE	99.3	99.4	99.3	99.4
Volume ratio (diam.)	Outside bark	$\bar{D}$	0.0054	−0.0010	0.0070	−0.0066
		s	0.0979	0.0746	0.0952	0.0702
		$\lvert\bar{D}\rvert$	0.0540	0.0455	0.0536	0.0440
		PVE	89.5	93.9	90.0	94.5
	Inside bark	$\bar{D}$	0.0101	−0.0037	0.0090	−0.0012
		s	0.0866	0.0741	0.0882	0.0735
		$\lvert\bar{D}\rvert$	0.0533	0.0485	0.0537	0.0483
		PVE	91.8	94.1	91.6	94.2

[a] Taper curve fit.
[b] Simultaneous fit.
[c] Average residual.
[d] Standard deviation of residuals.
[e] Average absolute residual.
[f] Percent variation explained.

fitting approach combination. For average residual, standard deviation of residuals, and average absolute residual the lower the value of the statistic, the better the equation predicts and it is therefore assigned a lower rank. But with PVE, a higher value indicates a better prediction (so a lower rank). In the case where the statistic values are the same for several systems, a mid-rank or average rank was assigned to each of these systems. By summing all of the ranks, a general guideline can be established to determine the best system. In general, the lower the rank sum, the better the predictive ability of the equation.

From these assigned rankings, the segmented models, Max and Burkhart's and Cao's (Models C, D, H, I), clearly were better fitting models. For red pine, any of the segmented models and either fitting procedure generally gave good rankings but for loblolly pine the clearly best ranked models were (I) and (H), the two

TABLE 6. Values of the test statistics for the validation data by % height classes for the best models—red pine, outside bark.

Equation	% of total height	N[a]	$\bar{D}$[b] MB-Sim[d]	$\bar{D}$[b] Cao-Sim[e]	s[c] MB-Sim	s[c] Cao-Sim
Taper (inches)	$0.0 < X < 0.1$	125	−0.26	−0.23	0.36	0.36
	$0.1 \le X < 0.2$	75	−0.13	−0.12	0.24	0.24
	$0.2 \le X < 0.3$	81	−0.17	−0.12	0.31	0.31
	$0.3 \le X < 0.4$	85	−0.05	0.02	0.27	0.27
	$0.4 \le X < 0.5$	77	0.07	0.15	0.27	0.27
	$0.5 \le X < 0.6$	91	0.12	0.18	0.34	0.35
	$0.6 \le X < 0.7$	79	0.28	0.29	0.40	0.41
	$0.7 \le X < 0.8$	87	0.21	0.13	0.39	0.39
	$0.8 \le X < 0.9$	85	0.09	0.07	0.38	0.37
	$0.9 \le X < 1.0$	122	0.00	0.03	0.22	0.22
Volume ratio (height)	$0.0 < X < 0.1$	125	−0.0078	−0.0082	0.0072	0.0072
	$0.1 \le X < 0.2$	75	−0.0096	−0.0113	0.0186	0.0187
	$0.2 \le X < 0.3$	81	−0.0172	−0.0192	0.0191	0.0191
	$0.3 \le X < 0.4$	85	−0.0183	−0.0194	0.0257	0.0257
	$0.4 \le X < 0.5$	77	−0.0197	−0.0193	0.0234	0.0234
	$0.5 \le X < 0.6$	91	−0.0130	−0.0113	0.0211	0.0211
	$0.6 \le X < 0.7$	79	−0.0119	−0.0097	0.0146	0.0146
	$0.7 \le X < 0.8$	87	−0.0050	−0.0038	0.0095	0.0094
	$0.8 \le X < 0.9$	85	−0.0011	−0.0013	0.0047	0.0047
	$0.9 \le X < 1.0$	122	0.0000	−0.0001	0.0008	0.0008
Volume ratio (diam.)	$0.0 < X < 0.1$	125	−0.0313	−0.0332	0.0295	0.0300
	$0.1 \le X < 0.2$	75	−0.0393	−0.0368	0.0873	0.0811
	$0.2 \le X < 0.3$	81	−0.0530	−0.0422	0.0706	0.0689
	$0.3 \le X < 0.4$	85	−0.0248	−0.0115	0.0494	0.0491
	$0.4 \le X < 0.5$	77	−0.0065	0.0061	0.0364	0.0369
	$0.5 \le X < 0.6$	91	0.0012	0.0103	0.0236	0.0252
	$0.6 \le X < 0.7$	79	0.0081	0.0128	0.0185	0.0206
	$0.7 \le X < 0.8$	87	0.0024	0.0022	0.0073	0.0084
	$0.8 \le X < 0.9$	85	0.0011	0.0006	0.0032	0.0033
	$0.9 \le X < 1.0$	122	0.0002	0.0002	0.0022	0.0021

[a] Number of height/diameter observations in a height class.
[b] Average residual.
[c] Standard deviation of residuals.
[d] Max and Burkhart—Simultaneous fit model.
[e] Cao—Simultaneous fit model.
Note: $X = h/H$.

segmented models with simultaneous fitting. The geometric and form factor systems were consistently the worst in rankings, with Bruce's model somewhere between these models and the segmented models. The geometric model, though not comparing closely to the segmented models, did compare favorably with the well-established form factor model when the *Rd* equation is not considered.

Summarizing the results to this point, the segmented models appear to be the best, especially for the volume ratio to a diameter limit equation. For red pine, the fitting approach doesn't seem to matter but for loblolly pine the simultaneous fitting procedure clearly produced more accurate and precise predictions. The test statistics for the four best models (Models C, D, H, I) are shown in Table 4 for red pine and Table 5 for loblolly pine.

To further explore the performance of the best models, Max and Burkhart's

TABLE 7. Values of the test statistics for the validation data by % height classes for the best models—loblolly pine, outside bark.

Equation	% of total height	N[a]	$\bar{D}$[b] MB-Sim[d]	$\bar{D}$[b] Cao-Sim[e]	s[c] MB-Sim	s[c] Cao-Sim
Taper (inches)	$0.0 < X < 0.1$	162	−0.20	−0.21	0.52	0.51
	$0.1 \leq X < 0.2$	132	0.01	−0.06	0.27	0.28
	$0.2 \leq X < 0.3$	123	−0.12	−0.14	0.32	0.32
	$0.3 \leq X < 0.4$	119	−0.08	−0.06	0.36	0.36
	$0.4 \leq X < 0.5$	112	−0.04	0.01	0.33	0.32
	$0.5 \leq X < 0.6$	131	0.03	0.10	0.33	0.34
	$0.6 \leq X < 0.7$	119	0.09	0.16	0.34	0.34
	$0.7 \leq X < 1.0$	141	0.09	0.11	0.34	0.34
Volume ratio (height)	$0.0 < X < 0.1$	162	−0.0041	−0.0043	0.0132	0.0131
	$0.1 \leq X < 0.2$	132	0.0136	0.0095	0.0290	0.0292
	$0.2 \leq X < 0.3$	123	0.0066	−0.0005	0.0327	0.0328
	$0.3 \leq X < 0.4$	119	0.0042	−0.0032	0.0316	0.0316
	$0.4 \leq X < 0.5$	112	−0.0018	−0.0079	0.0279	0.0278
	$0.5 \leq X < 0.6$	131	−0.0006	−0.0045	0.0227	0.0226
	$0.6 \leq X < 0.7$	119	−0.0009	−0.0023	0.0164	0.0164
	$0.7 \leq X < 1.0$	141	−0.0002	−0.0001	0.0067	0.0067
Volume ratio (diam.)	$0.0 < X < 0.1$	162	−0.0370	−0.0457	0.1044	0.0971
	$0.1 \leq X < 0.2$	132	0.0356	0.0075	0.1001	0.0936
	$0.2 \leq X < 0.3$	123	−0.0138	−0.0258	0.0869	0.0813
	$0.3 \leq X < 0.4$	119	−0.0028	−0.0069	0.0829	0.0777
	$0.4 \leq X < 0.5$	112	−0.0017	−0.0006	0.0552	0.0527
	$0.5 \leq X < 0.6$	131	0.0062	0.0098	0.0334	0.0327
	$0.6 \leq X < 0.7$	119	0.0094	0.0130	0.0198	0.0201
	$0.7 \leq X < 1.0$	141	0.0040	0.0053	0.0071	0.0080

[a] Number of height/diameter observations in a height class.
[b] Average residual.
[c] Standard deviation of residuals.
[d] Max and Burkhart—Simultaneous fit model.
[e] Cao—Simultaneous fit model.
Note: $X = h/H$.

and Cao's systems that have been simultaneously fit, test statistics were calculated for different portions of the stem. The height/diameter observations in each validation data set were split into relative height classes. The red pine data have 10 classes (each 10 percent of the total height) while the loblolly pine data have only 8 classes (the top 30 percent of the heights were grouped together because stem analysis was stopped at a 2-inch top thus providing few measurements in the top 20 percent of the stem). Average residual and standard deviation of the residuals were calculated at each height class and for each of the equations except total volume for the two models. The results of this analysis are given in Table 6 for red pine and Table 7 for loblolly pine, using outside bark data.

Several trends can be noted from Tables 6 and 7. For the taper equation, both models overpredict (negative average residuals) diameter in the lower bole and underpredict (positive average residuals) in the upper bole. The accuracy and precision were less at the bottom 10 percent for both species. This height level corresponds with the lower join point of the models (a_1 for Max and Burkhart's and $(1 - a_1)$ for Cao's in Table 2), which indicates a shift in the tree form. For loblolly pine, accuracy and precision were consistent up the rest of the stem but for red pine, the accuracy and precision were again reduced at about 60–80 percent of total height. This height range corresponds with the top join point of each model (a_2 for Max and Burkhart's and $(1 - a_2)$ for Cao's), which indicates the other shift in the form of the tree, somewhere in the crown. The upper join points occurred at greater percentages of total height for loblolly pine than red pine for each model. A form shift in the upper stem may have been more apparent for red pine than for loblolly pine because of the frequency of upper stem measurements. The loblolly stem measurements stopped at a 2-inch top diameter unlike the red pine data which continued to the top of the tree. The top loblolly measurements from 70 to 100 percent of total height were grouped together for analysis. Worse performance at the top of the tree may not be detectable in this type of data.

The height class statistics also show that the prediction of volume ratios to a height limit performs much better than the predictions of volume ratios to a diameter limit. For red pine the volume ratios to a height limit are overpredicted over the whole stem but for loblolly pine no pattern of over or underprediction is clearly apparent. In general, the volume ratios to a diameter limit are overpredicted at the bottom of the stem and underpredicted for the upper part of the stem for both models and species (the same pattern as the taper equation). Also, it can be noted that estimates of volume ratio to a diameter limit become more precise as one progresses up the stem. Though only the outside bark statistics are given, a similar analysis using inside bark data showed similar trends for both species.

In the previous comparison of the models, both predicted volume ratios were compared against the observed ratio calculated as actual merchantable volume divided by actual total volume. To provide further insight on how the models would compare in actual practice (where total volume is predicted), the predicted ratio was compared with the ratio calculated as actual merchantable volume divided by predicted total volume. In this comparison, the same two systems, the two segmented models with simultaneous fitting, proved to again be the best. But as expected the accuracy and precision were reduced when the predicted total volume was used.

One possible advantage of using Cao's system involves the computation of the form factor coefficient for the total volume equation. In the development of the system, the form factor total volume equation ($TV = cD^2H$) was assumed as the form of the total volume equation in the taper equation, where c is a fitted coefficient, usually about 0.002. Upon fitting the taper equation to the data, the regression coefficient c was found to be about 0.004, not very close to the known value of 0.002. After integration of the taper equation, the TV equation was in the form $TV = c\gamma D^2H$, where

$$\gamma = 1 + (b_2/3)(1 - a_1)^3 + (b_3/3)(1 - a_2)^3. \tag{21}$$

When calculated, the value of γ is about 0.6, which when multiplied by the fitted c of 0.004 produces a coefficient for the TV equation of 0.002, approximately the same as when the form factor TV equation is fit to the data. When the coefficient c was set at 0.00282 and only five coefficients used in fitting the taper equation, the value of γ was nearly 1 (0.97). So, if one had an established TV equation that needed to be retained, the TV coefficient could be used in the taper equation and the other coefficients developed for the volume estimation system.

Of all the volume estimation systems considered, the results indicate that Cao's segmented model with simultaneous fitting is the best. It ranked the highest for the sum of ranks for all four component equations for three out of the four species-bark combinations and a close third for the other. No other system predicted taper and volume as consistently well as this model. Potential retention of an established TV equation discussed above is also an advantage for the use of Cao's system.

SUMMARY

A comparison of five compatible volume estimation systems that have been derived from taper equations, both single and segmented, is carried out. One of the systems is based on a newly derived taper equation which utilizes the equations for the assumed geometric shapes which a tree stem takes on. Each volume estimation system consists of four component equations: a taper equation to predict stem diameter, a total cubic foot volume equation, and two volume ratio equations for use in predicting merchantable cubic-foot volume to any height or diameter limit. The taper equations were fit to stem analysis data for red and loblolly pine, both outside and inside bark, using nonlinear regression. The values of the coefficients of the other three equations are mathematically related to the taper equation coefficients. In an attempt to reduce the total error for a system, a simultaneous fitting procedure was also used in finding the taper equation coefficients. Using a reserved subset of the original data set, the fitted volume estimation systems were compared using four calculated test statistics. Results indicate that a system based on the segmented taper equation by Cao, and developed utilizing the simultaneous fitting procedure, is the most accurate and precise predictor of taper, total volume, and volume ratios for both red and loblolly pine. This segmented system shows a substantial improvement over previously defined simpler taper-volume estimation systems especially in the ability to predict volumes to a top diameter limit.

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APPENDIX: DESCRIPTION OF THREE VOLUME ESTIMATION SYSTEMS

VOLUME ESTIMATION SYSTEM BASED ON THE FORM FACTOR TAPER EQUATION (from Reed and Green 1984)

Taper Equation:

$$d^2 = b_1 D^2 z^{b_2}.$$

Total Volume Equation:

$$\mathrm{TV} = a_1 D^2 H,$$

where

$$a_1 = Kb_1/(b_2 + 1).$$

Volume Ratio Equation to a Height Limit:

$$Rh = 1 - z^{e_1},$$

where

$$e_1 = b_2 + 1.$$

Volume Ratio Equation to a Diameter Limit:

$$Rd = 1 + f_1(d/D)^{f_2}$$

where

$$f_1 = -b_1{}^{-[(b_2+1)/b_2]}$$
$$f_2 = 2[(b_2 + 1)/b_2].$$

VOLUME ESTIMATION SYSTEM BASED ON MAX AND BURKHART'S TAPER EQUATION (from Martin 1981 and Green and Reed 1985)

Taper Equation:

$$d^2 = D^2[b_1(h/H - 1) + b_2\{(h/H)^2 - 1\} + b_3(a_1 - h/H)^2 I_1 + b_4(a_2 - h/H)^2 I_2],$$

where

$$I_1 = \begin{cases} 1 \text{ if } h/H \le a_i \\ 0 \text{ if } h/H > a_i;\ i = 1, 2. \end{cases}$$

Total Volume Equation:

$$\mathrm{TV} = \alpha D^2 H,$$

where

$$\alpha = K[(b_2/3) + (b_1/2) - (b_1 + b_2) + (b_3/3)a_1^3 + (b_4/3)a_2^3].$$

Volume Ratio Equation to a Height Limit:

$$\begin{aligned} Rh = \frac{1}{\gamma}[&(b_2/3)(h/H)^3 + (b_1/2)(h/H)^2 - (b_1 + b_2)(h/H) \\ &- (b_3/3)\{(a_1 - h/H)^3 I_1 - a_1^3\} \\ &- (b_4/3)\{(a_2 - h/H)^3 I_2 - a_2^3\}], \end{aligned}$$

where

$$\gamma = (b_2/3) + (b_1/2) - (b_1 + b_2) + (b_3/3)a_1^3 + (b_4/3)a_2^3.$$

Volume Ratio Equation to a Diameter Limit:

An equation for predicting the height h to an upper diameter d must first be defined:

$$\begin{aligned} w &= \text{predicted partial height} \\ &= (H/2A)[-B - (B^2 - 4AC)^{1/2}], \end{aligned}$$

where

$$A = b_2 + b_3 J_1 + b_4 J_2,$$

$$B = b_1 - 2a_1 b_3 J_1 - 2a_2 b_4 J_2,$$

$$C = -(b_1 + b_2) + b_3 a_1^2 J_1 + b_4 a_2^2 J_2 - (d/D)^2,$$

$$J_i = \begin{cases} 1 \text{ if } d \ge M_i \\ 0 \text{ if } d < M_i;\ i = 1, 2, \end{cases}$$

$$\begin{aligned} M_i &= \text{estimated diameter at height } a_i H \\ &= D[b_1(a_i - 1) + b_2(a_i^2 - 1) + b_4(a_2 - a_i)^2]^{1/2}. \end{aligned}$$

The volume ratio equation to a diameter limit is

$$\begin{aligned} Rd = \frac{1}{\gamma}[&(b_2/3)(w/H)^3 + (b_1/2)(w/H)^2 - (b_1 + b_2)(w/H) \\ &- (b_3/3)\{(a_1 - w/H)^3 J_1 - a_1^3\} - (b_4/3)\{(a_2 - w/H)^3 J_2 - a_2^3\}]. \end{aligned}$$

Volume Estimation System Based on Bruce's Taper Equation
(taper and total volume equation from Martin 1981)

Taper Equation:

$$\begin{aligned} d^2 = D^2[&b_1 x^{1.5}(10^{-1}) + b_2(x^{1.5} - x^3)D(10^{-2}) + b_3(x^{1.5} - x^3)H(10^{-3}) \\ &+ b_4(x^{1.5} - x^{32})HD(10^{-5}) + b_5(x^{1.5} - x^{32})H^{1/2}(10^{-3}) \\ &+ b_6(x^{1.5} - x^{40})H^2(10^{-6})], \end{aligned}$$

where

$$x = (H - h)/(H - 4.5).$$

Total Volume Equation:

$$\mathrm{TV} = KD^2H[E_1H^{1.5} - E_2H^3 - E_3H^{32} - E_4H^{40}],$$

where

$$E_1 = \frac{b_1(10^{-1}) + b_2D(10^{-2}) + b_3H(10^{-3}) + b_4HD(10^{-5}) + b_5H^{1/2}(10^{-3}) + b_6H^2(10^{-6})}{2.5(H - 4.5)^{1.5}}$$

$$E_2 = \frac{b_2D(10^{-2}) + b_3H(10^{-3})}{4(H - 4.5)^3},$$

$$E_3 = \frac{b_4HD(10^{-5}) + b_5H^{1/2}(10^{-3})}{33(H - 4.5)^{32}},$$

$$E_4 = \frac{b_6H^2(10^{-6})}{41(H - 4.5)^{40}}.$$

Volume Ratio Equation to a Height Limit:

$$Rh = 1 - \left[\frac{E_1(H - h)^{2.5} - E_2(H - h)^4 - E_3(H - h)^{33} - E_4(H - h)^{41}}{E_1H^{2.5} - E_2H^4 - E_3H^{33} - E_4H^{41}}\right].$$

Volume Ratio Equation to a Diameter Limit:

In order to solve for a volume ratio equation to a diameter limit, an expression to predict h must be derived from the taper equation. Since this is not algebraically possible, no Rd equation is available.

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Site Quality Influences on Biomass Estimates for White Spruce Plantations

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ABSTRACT. Separate biomass estimation equations were developed for white spruce (*Picea glauca* (Moench) Voss) from plantations of low, medium, and high site quality in northern Minnesota. Site quality was defined by site index. Differences in regression coefficients among these quality-specific equations were statistically significant ($P < 0.005$). Estimates of aboveground biomass for a dominant tree based on those equations differed from an estimate based on an overall equation from all data. As tree size increased, estimates from the overall equation were less than those from the specific equation for high-quality sites, and greater than those from the equation for low-quality sites. Site quality differences can significantly affect biomass estimates for white spruce in plantations. In most cases, however, the practical significance of these effects should be small. FOREST SCI. 32:443–446.

ADDITIONAL KEY WORDS. *Picea glauca*, productivity, site index.

LOCALLY DEVELOPED BIOMASS EQUATIONS are considered the most accurate for predictions within specific geographic regions. Generalized equations have been suggested as offering

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