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ARTICLE

An antithetic variate to facilitate upper-stem height measurements for critical height sampling with importance sampling

Thomas B. Lynch and Jeffrey H. Gove

Abstract: Critical height sampling (CHS) estimates cubic volume per unit area by multiplying the sum of critical heights measured on trees tallied in a horizontal point sample (HPS) by the HPS basal area factor. One of the barriers to practical application of CHS is the fact that trees near the field location of the point-sampling sample point have critical heights that occur quite high on the stem, making them difficult to view from the sample point. To surmount this difficulty, use of the "antithetic variate" associated with the critical height together with importance sampling from the cylindrical shells integral is proposed. This antithetic variate will be $u = (1 - b/B)$, where b is the cross-sectional area at "borderline" condition and B is the tree's basal area. The cross-sectional area at borderline condition b can be determined with knowledge of the HPS gauge angle by measuring the distance to the sample tree. When the antithetic variate u is used in importance sampling, the upper-stem measurement will be low on tree stems close to the sample point and high on tree stems distant from the sample point, enhancing visibility and ease of measurement from the sample point. Computer simulations compared HPS, CHS, CHS with importance sampling (ICHS), ICHS and an antithetic variate (AICHS), and CHS with paired antithetic varariates (PAICHS) and found that HPS, ICHS, AICHS, and PAICHS were very nearly equally precise and were more precise than CHS. These results are favorable to AICHS, since it should require less time than either PAICHS or ICHS and is not subject to individual-tree volume equation bias.

Résumé : L'échantillonnage de la hauteur critique (EHC) permet d'estimer le volume de bois par unité de surface en multipliant la somme des hauteurs critiques mesurées sur les arbres inventoriés dans un échantillonnage horizontal par point (EHP) par le facteur de prisme de l'EHP. L'un des obstacles à l'application pratique de l'EHC vient du fait que les arbres proches de l'emplacement du point d'échantillonnage ont des hauteurs critiques qui se situent assez haut sur la tige. Ceci rend les hauteurs critiques difficilement visibles à partir du point d'échantillonnage. On suggère de surmonter cette difficulté en utilisant la « variable antithétique » associée à la hauteur critique dans l'échantillonnage par importance basée sur l'intégrale d'enveloppes cylindriques. Cette variable antithétique sera $u = (1 - b/B)$ où b est la section transversale à la limite de démarcation et B est la surface terrière de l'arbre. La section transversale à la limite de démarcation b peut être déterminée avec le facteur de prisme de l'EHP en mesurant la distance de l'arbre échantillonné. Lorsque la variable antithétique u est utilisée dans l'échantillonnage par importance, la mesure en haut de la tige est abaissée pour les arbres proches du point d'échantillonnage et haussée pour les arbres éloignés du point d'échantillonnage. La mesure à partir du point d'échantillonnage se trouve ainsi facilitée par une meilleure visibilité de la hauteur critique. Des simulations sur ordinateur ont permis de comparer l'EHP, l'échantillonnage de la hauteur critique (EHC), l'EHC avec l'échantillonnage par importance (EHCP), l'EHCP avec une variable antithétique (EHCPA) et l'échantillonnage de la hauteur critique avec des variables antithétiques appariées (EHCAA). Les comparaisons ont permis de constater que l'EHP, l'EHCP, l'EHCPA et l'EHCAA avaient pratiquement la même précision et qu'ils étaient plus précis que l'EHC. Ces résultats privilégient l'EHCPA qui requiert moins de temps que l'EHCAA ou l'EHCP et qui n'est pas sujet au biais de l'équation de volume des arbres individuels. [Traduit par la Rédaction]

Introduction

Critical height sampling (CHS), as introduced by Kitamura (1964), provides a method of unbiased estimation for tree volume per unit of land area that depends on measurements of critical height on sample trees selected by horizontal point sampling (HPS). The CHS estimator for one randomly located sample point is

$$\hat{V}_{h_c} = F \times A \sum_{i=1}^{N} h_{ci} \tag{1}$$

where A is land area (ha); $F = 10^4/\alpha^2$ (Gregoire and Valentine 2008, p. 249); $\alpha = 1/\sin(\varphi/2)$, where φ is the HPS gauge angle; h_{ci} is the critical height (m) at which the tree stem appears borderline with the HPS angle gauge for sample tree i selected at the point, otherwise zero; N is the total number of trees on land area A; and $\hat{V}_{h_c}$ is the CHS estimate of cubic volume (m^3) on land area A.

The notation used in eq. (1) derives from that used by Gregoire and Valentine (2008, p. 249) for HPS and McTague and Bailey (1985). Kitamura (1964) first proposed CHS, which was later independently conceived by Iles (1979*a*, 1979*b*). The developments of Kitamura (1964) and Iles (1979*a*, 1979*b*) indicated the unbiasedness of CHS. McTague and Bailey (1985) demonstrated the unbiasedness of estimator (1) for the taper functions of Clutter (1980) and Bitterlich (1976). Lynch (1986) and Van Deusen and Meerschaert (1986) showed that estimator (1) was unbiased for

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any taper function using the cylindrical shells integral for volume.

Although the CHS Estimator (1) is unbiased, it can be time consuming to perform the measurements required. The estimator requires an upper-stem measurement on all sample trees. Increasing the time required for this is the fact that the measurements nearest the plot center will be highest on the tree stem, requiring the individual performing the sampling to move away from the sample point for many of these trees. In fact, Husch et al. (1982, p. 37) stipulated that the optimal distance at which to measure tree height is a distance from the tree that is approximately equal to that height. For a forester who remains on the sample point when implementing CHS, trees near the sample point are too close, having very steep viewing angles, and those at a distance from the sample point are too far away, having very flat viewing angles. For the forester remaining on the sample point in CHS, the relationship between critical height and distance to the sample tree is just the reverse of that which would be desired. To correct this, the use of the "antithetic variate" associated with the ratio between the sample tree cross-sectional area at critical height and the sample tree basal area (or the cross-sectional area at groundline or stump) is proposed.

Monte Carlo integration and CHS

Antithetic variates were developed as a variance reduction technique in Monte Carlo integration (Hammersley and Morton (1956); also see Rubenstein (1981, p. 135)). Antithetic variates were applied to individual-tree volume estimation by Van Deusen and Lynch (1987). On an individual tree basis, CHS has a relationship to Monte Carlo integration (Lynch 1986). We can view each individual-tree volume as the following integral (Lynch et al. 1992):

$$v(b_i) = \int_0^{b_i} h(c)\mathrm{d}c \tag{2}$$

where $v(b_i)$ is the volume of the tree i stem within a cylinder of cross-sectional area b_i centered about the central axis of the tree stem and $h(c)$ is the upper-stem height of the tree to cross-sectional area c.

Using this expression, the total volume of the tree can be computed by

$$v(B_i) = \int_0^{B_i} h(c)\mathrm{d}c \tag{3}$$

where B_i may be the cross-sectional area of tree i stem at groundline or stump. If B_i is the stump cross-sectional area and the volume above stump is desired, then height measurements will need to be made from stump height to the appropriate upper-stem height. If we take B_i to be basal area, then integral (3) is close to total stem volume but excludes the volume in the stem that is outside a cylinder having height equal to breast height (BH) and cross-sectional area B_i.

Monte Carlo integration can be used to estimate individual-tree stem volume by obtaining random samples from integral (3). The crude Monte Carlo (CMC) sampling estimator associated with eq. (3) is obtained by selecting u_i for tree i at random from a uniform distribution on (0,1). Then a random cross-sectional area is obtained by the equation $c(u_i) = u_i \times B_i$. Finally, the height $h(c(u_i))$ is measured on the tree stem, and the resulting CMC estimator of tree volume is (Lynch et al. 1992)

$$\hat{v}_{CMi} = B_i \times h(c(u_i)) \tag{4}$$

On an individual tree basis, CHS is equivalent to CMC integration where $u_i^*(x) = b_{ci}(x)/B_i$ and $b_{ci}(x) = \pi R_i^2(x)/(200\kappa)^2$, where $R_i(x)$ is the distance from the sample point to the tree in metres, $\kappa = 1/(2F^{1/2})$, and x is the location of a sample point randomly drawn from a uniform distribution over the tract area. Here, the "plot radius factor" κ is defined using the notation and approach of Gregoire and Valentine (2008, p. 249, eq. (8.10)). For sample tree i, the diameter that appears "borderline" from the sample point at distance $R_i(x)$ (m) is defined by $d_{ci}(x) = R_i(x)/\kappa$, where $d_{ci}(x)$ is measured in centimetres (Gregoire and Valentine 2008 p. 249, eq. (8.8)), and its corresponding cross-sectional area is $b_{ci}(x)$ (m^2)

$$b_{ci}(x) = \pi[d_{ci}(x)/200]^2 = \pi[R_i(x)/(200\kappa)]^2 \tag{5}$$

The diameter corresponding to $b_{ci}(x)$ is the diameter to which critical height will be measured on any tree that is located at a distance $R_i(x)$ from the sample point. Thus, on an individual tree basis CHS is equivalent to the following CMC estimator:

$$\begin{aligned}\hat{v}_{CMi} &= B_i \times h\big(c\big(u_i^*(x)\big)\big) = B_i \times h\big(u_i^*(x)B_i\big) \\ &= B_i \times h((b_{ci}(x)/B_i)B_i) = B_i \times h(b_{ci}(x))\end{aligned} \tag{6}$$

Incorporating this into the CHS estimator and noting that $h_{ci}(x) = h(b_{ci}(x))I_i(x)$, where $I_i(x) = 1$ if tree i is selected by the HPS angle gauge and $I_i(x) = 0$ otherwise, we have

$$\hat{V}_{h_c} = F \times A \sum_{i=1}^{N} h_{ci}(x) = F \times A \sum_{i=1}^{N} \frac{B_i \times h_{ci}(x)}{B_i}$$

and noting that $h_{ci}(x_i) = h(b_{ci}(x))I_i(x)$, from eq. (6) we obtain

$$\Rightarrow \hat{V}_{h_c} = F \times A \sum_{i=1}^{N} \frac{B_i \times h(b_{ci}(x))}{B_i} I_i(x) = F \times A \sum_{i=1}^{N} \frac{\hat{v}_{CMi}(x)}{B_i} I_i(x)$$

From this we see that CHS is equivalent to HPS in which individual-tree volumes are estimated by CMC integration. Since CMC integration can have a high variance on an individual tree basis, variance reduction techniques appropriate to Monte Carlo integration have often been suggested. Gregoire et al. (1986) applied importance sampling to the estimation of tree volume. Van Deusen and Lynch (1987) first used antithetic variates to estimate tree volume. Van Deusen (1987) suggested the application of control variates to the CHS estimator. This reduces the variance of individual-tree volume estimates but requires measurement of total height on each sample tree, since most taper functions depend on tree diameter and height. An alternative that does not require the measurement of total tree heights would be the application of importance sampling from the cylindrical shells integral as described by Lynch et al. (1992; also see the discussion by Valentine et al. (1992)). They developed an importance sampling distribution function based on the use of the following power function solid of revolution as a proxy taper function:

$$h_k(b_i) = H_i\left[1 - \left(\frac{b_i}{B_i}\right)^k\right] \tag{7}$$

where $h_k(b_i)$ is the upper-stem height associated with tree i upper-stem cross-sectional area b_i, H_i is the total tree height for tree i, and k is a shape parameter. The shape parameter $k = 1$ for a paraboloid,

$k = ½$ for a cone, and, for a neiloid, k is between 0 and ½. This power function solid of revolution is very similar to simple taper functions that have often been used to describe tree stems, such as that of Ormerod (1973) or the taper functions used by Van Deusen and Lynch (1987). Integration yields the following volume function (Lynch et al. 1992):

$$(8)\qquad v_k(b_i) = H_i b_i\left[1 - \left(\frac{1}{1+k}\right)\left(\frac{b_i}{B_i}\right)^k\right]$$

where $v_k(b_i)$ is the volume inside a cylinder having cross-sectional area b_i for tree i according to eq. (7) with parameter value k. Lynch et al. (1992) used this function to develop the following importance sampling distribution function:

$$(9)\qquad F_k(b_i) = \frac{v_k(b_i)}{v_k(B_i)} = \frac{b_i}{kB_i}\left[(k+1) - \left(\frac{b_i}{B_i}\right)^k\right]$$

It is important to note that eq. (9) does not require any total height measurements, because the variable H_i is eliminated when forming the ratio required for the distribution function. Importance sampling is implemented by selecting a uniform random variate u_i defined on (0,1) and equating this to the distribution function: $u_i = F_k(b_i)$. The solution of this equation yields an upper-stem cross-sectional area b_i, which is used to sample $h(b_i)$, the height to that cross-sectional area for tree i associated with u_i. For most values of k this requires an iterative solution to eq. (8); however, for $k = 1$ (paraboloid), Lynch et al. (1992) gave the following solution:

$$(10)\qquad b_i = B_i(1 - \sqrt{1-u_i})$$

To apply importance sampling from the cylindrical shells integral to CHS, we used $u_i = u_i^*$ in eq. (10)

$$(11)\qquad b_i(x) = B_i\left(1 - \sqrt{1-u_i^*(x)}\right) = B_i\left(1 - \sqrt{1 - \frac{b_{ci}(x)}{B_i}}\right)$$

Note that $b_i(x)$ does not equal $b_{ci}(x)$ because the latter is the upper-stem cross-sectional area that would have been associated with the tree's critical height, whereas the former is a cross-sectional area associated with importance sampling. In the formulation of eq. (11), the ratio between $b_{ci}(x)$ and B_i is used as the uniform random variate. In this case, of course, randomness arises from the random location x of the point sample in the forested tract of interest. Heights sampled by using eq. (11), of course, retain the characteristic of CHS that upper-stem heights sampled on trees close to the sample point on the ground would be quite high, approaching total tree height as the tree location approaches the location of the sample point on the ground. For an individual tree, the importance sampling volume estimator associated with use of eq. (11) ($k = 1$) would be (Lynch et al. 1992)

$$(12)\qquad \hat{v}_i(x) = \frac{h(b_i(x))}{h_k(b_i(x))}V_k(B) = \frac{h(b_i(x))B_i}{2\sqrt{1-u_i^*(x)}} = \frac{h(b_i(x))B_i}{2\sqrt{1-\dfrac{b_{ci}(x)}{B_i}}}$$

Since it is frequently desirable to select trees at DBH or some other point above stump diameter, we can alter the previous equation. We use importance sampling as indicated previously to estimate the volume above a selected cross-sectional stem area B, and estimate the volume below that cross-sectional area by measuring a cross-sectional area $C_{\text{base}}(x)$ at a randomly selected height $h_{\text{samp}}(x) = u_i^*(x) \times h(B_i)$ chosen from a uniform distribution on the interval between groundline and $h(B_i)$. The resulting individual-tree volume estimator is

$$(13)\qquad \hat{v}_i(x) = \frac{[h(b_i(x)) - h(B_i)]B_i}{2\sqrt{1-\dfrac{b_{ci}(x)}{B_i}}} + C_{\text{base}}(x)h(B_i)$$

where $h(B_i)$ is the height (m) at which B_i (m²) is measured, $C_{\text{base}}(x) = \frac{\pi}{40\,000}D^2_{\text{base}}(x)$ with units of (m²), $D_{\text{base}}(x)$ is the diameter (cm) at $h_{\text{samp}}(x) = u_i^*(x) \times h(B_i)$, and $\hat{v}_i(x)$ is the estimated volume (m³) for tree i.

$C_{\text{base}}(x)h(B_i)$ is a CMC integration estimator for the volume between groundline and $h(B_i)$ of a type that is well-known to be unbiased. If the location of B_i corresponds to BH, then the difference between DBH and stump diameter may be small enough so that the variance of this estimator may not be excessive within the context of total stem volume estimation. In any case, the results of our simulations (see the following) indicate that it is adequate for these purposes. Note that if we are willing to ignore the often modest amount of wood volume on the lower stem outside the cylinder defined by $B_i = \pi(\text{DBH}/200)^2$, then we can simply set $C_{\text{base}}(x) = \pi(\text{DBH}/200)^2$ and avoid sampling the lower stem below DBH.

With a generalized proxy taper function having distribution function $F_p(b_i)$, we can find a sample upper-stem height, $h(b_i(x))$, corresponding to $b_i(x) = F_p^{-1}(u_i^*(x))$. For an upper-stem height, $h_p(b_i(x))$, corresponding to the generalized proxy function with proxy volume $V_p(B_i)$, and the corresponding estimate of volume contained on land area, A, associated with one randomly located point sample is

$$(14)\qquad \hat{V}_{u^*} = F \times A\sum_{i=1}^{N}\frac{\hat{v}_i(x)}{B_i}I_i(x)$$
$$= F \times A\left(\sum_{i=1}^{N} I_i(x)\left\{\frac{[h(b_i(x)) - h(B_i)]}{[h_p(b_i(x)) - h(B_i)]}\frac{V_p(B_i)}{B_i} + \frac{C_{\text{base}}(x)h(B_i)}{B_i}\right\}\right)$$

where $\hat{V}_{u^*}$ is the estimated cubic metre volume contained on A by using importance sampling with u^*, $h_p(b_i(x))$ is the height predicted by the proxy taper function at stem cross-sectional area b_i for tree i, and $I_i(x) = 1$ if tree i is selected by the angle gauge and $I_i(x) = 0$ if tree i is not selected by the angle gauge. Here, we used the well-known HPS estimator in a form similar to that presented by Gregoire and Valentine (2008, p. 255, eq. (8.21b)), except that we are using the Bernoulli random variable $I_i(x)$ to indicate the presence of tree i in the point sample. We replaced actual tree volume with estimated tree volume. Based on the well-known proof that the HPS estimator is unbiased given by Palley and Horwitz (1961), it is easy to show that this estimator is unbiased if the individual-tree volume estimate is unbiased. Unbiasedness of the individual-tree volume estimator with this type of importance sampling has been shown by Lynch et al. (1992). Let it be given that

$$E(\hat{v}_i(x)|I_i(x) = 1) = E(\hat{v}_i(x)I_i(x)|I_i(x) = 1) = v_i$$

then

$$
\begin{aligned}
E(\hat{V}_{\hat{v}}) &= E\left\{F \times A \sum_{i=1}^{N} \frac{\hat{v}_i(x)}{B_i} I_i(x)\right\} \\
&= F \times A \sum_{i=1}^{N} \frac{1}{B_i}\left\{E(\hat{v}_i(x)I_i(x)\big|I_i(x) = 1)P(I_i(x) = 1) + E(\hat{v}_i(x)I_i(x)\big|I_i(x) = 0)P(I_i(x) = 0)\right\} \\
&= F \times A \sum_{i=1}^{N} \frac{v_i}{B_i} P(I_i(x) = 1) = F \times A \sum_{i=1}^{N} \frac{v_i}{B_i} \frac{\pi(\kappa \times \mathrm{DBH})^2}{A \times 10^4}
\end{aligned} \tag{15}
$$

and since

$$\kappa = \frac{1}{2\sqrt{F}}$$

and

$$B_i = \pi\left(\frac{\mathrm{DBH}}{200}\right)^2 \Rightarrow E(\hat{V}_{\hat{v}}) = \sum_{i=1}^{N} v_i = V$$

where $\hat{V}_{\hat{v}}$ is the point-sampling volume estimate using estimated individual-tree volumes and V is the total volume on a land area of size A ha. Thus, HPS is, in general, unbiased when actual volume is replaced by an unbiased estimate of volume.

Because we use the parabolic proxy function in our simulations (see the following), we also present the form of the tract volume estimator when this proxy function is used. Our simulation results also lead us to believe that the parabolic proxy function will be entirely adequate for many purposes, and it is simple to use and apply. Thus, using individual-tree volume estimator (13) we obtain the following:

$$
\begin{aligned}
\hat{V}_{u^*} &= F \times A \sum_{i=1}^{N} \frac{\hat{v}_i(x)}{B_i} I_i(x) \\
&= F \times A\left(\sum_{i=1}^{N} I_i(x)\left\{\frac{[h(b_i(x)) - h(B_i)]}{2\sqrt{1 - \dfrac{b_{ci}(x)}{B_i}}} + \frac{C_{\mathrm{base}}(x)h(B_i)}{B_i}\right\}\right)
\end{aligned} \tag{16}
$$

where $\hat{V}_{u^*}$ is the estimated cubic metre volume contained on A by using importance sampling with $u^*(x)$.

We will refer to estimator (16) as the importance sampling CHS (ICHS) estimator. Estimators for use when random b_i are chosen using values of k other than the paraboloidal $k = 1$ can be obtained by substituting for $\hat{v}_i$ in eq. (16) from equations presented by Lynch et al. (1992). It should be noted that estimator (16) does not require any tree total-height measurements, however, measurements of B_i (where B_i could be the groundline cross-sectional area, stump cross-sectional area, or basal area) are required. Only distance measurements $R_i(x)$ are needed to compute b_{ci} and, subsequently, b_i. However, modern mensurational instruments, which make measurements of distance based on laser or sonic principles, should make it possible to measure distance with little movement from the sample point in most cases. Measurements of diameter at BH and other lower stem diameters can be made from the sample point using calipers tipped with small lasers.

Antithetic variate for easier upper-stem height measurement

ICHS estimator (16) shares with the ordinary CHS estimator the property that upper-stem measurement heights will be high on the stem for trees close to the sample point and lower on the stem for trees more distant from the sample point. This property can be altered by the use of an antithetic variate. Antithetic variates can be obtained from the importance sampling distributions (9) or (10) by using the uniform random variate, u, and to generate an antithetic variate $(1 - u)$. As indicated by Lynch et al. (1992), for integrands that are monotonic and fairly close to linearity, this procedure usually generates pairs of negatively correlated estimates. Using the average of this antithetic pair, estimation variance is then reduced in comparison with the variance that would be obtained by averaging two independent samples. Here, it is desired to use an antithetic variate for a different purpose, that of shifting the upper-stem height measurement point so that it will be low for tree stems close to the sample point in the field and high for stems more distant from that sample point. Specifically, we will use $(1 - u_i^*(x))$, which produces the variate antithetic to that obtained by using $u_i^*(x)$. When used in eq. (11) we obtain

$$b_{ai}(x) = B_i\left[1 - \sqrt{1 - (1 - u_i^*(x))}\right] = B_i\left(1 - \sqrt{\frac{b_{ci}(x)}{B_i}}\right) \tag{17}$$

where b_{ai} is the upper-stem cross-sectional area associated with the antithetic variate for tree i. Inspection of eq. (17) indicates that small values of $b_{ci}(x)$ are associated with large values of $b_{ai}(x)$. Since small values of $b_{ci}(x)$ occur near the sample point, corresponding values of $b_{ai}(x)$ will be large and relatively low on the stem. The corresponding individual-tree volume estimator will be

$$\hat{v}_i^a(x) = \frac{h(b_{ai}(x))B_i}{2\sqrt{1 - \left(1 - u_i^*(x)\right)}} = \frac{h(b_{ai}(x))B_i}{2\sqrt{\dfrac{b_{ci}(x)}{B_i}}} = \frac{h(b_{ai}(x))B_i^{3/2}}{2\sqrt{b_{ci}(x)}} \tag{18}$$

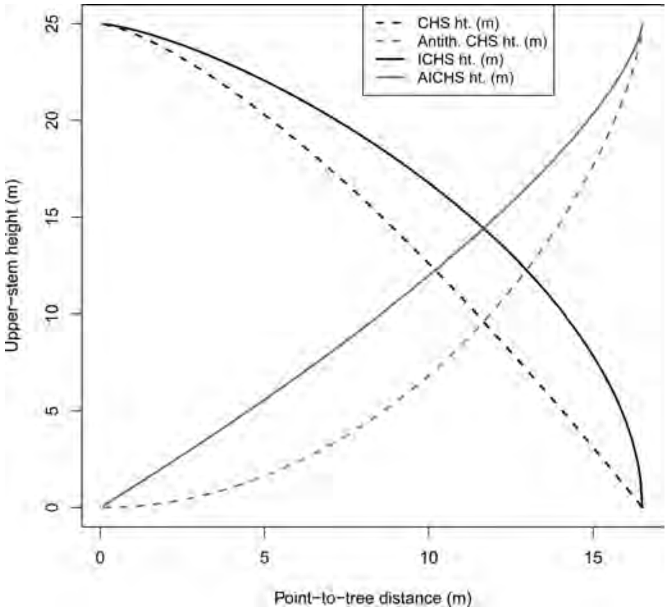
Note the difference in $b_{ai}(x)$ and $b_{ci}(x)$ in this estimator owing to the use of the antithetic variate. When $b_{ci}(x)$ is small, $b_{ai}(x)$ will be large and $h(b_{ai}(x))$ will be correspondingly small. When $b_{ci}(x)$ is large, $b_{ai}(x)$ will be small and $h(b_{ai}(x))$ will be large. When adjusted for tree selection at DBH or another cross-sectional area above groundline, estimator (18) becomes

$$\hat{v}_i^a(x) = \frac{[h(b_{ai}(x)) - h(B_i)]B_i^{3/2}}{2\sqrt{b_{ci}(x)}} + C_{\mathrm{base}}(x)h(B_i) \tag{19}$$

The volume per acre estimate when eq. (19) is used in a HPS scheme will then be

$$
\begin{aligned}
\hat{V}_{u^*}^a &= F \times A \sum_{i=1}^{N} \frac{\hat{v}_i^a(x)}{B_i} I_i(x) \\
&= F \times A \sum_{i=1}^{N} \left\{([h(b_{ai}(x)) - h(B_i)]/2)\sqrt{\frac{B_i}{b_{ci}(x)}} + \frac{C_{base}(x)h(B_i)}{B_i}\right\} I_i(x)
\end{aligned} \tag{20}
$$

Fig. 1. Upper-stem height measurement points for critical height sampling (CHS), antithetic CHS, CHS with importance sampling (ICHS), and CHS with antithetic variates and importance sampling (AICHS) for a tree with a basal diameter of 0.5 m and a total height of 25 m using an F = 2.296 metric angle gauge (English, F_E = 10).



where $\hat{V}^a_{u^*}$ is the estimate of cubic metre volume on A based on using the antithetic variate associated with $u^*(x)$.

We will refer to eq. (20) as the antithetic ICHS (AICHS) estimator. The measurements required to use estimator (20) are the same as those needed for estimator (16). However, when estimator (20) is used sampled upper-stem heights $h(b_{ai}(x))$ will be relatively low on the stem for trees near the sample point on the ground and relatively high on the stem for trees distant from that point. The behavior of eqs. (17)–(20) as $b_{ci}(x)$ goes to zero could be of interest. However, as $b_{ci}(x)$ goes to zero, $h(b_{ai}(x))$ also goes to zero. Lynch et al. (1992) commented on this issue for the importance sampling estimator having the same proxy taper function. They commented that if a proxy taper function is chosen that slopes upward more steeply than the typical neiloidal shape of the tree stem base, the numerator of eq. (12) (in this article) should approach zero more rapidly than the denominator. Equations (18) and (19) are derived from eq. (12) and share this property (Lynch et al. (1992) investigated this further by applying l'Hospital's rule with some example taper and proxy taper functions). Furthermore, in actual applications of eq. (18) and (19), it should be noted that field locations of sample points are not located within tree stems. Thus, the minimum value of $b_{ci}(x)$ in actual applications is limited by the minimum possible distance between the sample point and the center of a sample tree, which would be the radius of the smallest merchantable tree, or the smallest diameter considered in sampling. This typically might be in the neighborhood of 5 cm. Thus, in actual field applications, the denominators in eqs. (18)–(20) cannot go to zero. This could be thought to result in a very minor bias, which would also be present in the traditional form of CHS (without importance sampling).

Figure 1 illustrates the differences between upper-stem height measurement points among sampling methods for a tree having a basal diameter of 0.5 m and a total height of 25 m when using a HPS angle gauge having F = 2.296 metric units (equivalent to F_E = 10 English units). The tree in Fig. 1 uses taper function eq. (7) with k = 0.7, which is a shape approximately midway between a cone and a parabola. The graph compares classic CHS, antithetic CHS, ICHS, and AICHS (using $1 - u^*(x)$). As expected, heights using classic CHS or ICHS are very high on the upper stem for tree locations close to the horizontal-point sample field location and decrease with distances farther away. The borderline distance for a tree with a basal diameter of 0.5 m is 16.5 m, so critical heights go to zero at that point. Use of the AICHS reverses the situation so that upper-stem heights are low on the stem close to the field location of the point sample and higher for more distant locations up to a maximum of the 25 m tree total height at the limiting distance of 16.5 m. The upper-stem measurement point for AICHS is only slightly higher than the 45° ideal up to a 10 m distance from the sample point and then increases so that the height is about 20 m at a 15 m distance. This should still be practicable. Of course, mathematically, the height must increase to 25 m at the limiting distance of 16 m. The upper-stem measurement point for antithetic CHS without importance sampling increases more slowly with distance from the point sample location so that it stays below the height required with importance sampling as can be seen in Fig. 1.

Johnson (1988) investigated the performance of antithetic CHS (without importance sampling) but found that it was not as good as classical CHS. This may be due to the CHS characteristic that the magnitude of the critical heights gradually increases at the edge of the plot so that, intuitively, we might imagine the volume estimate changing gradually with translation of the point sample location. This latter property may cause classical CHS to perform better than we might expect given the high variance of CMC integration on an individual tree basis. Since large critical heights occur at the plot edge for antithetic CHS, the volume per hectare estimate tends to change in "jumps" with a translation of the sample point as "new" trees are encountered and "old" trees dropped. Of course, this is also the case with classical HPS in which individual-tree volume tables or equations are usually used. AICHS should have substantially better estimation variance than anthithetic CHS because, as Lynch et al. (1992) showed, importance sampling substantially lowers the variance of the individual-tree volume estimate.

Estimation with a pair of antithetic variates

Previously, only one of a pair of antithetic variates was used to obtain optimal upper-stem height viewing from the sample point on the ground. However, Lynch et al. (1992) found that use of an anthithetic pair of variates with importance sampling was much better for individual-tree volume estimation. This leads to the question of what would be the best arrangement for measuring upper-stem heights within a point sample when both antithetic variates are used. When u is near 1 then $(1 - u)$ will be near zero and vice versa, so when u is near 1 or zero, upper-stem heights will be widely separated on the stem such that one of the pair will be high on the stem and the other will be low on the stem. This viewing situation would most likely be optimal for trees relatively distant from the field location of the sample point. The best possible position on the stem for viewing a pair of antithetic variates when trees are quite near the sample point would most likely be when $u = (1 - u)$ or u = 0.5. For u = 0.5, the antithetic pair are coincident. Thus, we would like to use CHS parameters to obtain u = 0.5 near the sample point while departing towards 0 or 1 as the distance between the tree and the sample point increases. This can be accomplished by defining a pair of uniform random variates as discussed in the Supplementary data.[1] We will refer to estimator (S6)[1] as the paired AICHS (PAICHS) estimator.

[1]Supplementary data are available with the article through the journal Web site at http://nrcresearchpress.com/doi/suppl/10.1139/cjfr-2013-0279.

Fig. 2. Upper-stem height measurement points for importance sampling with an antithetic pair of variates (PAICHS) based on cross-sectional areas at critical heights for a tree with a basal diameter of 0.5 m and a total height of 25 m using an $F = 2.296$ metric angle gauge (English, $F_E = 10$).

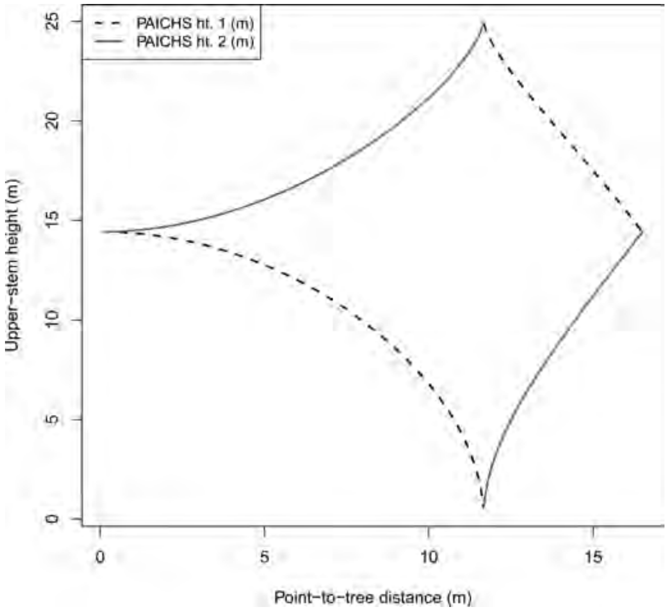


Figure 2 illustrates the relationship between upper-stem measurement points for each of an antithetic pair of variates using importance sampling as indicated previously, for a tree having a basal diameter of 0.5 m and a total height of 25 m. Under this scenario at the sample point, both measurements start at just under 15 m and the difference between them increases until one is at zero and the other at the total height of 25 m at a distance of about 11.66 m from the sample point. At this point the position of the measurement points "switch" such that the upper antithetic height switches to the lower position and vice versa. The measurement points then reconverge to just under 15 m at the limiting distance for this tree of 16.5 m. Unfortunately, this arrangement is not optimal in terms of obtaining a nearly 45° viewing angle for upper-stem heights, but it is better than the alternative of having a measurement at 25 m at the sample point. For upper-stem height and diameter measurement devices that require a tripod, fewer instrument setups would be required as many trees at similar distances from the sample point would have similar upper-stem height characteristics.

Simulation methods

A comparison of the precision of the proposed new estimators, ICHS, AICHS, and PAICHS with CHS and HPS was desired. A purely mathematical ranking of the variances of these estimators appears to be intractable (e.g., see comparison of the variances for HPS and CHS in Van Deusen and Meerschaert (1986)). Therefore, we used computer simulation to compare these estimators with regard to precision. Simulation has often been used to assess the bias and precision of the estimators used for forest sampling. Sterba (1982; see discussion in Bitterlich (1984), p. 139–141) used computer simulation on a mapped forest stand to compare CHS and HPS with some other alternatives, but did not consider any of the importance sampling estimators developed here. Sterba (1982) indicated that HPS was more precise than CHS for $F = 4$, but HPS and CHS were equally precise for $F = 2$ on the 110-year-old mapped spruce stand used in simulations. It is known a priori that both the classic CHS estimator (1) and the proposed importance sampling estimators (16), (20), and (S6)[1] with "reference height" (here and in the following reference height refers to the stem height to which the angle gauge refers for sample tree selection) at the tree base (tree selection with angle gauge at the tree base so that $h(B_i) = 0$ in the estimators) will be unbiased, because both the CMC integration and importance sampling methods on which they are based are unbiased. In the alternate case where sampling from a BH reference (tree selection with the angle gauge at DBH rather than groundline), a small amount of bias will accrue (when $C_{\text{base}}(x)$ in estimators (16)–(20) and (S6)[1] is set to the cross-sectional area at DBH), unless the subsampling methods described earlier are employed below DBH (estimators (16), (20), and (S6)[1] with $C_{\text{base}}(x)$ being a randomly selected cross-sectional area below DBH) for the importance-sampling protocols. A similar correction can be employed for CHS in this case, where the volume of a cylinder with dimensions of basal area and BH is subtracted from CHS with the BH reference, and a CMC estimate taken in the section of the tree bole below DBH. In the simulations presented, the corrected estimators (obtained by the selection of sample $C_{\text{base}}(x)$) are used in the importance protocols for the BH reference, but not in the CHS protocol, to illustrate the magnitude of the bias in this case. The main purpose for the simulations, therefore, is to assess the comparative variability of the estimators.

For the simulations, a square tract with area A is tessellated into grid cells of half-metre resolution. The center of each grid cell becomes a sample point within the tract. For each of the different protocols, an estimate of the volume on the tract using the appropriate estimator is established for each grid cell based on the individual-tree inclusion zones containing the cell center. This method has some advantages compared with a purely random selection of point samples. For example, a spatial map of the estimates is created over the tract using this method that would not be possible under purely random sampling unless many more sample points were taken and, even then, the coverage could not be guaranteed to be uniform enough to interpolate the estimation response surface. Williams (2001*a*, 2001*b*) termed this method sampling surface simulation. He demonstrated that the variance of the estimator is simply the variance of the grid cell estimates in the surface about the mean plane over the surface. Thus, a less variable surface translates into the same property for the estimator. The added benefit of the method is that visual comparisons also can be made. In both CHS and the proposed estimators, the surface height varies within an individual tree's inclusion zone in a predictable manner. When inclusion zones overlap, the estimates for the cells within the intersections of the zones are summed. Thus, both the size and spatial distributions of the trees will affect the final variance estimate. Lastly, increasing the inclusion zone radius by use of a smaller F will also decrease the variance, because the attribute density for an individual tree is spread over a larger inclusion area. If there was only one tree in the population, the lowest variance possible would be a flat surface (deviation from the mean plane), so that HPS with a known tree volume would be the target or lowest variance possible. However, for populations with more than one tree that have overlapping inclusion zones, it is no longer obvious that a flat-topped surface provides the lowest variance. The new estimators introduced here, as well as others for both standing trees and downed coarse woody debris (CWD), are available in the R statistical language (R Development Core Team 2012) package sampSurf (Gove 2012).

The tract area for the simulations was $A = 1$ ha, with $m = 40\,000$ sample points (grid cell centers) at half-metre resolution. A 12.5 m internal buffer was established so that all inclusion zones in the synthetic tree population would be completely contained within the tract, obviating the need for any boundary correction method. This is essentially Masuyama's (1953) method. The tract including the external buffer is square. A synthetic population of $N = 235$ shortleaf pine trees was generated using a spatial inhibition process with 3 m inhibition distance (Venables and Ripley (2002), p. 434). Tree diameters were randomly assigned from a Weibull

Table 1. Results of the simulations for a population of synthetic shortleaf pines of size N = 235 trees with a total volume of 178.84 m^3.

Protocol	Sampling surface Estimate (m^3)	Bias (m^3)	Bias (%)	SD (m^3)	Max. (m^3)	m_e[a]
Tree base (0 m)[b]						
CHS	178.84	−0.00	−0.00	135.89	468	34 024
ICHS	178.73	−0.11	−0.06	128.04	469	
AICHS	178.64	−0.20	−0.11	127.22	522	
PAICHS	178.70	−0.14	−0.08	127.35	492	
DBH (1.37 m)[c]						
CHS	176.77	−2.07	−1.16	134.87	468	32 962
ICHS	178.70	−0.14	−0.08	130.81	493	
AICHS	178.62	−0.22	−0.12	130.34	542	
PAICHS	178.67	−0.17	−0.09	130.38	504	
HPS	178.67	−0.17	−0.10	130.37	508	

Note: CHS, critical height sampling; ICHS, importance sampling CHS; AICHS, antithetic ICHS; and PAICHS, paired AICHS.

[a]Values for m_e, the number of sampled grid cell centers, are the same for each protocol.

[b]Angle gauge at tree stem groundline.

[c]Angle gauge at tree stem BH.

distribution that was fitted to an 80-year-old shortleaf pine stand with a basal area of 20.7 m^2/ha (90 ft^2/acre) on management area 22 of the Ouachita National Forest (Huebschmann (2000), Table 3.2). The heights of all trees were predicted using the total height equations for a site index of 15.2 m (50 ft at base age 50 years) from Lynch and Murphy (1995). Volumes were generated for each tree using a reparameterization of eq. (8) that is the default equation in "sampSurf", where shape parameter $k_G = 2$ is a cone, $k_G < 2$ is neiloid, and $k_G > 2$ is paraboloid (Gove 2011). Geometric shapes for the trees for the volume and associated taper equation were randomly assigned to each tree from a uniform distribution in the range $k_G \in [1.5, 3]$. Once generated, the forest used for simulations remained constant for all methods and all simulations. A metric F = 4 angle gauge was used for inclusion zone determination in all simulations. This resulted in 85% of the grid cells including at least one tree when selection reference height was at the base of the tree, and 82% of the grid cells when the selection was at BH. This can be thought of as the effective sample size, m_e, for the simulation (Table 1). Figure 3 presents a set of sampling surfaces for each of the methods based on sampling at the BH reference, using a randomly chosen subset of 20 trees from the synthetic population, illustrating both the individual-tree and conglomerate surfaces.

Simulation results

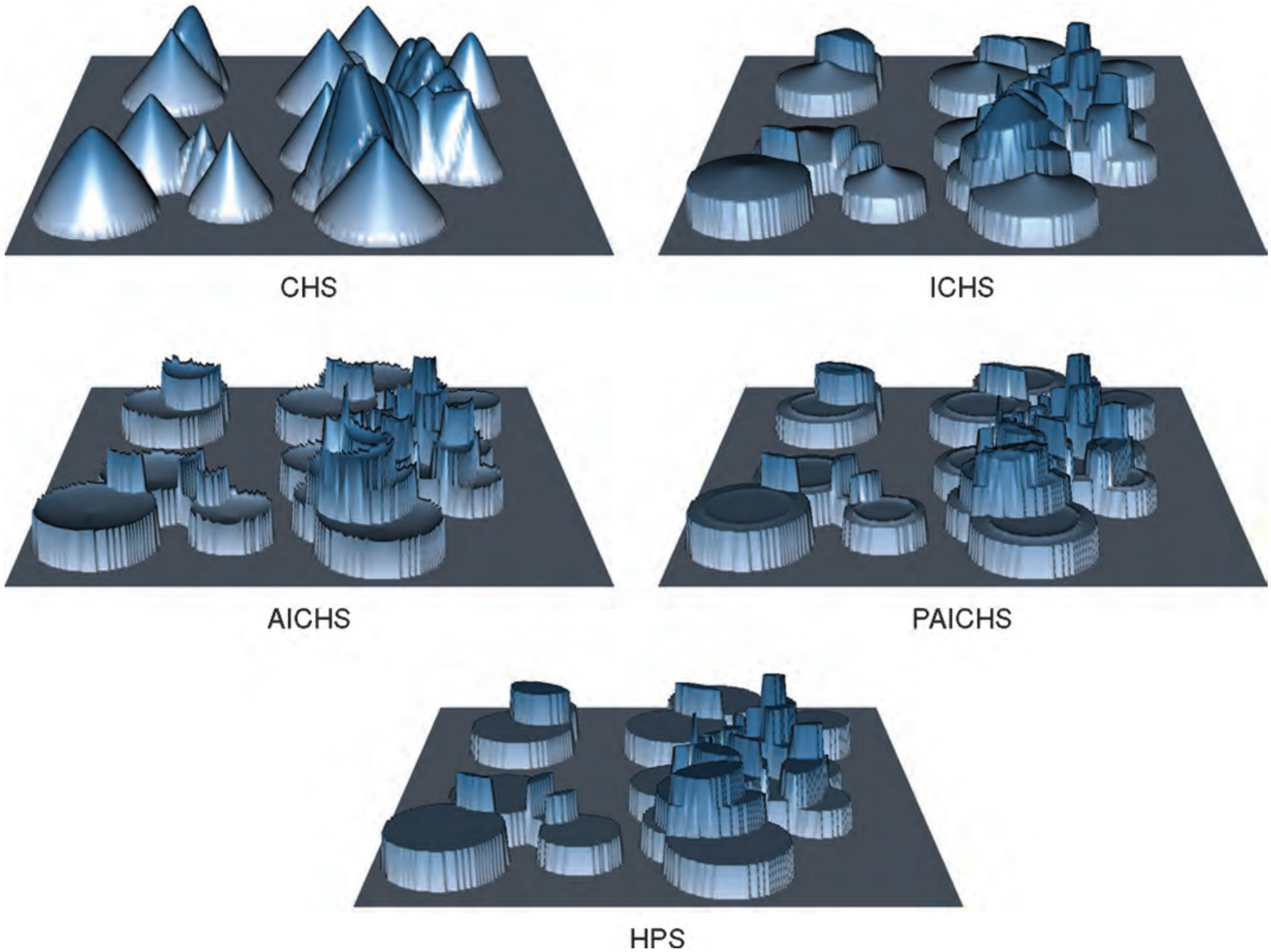
The results of the simulations in Table 1 confirm that all of the methods are unbiased, with the exception of CHS with selection at BH, and the reasons for its 2% bias were discussed already. All methods except CHS used parabolic proxy taper functions; however, it is important to note that they are all design-unbiased for any proxy taper function. This is due to the fact that importance sampling individual-tree volume estimators are design-unbiased (e.g., Lynch et al. (1992)). The very small amount of bias reported in all other methods is an "apparent bias" and is typical in any simulation study where a finite number of sample points is used. This bias will decrease towards zero as $m \rightarrow \infty$ (i.e., the grid cell resolution increases). The variances of the different measurement protocols can be compared within each of two groups based on the reference height selection protocol (i.e., whether the angle gauge selects sample trees using the tree base or DBH). However, one should be cautious when making comparisons between the two protocols. The reason for this is that m_e differs so that there is more sampling effort when tree selection with the angle gauge is at the ground level rather than at BH. This is obviously a consequence of the larger inclusion zone corresponding to the larger base diameter in the former case, such that the differing sample sizes in the variance calculation preclude exact comparison. The second reason that the variance is lower for the ground than for BH reference, as noted above, is due to the enlarged inclusion zones, allowing the volume attribute density to be spread over a larger region of the tract, thereby decreasing the variance. The exception is CHS again, and the reason is again because of the unestimated volume that is lost through excluding the butt swell taper of the tree outside the cylinder defined by DBH. Because the volume estimate is directly related to the critical height (which in turn depends on taper), this can be a very variable section of the stem that contributes higher variance in the ground-level selection estimator, even when the increase sampling effort is taken into consideration.

The results in both Table 1 and Fig. 3 demonstrate that, in this case, the importance sampling estimators ICHS, AICHS, and PAICHS did have lower variance than CHS, even in the test population that had a significant degree of overlapping in tree inclusion zones, as would be typical in forested populations. At the population level (Table 1), they all show an improvement on CHS in terms of standard deviation (and of course standard error), and are essentially equivalent to HPS with regard to precision for all practical purposes; the antithetic estimator actually shows a slight improvement over HPS. Since the importance-based estimators evidently improve on CHS, the slightly lower variance reported for AICHS over HPS is not unreasonable. The same trends seen for sampling with the DBH reference are evident for the estimators using the ground-based selection protocol in Table 1.

Discussion

The results of the previous section indicate that AICHS has essentially the same precision on the mapped forest stands selected for simulation as HPS and is more precise that CHS. It was expected that the importance sampling estimator employed by AICHS would improve precision relative to CHS, but we did not necessarily expect a priori that AICHS would be equally as precise as HPS. Prior to conducting simulations, we surmised that PAICHS would most likely be more precise than AICHS. This was due to the fact that on an individual tree basis, a combination of antithetic variates and importance sampling has been found to provide a more precise estimate of individual-tree volume than importance sampling alone in (e.g., Lynch et al. (1992) and Van Deusen and Lynch (1987)). However, our simulation results indicated that PAICHS was no more precise than AICHS (or ICHS) for our test population. This implies that there is no good reason to apply PAICHS if the goal is to estimate cubic volume per hectare within the context of HPS. The result is fortuitous in that PAICHS

Fig. 3. Sampling surfaces for a subpopulation of 20 shortleaf pine trees sampled at breast height. (Note: the surfaces are slightly distorted to facilitate visualization. Please see Table 1 for maximum surface heights.)



is more time-consuming because it requires two upper-stem measurements and they are still relatively high on the stem, being located in the mid-stem region, for trees located close to the sample point. It seems somewhat counterintuitive to say that more precise estimation of individual-tree volume does not necessarily translate into more precise estimation of cubic volume per hectare or per tract. Reflecting on the analysis of positive and negative correlations among HPS and CHS sample trees given by Van Deusen and Meerschaert (1986), it may be that the nature of the correlations between individual trees in AICHS (and also ICHS) relative to PAICHS is favorable to AICHS so that the AICHS per hectare volume estimator is unexpectedly nearly equivalent to PAICHS, even though AICHS requires substantially less effort in the field. Another consideration is that individual-tree volume estimation is only a portion of the total estimation variance, which also includes sampling error owing to random tree inclusions based on random sample point locations. Evidently, the component of sampling error was substantially greater than the difference between individual-tree volume errors between AICHS and PAICHS. Figure 3 demonstrates that the difference in variation on the top of the sample surfaces between AICHS and PAICHS was not great compared with the average height of the total, nearly cylindrical sample surface. The result that ICHS was essentially equivalent to AICHS was expected. Consequently, we would clearly choose to implement AICHS rather than ICHS because of the advantages of having upper-stem height measurements low on the stem for trees that are located close to the sample point.

Perhaps even more surprising is the simulation result that HPS with known individual-tree volumes was not more precise in the simulations than AICHS, ICHS, or PAICHS. A priori we would have thought that having known individual-tree volumes would translate into an estimator that was more precise than one that uses estimated individual-tree volumes. Van Deusen and Meerschaert (1986) in their mathematical comparison of the variances of CHS and HPS found that, perhaps surprisingly, it is not easy to prove that HPS is more precise than CHS because of the nature of the correlations among the critical heights of selected trees, some of which are positive, tending to increase CHS variances, but others are negative, tending to decrease CHS variance. It is nearly certain that in the case of CHS for typical forest spatial populations, negative correlations between critical heights do reduce the variance of the volume per hectare estimate substantially below what would be obtained by using HPS with CMC integration based on independent random variates for each tree. For CHS, the correlations induced by using a common uniform random variate for all trees selected at an HPS sample point are, therefore, advantageous. In a similar way, positive and negative correlations are also occurring between individual-tree volume estimates of trees selected in AICHS, ICHS, and PAICHS. Since these correlations depend on several factors, including the spatial distribution of the forest population and individual-tree taper, it seems that it would be very difficult to rank their precision relative to HPS using pure mathematics. Also relevant here is the previous discussion about the contribution of individual-tree estimation error and sample-tree selection error to total error of the estimator. Visually, sample surfaces for the AICHS, ICHS, and PAICHS for individual trees without overlapping inclusion zones are nearly cylindrical, although their tops are not completely smooth and flat as is the case for HPS. Since we could not use pure mathematics to rank the estimators, we used simulations to investigate these matters. Of course, as in any simulation, the results are to some extent dependent on the particular population with which we worked.

We stated that for the CHS, ICHS, AICHS and PAICHS estimators, all of the individual-tree CMC or importance-sampling estimators for a given sample point are based on a common uniform variate that is a function of distance between the sample tree and the sample point. This is what induces the individual-tree covariances that were noted by Van Deusen and Meerschaert (1986). The consequences of this can also be viewed graphically in Fig. 3. Note that the top surfaces of the inclusion zones for all of these methods are smooth. If the individual-tree CMC or importance-sampling estimators were based instead on uniform random variates generated independently for each tree, the tops of the sample surfaces would be quite jagged and would differ with each simulation "run" because they would be independent of each other and the distance to the sample point. The same kind of property is found in several recent estimators of downed woody debris volume (e.g., Gove et al. (2012) and Williams and Gove (2003)), i.e., estimators of individual debris particle volume can be thought to be CMC functions of the distance along the debris particle determined by the sample point (the location of which is random). As a consequence, visual representations of these estimators produced by sampSurf (Gove 2012) are similar in some ways to those in Fig. 3. In Fig. 3 the visual impact of the importance sampling estimators relative to the CHS estimator can be seen readily. The importance-sampling estimator "levels off" the top of the sample surface, depressing the high center found in CHS and pushing volume out to the edges to make the surface more similar to the cylinders found in HPS. This process would clearly reduce the variance for the estimator in a tree population in which tree HPS inclusion zones did not overlap. In a population with a nonoverlapping inclusion zone, the importance-sampling variants should, therefore, be bounded in the variance from below by HPS (with known tree volume) and from above by CHS for individual trees. However, it is not necessarily clear that variance reduction would ensue from importance sampling in a population that has overlapping HPS inclusion zones as is typically the case. A "thought experiment" indicates that for inclusion zones that overlap at the edge, the height of the overlap would be higher for HPS than for CHS. However, for portions of overlapping inclusion zones near the centers of these zones, CHS could have a higher height that HPS. It is not clear how these would "balance out" in the final variance of the estimators. Incidentally, the shapes of overlapping HPS inclusion zones are indicative of the contribution of the correlation between sample trees to the overall variance of the tract volume estimator. This was demonstrated more formally by Van Deusen and Meerschaert (1986) in a comparison of the variance between CHS and HPS, as previously discussed.

An importance-sampling estimator with random variates based on the tree distance to the sample point could be developed using the upper-stem diameters selected at randomly chosen heights as suggested by Gregoire et al. (1986). Lynch et al. (1992) showed that for the taper equation developed by Max and Burkhart (1976), importance sampling of upper-stem diameters at randomly selected heights as proposed by Gregoire et al. (1986) is more precise than importance sampling of upper-stem heights at randomly selected cross-sectional areas as was utilized in the development above. A similar result was obtained by Valentine et al. (1992), who compared vertical versus horizontal approaches to importance sampling. However, the Gregoire et al. (1986) approach requires that total tree height be measured on each individual tree. This requirement would obviate the advantageous placement of upper-stem height measurements as previously outlined, because total tree height measurement for trees close to the sample point would likely require a viewing angle greatly in excess of 45°. On the other hand, measurement of upper-stem heights to randomly selected upper-stem cross-sectional areas using importance sampling as previously outlined does not require measurement of the total height for any of the sample trees. It does require measurement of basal cross-sectional area, B, but this can be done using calipers, or may even be done without leaving the sample plot center using laser-tipped calipers. However, if both of an antithetic pair of variates are used on each tree, Lynch et al. (1992) showed that antithetic variates with importance sampling of upper-stem heights at randomly selected upper-stem cross-sectional areas was more precise (for the Max–Burkhart taper function) than antithetic variates using importance sampling of upper-stem diameters at randomly selected upper-stem heights. Unfortunately, the developed scheme for use of both of an antithetic pair of variates still would require a rather substantial viewing angle from the sample point to upper-stem sample heights on trees close to the sample point. Furthermore, the simulations indicate that the measurement of a pair of antithetic variates on each tree (PAICHS) does not improve the volume per hectare estimate obtained by AICHS, which measures only one upper-stem height per sample tree. Van Deusen (1987) suggested a control variate in the context of CHS. Potentially, a control variate could be based on a uniform random variate that is a function of the distance from tree to sample point, although Van Deusen (1987) appears to suggest independent random variates for each tree. However, application of a control variate would almost certainly require measurement of individual-tree total heights, which would be time-consuming, and also the total heights for trees located near the sample point could not be measured from the sample point.

Valentine et al. (1992) suggested a solution for the problem of nonmonotone taper or "reverse taper", which can also affect the estimators developed here. In this situation, the same upper-stem diameter may occur more than once on trees sampled in the field. Valentine et al. (1992) proposed that, in this case, the height to each occurrence of a particular diameter be measured so that the lengths of the stem segments that are wider than the sample diameter can be determined and summed in the volume estimator.

Edge effect: sampling near the tract boundary

The sampling surfaces associated with each of the estimation methods (CHS, ICHS, AICHS, and PAICHS) are radially symmetric in all directions. Therefore, the same critical height will be obtained at any given radial distance, R_i, from the sample point to the subject tree i, regardless of the direction of a line between these two locations (and recall that the importance-sampling methods implicitly depend on critical height to obtain the appropriate uniform random variate for sampling). This produces individual-tree sampling surfaces that have circular cross sections centered about the subject tree. Thus, in selecting a boundary correction method the distance R_i from the sample point to the subject tree is crucial because this distance determines the individual-tree volume estimate for that particular tree to be used in the stand volume estimator.

When applying the mirage method (Schmid-Haas 1969; Gregoire 1982) to CHS and the other methods developed in this paper, the "double counting" of volume for trees (e.g., Gregoire and Valentine 2008, page 228, eq. 7.23) tallied from the mirage method will give incorrect results. This happens because the portion of the tree inclusion zone falling outside the tract boundary is folded back into the tree inclusion zone by hinging on the border (assuming a straight line as in fig. 2 from Gregoire (1982)) so that the radial distance from the mirage point to the mirage sample trees is greater than the radial distance from the original sample point to these same sample trees (unless a sample tree and (or) the sample point falls exactly onto the boundary). However, the mirage method can give correct results for CHS if, instead of double counting, *additional samples* of critical height are obtained from the *mirage point*. For each tree tallied on the mirage point, h_{ci}^{mirage} will need to be measured and added to the sum of critical heights in the CHS estimator. For these trees subject to edge effect, the CHS sampling surface (Williams 2001*a*, 2001*b*) extends over the boundary. The mirage points effectively sample outside the boundary within a portion of the sampling surface that extends over the boundary, "completing" the surface. A proof (available from the first author on request, part of a manuscript in preparation) of the unbiasedness of the mirage method for CHS follows the approach of Gregoire (1982), but is somewhat more complex because new mirage critical heights are measured instead of using double counting. For sample points that are located in a boundary "corner" instead of next to a straight-line boundary, as many as three mirage points may need to be established outside the boundary as discussed in Gregoire and Valentine (2008, p. 228, Fig. 7.9), and mirage critical heights for each mirage sample tree would need to be obtained from each of these points and then added to the total for the estimator at that point. Application of the mirage method to ICHS, AICHS, and PAICHS is very similar to the procedure for CHS. For each of these methods, at each mirage point one must measure $\hat{v}_i^{\text{mirage}}$, $\hat{v}_i^{a\text{: mirage}}$, or $\hat{v}_i^{a1,a2\text{: mirage}}$ as appropriate to the method for each mirage sample tree, and these individual-tree volume estimates would be added to those obtained at the "original" point for volume estimation at that point. Note that the mirage individual-tree volume estimate for tree *i* will not be the same as the volume estimate for tree *i* from the original point because the distance from the mirage point to tree *i* will be different than the distance from the original point to tree *i* unless the tree is located exactly on the tract boundary. This is why double counting of individual-tree volume as done in the original mirage method cannot be used here.

There are a number of situations in which the mirage method cannot be applied; for example, for boundaries that cannot be crossed such as a boundary at the edge of a sheer cliff. Gove et al. (1999) developed the boundary reflection method to address these problems for nonconvex inclusion zones that arise in sample downed woody debris. Ducey et al. (2004) generalized the method and developed a simple field protocol known as the "walk-through" method. The walk-through method rotates the portion of an inclusion zone falling outside the tract by 180° so that it overlaps the portion of the inclusion zone inside the tract. If the randomly located sample point falls in this overlap region, the subject tree will be effectively counted twice. Because of the radial symmetry of sampling surfaces associated with CHS, as well as the other new methods developed here, when the walk-through method is applied, the rotated portion of the inclusion zone aligns perfectly to the same radii between tree and sample point with the inclusion zone (see Fig. 6 from Ducey et al. (2004)). Thus, the same critical heights will be obtained in the overlapping portion of the inclusion zone as in the original inclusion zone so that individual-tree volume estimates may be "counted twice" for sample points located in the inclusion zone overlap area, as is the case for HPS with the walk-through method. Applying the walk-through method in the field, one walks a distance of R_i between the sample point and the subject tree *i* and then proceeds an additional distance R_i on the same azimuth in the direction of the boundary. If the boundary line is encountered before walking the additional distance R_i, then the volume estimate for the subject tree is counted twice. The correction provides an unbiased estimate of volume. Double counting does, in fact, work with the walk-through method because the distance between tree *i* and the sample point is the same for the original inclusion zone and the rotated portion of the inclusion zone. The mirage method in which the inclusion zone is folded back over the boundary is not equivalent to the walk-through method, which uses a 180° rotation instead, yet both methods are unbiased. Consult Ducey et al. (2004) for more information concerning matters such as various nonstraight-line boundary configurations and possible limitations. For applications to the methods presented here, the walk-through method may have advantages over the mirage method because, with the walk-through method, one may use double counting and one does not need to make any additional tree measurements. As well, the walk-through method appears to be more robust with respect to various boundary configurations and limitations.

Conclusion

The use of one antithetic variate with importance sampling overcomes a long-standing difficulty of CHS by obtaining upper-stem sample points that can, in most cases, be viewed much more easily from the sample point. Although some of the advantages of CHS are lost in terms of "smooth" entry and exit of sample trees with translation of the sample point, on the other hand, importance sampling improves the variance of the volume estimate on an individual tree basis, which should, to a great extent, compensate or more than compensate for what is lost. The simulations indicated that AICHS is more precise than CHS and equally as precise as ICHS, PAICHS, and HPS. HPS requires total height measurements, which are time-consuming, although admittedly not quite as time-consuming as upper-stem height samples. Although AICHS requires upper-stem height measurements, it does not require total height measurements. In actual applications, HPS is subject to volume table bias that cannot be eliminated by the adjustment of sample size. Therefore, we propose that AICHS would be advantageous in situations where there is no volume table or equation available, or where one wishes to eliminate the bias inherent in volume tables, equations, or published taper functions.

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