

The unbiasedness of a generalized mirage boundary correction method for Monte Carlo integration estimators of volume

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Abstract: The typical “double counting” application of the mirage method of boundary correction cannot be applied to sampling systems such as critical height sampling (CHS) that are based on a Monte Carlo sample of a tree (or debris) attribute because the critical height (or other random attribute) sampled from a mirage point is generally not equal to the critical height measured from the original sample point. A generalization of the mirage method is proposed for CHS and related techniques in which new samples of critical heights or other dimensions are obtained from mirage points outside the tract boundary. This is necessary because, in the case of CHS, the critical height actually depends on the distance between the tree and a randomly located sample point. Other spatially referenced individual tree attribute or coarse woody debris (CWD) estimators that use Monte Carlo integration with importance sampling have been developed in which the tree or CWD attribute estimate also depends on the distance between the tree and the sample point. The proposed modified mirage method is shown to be design unbiased. The proof includes general application to Monte Carlo integration estimators for objects such as CWD sampled from points.

Key words: critical height sampling, importance sampling, Monte Carlo integration.

Résumé : L'application typique de la « preuve par double dénombrement » que représente la correction du chevauchement des contours par la méthode du mirage ne peut pas s'appliquer à des systèmes d'échantillonnage qui sont basés sur un échantillon d'attributs d'un arbre (ou d'un débris ligneux) selon la méthode Monte Carlo comme celui de la hauteur critique. Cela, en raison du fait que la hauteur critique (ou tout autre attribut aléatoire) échantillonnée à partir d'un point mirage n'est généralement pas égale à celle qui est mesurée à partir du point d'échantillonnage d'origine. Une généralisation de la méthode du mirage est proposée, pour l'échantillonnage de la hauteur critique et des techniques qui lui sont apparentées, dans laquelle de nouveaux échantillons de la hauteur critique ou d'autres attributs sont mesurés à partir d'un point mirage situé à l'extérieur des limites de la parcelle échantillonnée. Cela est nécessaire dans le cas de l'échantillonnage de la hauteur critique car cette hauteur critique dépend en fait de la distance entre l'arbre et un point d'échantillonnage dont la localisation est aléatoire. D'autres estimateurs d'attributs à référence spatiale d'un arbre individuel ou de gros débris ligneux (GDL) qui utilisent une méthode de Monte Carlo avec un échantillonnage préférentiel, pour lesquels l'attribut d'un arbre ou d'un GDL dépend aussi de la distance entre l'arbre et le point d'échantillonnage, ont été développés. Nous démontrons que la méthode du mirage modifiée qui est proposée dans cet article est un échantillon sans biais en raison de sa conception même. La preuve comprend une application générale des estimateurs d'intégration de Monte Carlo pour des objets comme des GDL échantillonnés lors de sondages par points. [Traduit par la Rédaction]

Mots-clés : échantillonnage de la hauteur critique, échantillonnage préférentiel, intégration de Monte Carlo.

Introduction

Critical height sampling (CHS), developed by Kitamura (1964), is perhaps the best-known variant of horizontal point sampling (HPS) in which individual tree volumes are estimated from a sample of upper-stem dimensions rather than a volume table or equation. The CHS stand volume estimator uses the product of the HPS basal area factor and the sum of critical heights to provide a design-unbiased estimate of stand volume. The critical height is defined as the height at which the HPS gauge angle is “borderline” on the upper stem of HPS sample trees. We can interpret CHS as HPS in which Monte Carlo integration is used to estimate the individual volumes of the HPS sample trees (Lynch and Gove 2013). Iles (1979a, 1979b) described CHS and advocated its use in growth estimation. Bitterlich (1984, pp. 48–50) also discussed CHS and the computer simulations of CHS conducted by Sterba (1982; Bitterlich 1984, pp. 139–141). He described volume estimators using indirectly estimated critical heights (Bitterlich 1976). The un-

biasedness of CHS was evident from the discussions of Kitamura (1964) and Iles (1979a, 1979b). McTague and Bailey (1985) discussed the unbiasedness of CHS for certain individual tree taper functions. Lynch (1986) and Van Deusen and Meerschaert (1986) developed the approach of cylindrical shells as a way to demonstrate the unbiasedness of CHS. Lynch and Gove (2013) also developed related estimators that add importance sampling to CHS for variance reduction. Importance sampling uses a proxy taper function to develop a more efficient sampling distribution for upper-stem sample heights or diameter when estimating individual tree attributes (Gregoire and Valentine 2008, pp. 111–113). Bitterlich (1976) also used a taper function in CHS, but used it to find a substitute for measured critical height rather than to develop an importance sampling distribution function.

Although several authors have discussed CHS over a period of decades, we are not aware of significant discussion except for the unpublished thesis of Johnson (1988) regarding the issue of edge effect or boundary overlap with CHS, which occurs when a por-

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tion of the inclusion zone for a sample tree close to the tract boundary falls outside of the boundary (Gregoire and Valentine 2008, pp. 223–224). The mirage method has been widely advocated for HPS, as well as for fixed-radius plot sampling, since its development by Schmid-Haas (1969). In the mirage procedure, a mirage point is established outside the tract boundary on a line perpendicular to the boundary passing through both the original sample point and the mirage point. The distance between the mirage point and tract boundary is the same as the distance between the original sample point and the tract boundary. Upon completion of the tree tally at the original sample point, one conducts a tally from the mirage point using either the HPS angle gauge or the plot radius in the case of fixed-radius plot sampling. When applied to fixed-radius plot or HPS sampling, all of the trees that are tallied from the mirage point will have been also tallied from the original sample point. The mirage estimator simply uses a double count of volume for trees that are sampled from the mirage point (Gregoire and Valentine 2008, eq. 7.23, p. 228). Where double counting is referenced, it should be understood that multiple counts will be required at corners where as many as three mirage points may be required (Gregoire and Valentine 2008, figs. 10.9 and 10.10, p. 345).

Gregoire (1982) published a proof that the mirage method is design unbiased for ordinary HPS and fixed-radius plot sampling. However, it is not obvious that the method can be applied to CHS or related methods because the critical height of a sample tree from the mirage point will be different from the critical height of a sample tree measured from the original sample point within the tract. In general, if the estimator of individual tree volume (or other attribute) depends on the distance between the sample point and tree i , then it is not clear that the mirage method can be used in the typically prescribed manner. This problem does not arise in ordinary HPS because any sample tree is assigned the same volume irrespective of its distance from the HPS sample point. Lynch and Gove (2013) asserted that a modified form of the mirage method can be applied in which double counting is not used, but the tree volume estimate (e.g., critical height for CHS) is measured from the mirage point for trees tallied at the mirage point and added to the tally from the original sample point. Johnson (1988) applied this method to simulations of CHS and found it to be unbiased in simulations. Lynch (1995) stated that the mirage method was used in simulations of Ueno's method. Because Ueno's (1979) method (also see Bitterlich (1984, pp. 50–52, 140–141)) compares randomly generated heights to critical height, the generalization of the mirage method stated previously would have to have been used to obtain unbiased volume estimates in simulations, but no details regarding the application of the mirage method were given. Measurement of critical height trees from the mirage point (rather than double counting) has been recommended to practitioners in the past (K. Iles, personal communication, 23 February 2014). However, none of these authors provided a proof of unbiasedness for the modified mirage method. Of course, the critical height or other upper-stem height obtained at the mirage point that is needed for individual tree volume estimation will be different from the critical height obtained from the original sample point unless the tree is located exactly on the tract boundary so that it is equidistant from the mirage point and the original sample point.

It should be noted that the walkthrough method (Ducey et al. 2004) will handle all of the situations described herein without modification. The toss-back method (Iles 2003, p. 641; Gregoire and Valentine 2008, p. 344) will handle these situations by using critical height measurements outside the tract boundary. However, the mirage method is often used in field surveys and has been described and recommended by widely read and highly cited forest measurements texts such as Avery and Burkhart (2002, pp. 221, 241–242). Thus, we feel that a generalization of this method that is widely used by practitioners is warranted. A possi-

ble technical advantage of the generalized mirage method is that it can be applied to nonsymmetric objects. However, at the current time, most objects sampled in forestry such as tree stems or CWD are assumed to be symmetrical for sampling purposes.

The objective of this paper is to prove mathematically that a generalized mirage method is design unbiased. This generalized method contains the traditional double count method as a special case, but also includes the case in which one needs to obtain differing tree stem or CWD measurements from the mirage point and the original sample point. Thus, the new generalized method can be used for CHS, importance sampling estimators that depend on distance from tree to sample point, and CWD or standing tree estimators based on Monte Carlo integration principles.

Unbiasedness of a generalized mirage method

It is desirable to formulate a generalized mirage estimator that will be appropriate for CHS, the importance sampling estimators developed by Lynch and Gove (2013) and Lynch (2014), as well as several CWD estimators. We want to estimate the quantity T , which is the sum of individual tree attributes t_i . T could represent tree stem volume or weight contained on a forested tract or possibly the sum of individual tree surface areas (Lynch 1986). T could also be the biomass, volume, or surface area of CWD. A flat tract A with area $|A|$ is considered in the subsequent proof. In addition, any point within the land area can be identified by its coordinates $(x, y) \in A$. We also assume that an extended zone A' with area $|A'|$ surrounding the tract is flat and accessible with coordinates of external points $(x', y') \in A'$. The situation addressed by a generalized mirage method is illustrated in for CHS. In Fig. 1, the CHS inclusion zone is circular; however, in the proof that follows, we define the inclusion zone as a generalized set of points that does not have to be circular. Examples of noncircular inclusion zones include the rectangular inclusion zones induced by horizontal line sampling (HLS) (Gregoire and Valentine 2008, p. 360) and inclusion zones of various shapes induced by various forms of CWD sampling. In Fig. 1, note that the inclusion zone Θ_i for tree i overlaps the boundary. The area of the inclusion zone that overlaps the boundary is defined as Ω_i and the reflection of this overlap area within the boundary is Ψ_i . The original sample point is assumed to be uniformly distributed over the tract of area $|A|$ within the tract boundary.

Clearly, the area measure for Ω_i is equal to the area measure of Ψ_i . When the sample point $(x, y) \in \Psi_i$, this induces a mirage point $(x', y') \in \Omega_i$. The mirage point is uniformly distributed over Ω_i conditional on $(x, y) \in \Psi_i$. When this happens, we obtain $\hat{t}_{i,\omega'}$, an estimate of t_i from the mirage point, as well as $\hat{t}_{i,\delta}$, an estimate of t_i from the original sample point. Examples of $\hat{t}_{i,\delta}$ could be the product of critical height and individual tree basal area or the spatially referenced individual tree attribute (SRIA) estimators developed by Lynch and Gove (2013) that use importance sampling. We define Θ'_i as the portion of the inclusion zone within the boundary that does not include the reflection Ψ_i , so we have $\Theta_i = \Theta'_i \cup \Psi_i \cup \Omega_i$.

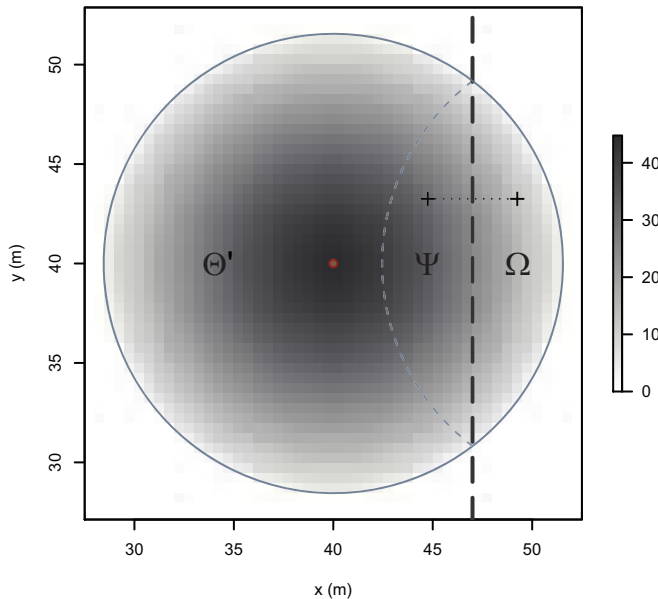
If no mirage or other boundary correction method is employed, the following estimator using actual sample tree volumes t_i would be design unbiased:

$$\hat{T}' = \sum_{i=1}^N t_i \frac{\delta_i}{\pi'_i}$$

where

$$\pi'_i = \frac{|\Theta'_i| + |\Psi_i|}{|A|}$$

Fig. 1. The inclusion zone $\Theta = \Theta' \cup \Psi \cup \Omega$ under CHS showing the actual volume surface estimate (gray shade) at sample points of 0.5 m resolution. Note that the estimates in the mirage zone Ω differ from those within tract $\Theta' \cup \Psi$. A sample point (+) is shown within the tract and associated mirage point across the boundary (dashed line) in Ω .



$\delta_i = 1$ if the random sample point $(x, y) \in \{\Theta'_i \cup \Psi_i\}$, otherwise 0; $|\Omega_i| = |\Psi_i|$, the area of the overlap zone and its reflection; $|\Theta_i|$ is the area of the inclusion zone for tree i ; $\Theta_i = \Theta'_i \cup \Psi_i \cup \Omega_i$ and, therefore, $|\Theta_i| = |\Theta'_i| + |\Psi_i| + |\Omega_i|$; $|\Theta'_i| + |\Psi_i| = |\Theta_i| - |\Omega_i|$; and $|A|$ is the area of the tract.

Inspecting Fig. 1, it is clear that the difficulty encountered when using this estimator is that one would have to measure the area $|\Theta'_i| + |\Psi_i|$ for all sample trees with inclusion zones overlapping the tract boundary. For straight boundaries, this could be accomplished by measuring the perpendicular distance between trees near the boundary and the boundary line. This estimator is essentially a tree-centered area correction method discussed by Iles (2003, p. 627), Penner and Otukol (2000), Gregoire and Scott (1990), and Gregoire and Valentine (2008, p. 224 (measure π method) and p. 344 (direct measurement)). However, this estimator would not be unbiased for techniques such as CHS that use an estimate of tree volume, which is unbiased for actual tree volume over the inclusion zone Θ_i . This follows because truncation of the inclusion zone by the boundary does not allow for the possibility of sampling all of the critical heights on Θ_i , but only a biased portion of them. The direct weighting procedure suggested by Beers (1966) is similar to the one described previously but is based on truncated plot circles centered on the sample point.

The mirage estimator provides a solution to some of the problems indicated previously because it does not require the measurement of $|\Theta'_i| + |\Psi_i|$. Furthermore, the mirage method can be generalized for use with CHS or other methods that use Monte Carlo integration to estimate individual sample tree volumes. A generalization of the mirage estimator that includes the use of estimated tree or CWD volumes will be the following estimator of T :

$$(1) \quad \hat{T} = \sum_{i=1}^N \frac{\hat{t}_{i,\delta} \delta_i + \hat{t}_{i,\omega} \omega_i}{\pi_i}$$

where

$$\pi_i = \frac{|\Theta_i|}{|A|}$$

$\hat{t}_{i,\delta}$ is an estimate of t_i at the random sample point $(x, y) \in \Theta'_i \cup \Psi_i$, the inclusion zone minus the overlap zone; $\hat{t}_{i,\omega}$ is an estimate of t_i at the mirage point in Ω_i corresponding to the sample point; and $\omega_i = 1$ if the mirage point $(x', y') \in \Omega_i$, the overlap zone, or, equivalently, that $(x, y) \in \Psi_i$, the reflection of the overlap zone, otherwise 0.

We want to prove that

$$E[\hat{T}] = \sum_{i=1}^N t_i = T$$

subject to the condition that the individual tree volume estimators are conditionally unbiased on the inclusion zone Θ_i . This means that the conditional expected value of $\hat{t}_{i,e}$ over the set of points in the inclusion zone Θ_i is equal to the actual value of the individual tree attribute t_i :

$$\begin{aligned} E[\hat{t}_{i,e}(z, w) | \gamma_i = 1] &= t_i \\ \Rightarrow \iint_{(z, w) \in \Theta_i} \hat{t}_{i,e}(z, w) p(z, w | \gamma_i = 1) dz dw &= t_i \end{aligned}$$

Because the sample point (z, w) is distributed uniformly with the following conditional probability density:

$$p(z, w | \gamma_i = 1) = \frac{1}{|\Theta_i|}, \quad (z, w) \in \Theta$$

Using this probability density and the definition of conditional expectation,

$$(2) \quad E[\hat{t}_{i,e}(z, w) | \gamma_i = 1] = \frac{1}{|\Theta_i|} \iint_{(z, w) \in \Theta_i} \hat{t}_{i,e}(z, w) dz dw = t_i$$

where $\hat{t}_{i,e}(z, w)$ is an estimate of t_i at spatial location (z, w) , and $\gamma_i = 1$ if $(z, w) \in \Theta_i$, otherwise zero.

We postulate this conditional expected value on an area that is enlarged so that none of the inclusion zones overlaps the boundary. Basically, the condition stipulates the conditional unbiasedness that would occur if the tree inclusion zone did not overlap the boundary. We now return to the use of area such that some inclusion zones do overlap the boundary and will use the condition stipulated by eq. 2 (hereafter referred to as the condition of eq. 2) in the last step of the subsequent proof. It is important to note that $\hat{t}_{i,e}(z, w)$ is an estimate that may depend on spatial location (z, w) . An example would be CHS, in which critical height varies depending on the distance between the sample point and tree i . It can be shown (see example following proof) that the condition of eq. 2 is satisfied by CHS.

Now,

$$\begin{aligned} (3) \quad \hat{T} &= |A| \sum_{i=1}^N \frac{\hat{t}_{i,\delta} \delta_i + \hat{t}_{i,\omega} \omega_i}{|\Theta_i|} \Rightarrow \\ E(\hat{T}) &= |A| \sum_{i=1}^N \frac{E[\hat{t}_{i,\delta} \delta_i | \delta_i = 1] P(\delta_i = 1) + E[\hat{t}_{i,\omega} \omega_i | \omega_i = 1] P(\omega_i = 1)}{|\Theta_i|} \end{aligned}$$

Note that $E[\hat{t}_{i,\delta}|\delta_i = 1] \neq t_i$ because $\delta_i = 1$ only for $\Theta'_i \cup \Psi_i$, which is just a subset of Θ_i , so the expectation is not over the complete domain as required by the condition of eq. 2. The same is true for the other conditional expectation in eq. 3. A good example would be application to CHS, where the expected value of critical height within the portion of the inclusion zone inside the boundary will be different than the expected value of critical height in the portion of the inclusion zone overlapping the boundary (e.g., Fig. 1). We first let $\hat{t}_{i,\delta} \equiv \hat{t}_{i,e}(x, y)$ for points $(x, y) \in \{\Theta'_i \cup \Psi_i\}$ inside the tract, then

$$E[\hat{t}_{i,\delta}|\delta_i = 1] = \iint_{(x,y) \in \Theta'_i \cup \Psi_i} \hat{t}_{i,e}(x, y) p(x, y | \delta_i = 1) dx dy$$

Because the random point location (x, y) is distributed uniformly over the area $|\Theta'_i| + |\Psi_i|$, we have the following conditional probability density:

$$p(x, y | \delta_i = 1) = \frac{1}{|\Theta'_i| + |\Psi_i|}, \quad (x, y) \in \{\Theta'_i \cup \Psi_i\}$$

which leads to the following conditional expectation:

$$(4) \quad E[\hat{t}_{i,\delta}|\delta_i = 1] = \frac{1}{|\Theta'_i| + |\Psi_i|} \iint_{(x,y) \in \Theta'_i \cup \Psi_i} \hat{t}_{i,e}(x, y) dx dy$$

Similarly, for the mirage points (x', y') , let $\hat{t}_{i,\omega} \equiv \hat{t}_{i,e}(x', y')$, $(x', y') \in \Omega_i$, and

$$E[\hat{t}_{i,\omega}|\omega_i = 1] = \iint_{(x',y') \in \Omega_i} \hat{t}_{i,e}(x', y') p(x', y' | \omega_i = 1) dx' dy'$$

Because the random point location (x', y') is distributed uniformly over the area $|\Omega_i|$ with the following probability density:

$$p(x', y' | \omega_i = 1) = \frac{1}{|\Omega_i|}, \quad (x', y') \in \{\Omega_i\}$$

we obtain the following conditional expectation:

$$(5) \quad E[\hat{t}_{i,\omega}|\omega_i = 1] = \frac{1}{|\Omega_i|} \iint_{(x',y') \in \Omega_i} \hat{t}_{i,e}(x', y') dx' dy'$$

We define $z \equiv x, w \equiv y$ if $(x, y) \in \Theta'_i \cup \Psi_i$ and $z \equiv x', w \equiv y'$ if $(x', y') \in \Omega_i$, and substitute eqs. 4 and 5 into eq. 3 to obtain

$$(6) \quad E(\hat{T}) = |A| \sum_{i=1}^N \frac{1}{|\Theta_i|} \left\{ \frac{1}{|\Theta'_i| + |\Psi_i|} \iint_{(z,w) \in \Theta'_i \cup \Psi_i} \hat{t}_{i,e}(z, w) dz dw \right. \\ \left. \times (P(\delta_i = 1)) + \frac{1}{|\Omega_i|} \iint_{(z,w) \in \Omega_i} \hat{t}_{i,e}(z, w) dz dw (P(\omega_i = 1)) \right\}$$

and we have the following probabilities:

$$(7a) \quad P(\delta_i = 1) = \frac{|\Theta'_i| + |\Psi_i|}{|A|}$$

which is the probability that $\delta_i = 1$, and

$$(7b) \quad P(\omega_i = 1) = \frac{|\Psi_i|}{|A|} = \frac{|\Omega_i|}{|A|}$$

which is the probability that $\omega_i = 1$, because $|\Omega_i| = |\Psi_i|$.

By substituting eqs. 7a and 7b into eq. 6, we obtain

$$(8) \quad E(\hat{T}) = \sum_{i=1}^N \frac{1}{|\Theta_i|} \left\{ \iint_{(z,w) \in \Theta'_i \cup \Psi_i} \hat{t}_{i,e}(z, w) dz dw + \iint_{(z,w) \in \Omega_i} \hat{t}_{i,e}(z, w) dz dw \right\}$$

Finally, we observe that $\Theta_i = \Theta'_i \cup \Psi_i \cup \Omega_i$ so we can combine the integrals in eq. 8 to obtain

$$E(\hat{T}) = \sum_{i=1}^N \frac{1}{|\Theta_i|} \left\{ \iint_{(z,w) \in \Theta_i} \hat{t}_{i,e}(z, w) dz dw \right\}$$

We then use the condition of eq. 2 to make the substitution that completes the proof:

$$(9) \quad E(\hat{T}) = \sum_{i=1}^N \frac{1}{|\Theta_i|} \left\{ \iint_{(z,w) \in \Theta_i} \hat{t}_{i,e}(z, w) dz dw \right\} = \sum_{i=1}^N t_i = T$$

Thus, estimator 1 (eq. 1) is design unbiased because the proof was not dependent on any model-based properties of the population or the estimator. Examples that fit into this framework include the CHS volume estimator, the CHS surface area estimator (Lynch 1986), and the SRIA importance sampling estimators of Lynch and Gove (2013). This framework could also include certain estimators of attributes for CWD. The spatially referenced line sampling estimators developed by Lynch (2014) can also fit into this framework if we establish a mirage line based on reflecting the midpoint of the original sample line over the tract boundary.

SRIA example applications

CHS

To prove that the generalized mirage method is unbiased for CHS, we need only show that the condition of eq. 2 holds. That is, that the conditional expectation of the CHS estimator given that the sample point is in the inclusion zone is equal to individual tree volume. For typical CHS formulations,

$$(10) \quad \hat{t}_{i,e}(x, y) = B_i h_{Ci}$$

where B_i represents the groundline cross-sectional area (m^2) for tree i ; h_{Ci} is critical height (m) for tree i , the height at which the HPS angle gauge is borderline; $\pi_i = \frac{|\Theta_i|}{|A_{hal}| \times 10^4}$, where $|\Theta_i| = \pi R_i^2 (m^2)$, the HPS inclusion zone area, R_i is the HPS borderline distance for tree i , and $|A_{hal}|$ is the tract area in hectares; $F = 10^4/\alpha^2$, HPS basal area factor (Gregoire and Valentine 2008, p. 249); $\alpha = 1/\sin(\varphi/2)$; and φ is the HPS gauge angle.

Taking the expected value,

$$(11) \quad E[\hat{t}_{i,e}(z, w) | \gamma_i = 1] = \iint_{(z,w) \in \Theta_i} B_i h_{Ci} \frac{1}{|\Theta_i|} dz dw \\ = \iint_{(z,w) \in \Theta_i} \frac{1}{\pi R_i^2} B_i h_{Ci} dz dw = \frac{B_i}{\pi R_i^2} \iint_{(z,w) \in \Theta_i} h_{Ci} dz dw$$

Using the definition of the plot radius factor (PRF) from [Gregoire and Valentine \(2008, p. 249\)](#), we have $\frac{B_i}{\pi R^2} = \frac{1}{40\,000\kappa^2}$, where $\kappa = 1/(2F^{1/2})$. This leads to the following:

$$(12) \quad E[\hat{t}_{i,c}(z, w) | \gamma_i = 1] = \frac{1}{40\,000\kappa^2} \iint_{(z,w) \in \Theta_i} h_c dz dw = t_i$$

The integral in [eq. 12](#) is the sum of all critical heights over the inclusion area. This is equivalent to the volume of expanded tree stem of [Iles \(1979a, 1979b\)](#). Because the volume of the expanded tree stem is proportional to the volume of the actual tree stem with a proportionality constant equal to $1/(40\,000\kappa^2)$, the result follows. This shows that CHS is conditionally unbiased for tree volume on the inclusion area fulfilling the requirements of [eq. 2](#). A more detailed derivation using the concept of the cylindrical shells is provided in [Appendix A](#).

Importance sampling

We can provide a similar example of an estimator that satisfies the condition of [eq. 2](#) with an estimated volume using importance sampling. This application is termed importance sampling with critical height sampling (ICHS) by [Lynch and Gove \(2013\)](#). This method uses the following estimator from [eq. 11](#) in [Lynch et al. \(1992\)](#) with $n = 1$ for one upper-stem height measurement:

$$(13) \quad \hat{t}_{i,c} = \frac{h(c)v_k(B_i)}{h_k(c)}, \quad 0 \leq c \leq B_i$$

where $h(c)$ is upper-stem height at cross-sectional area c ; $v_k(B_i)$ is proxy tree volume according to a proxy taper function such as a paraboloid; and $h_k(c)$ is upper-stem height at cross-sectional area c according to the proxy taper function.

As for CHS, the sample tree measurement height for [eq. 13](#) varies depending on the location of the sample point within the inclusion zone because the importance sampling distribution function is evaluated on the basis of distance from tree to sample point. However, due to the use of importance sampling, the variation in the value of [eq. 13](#) across the inclusion zone is not as great as the variation among the critical height volume estimates in the inclusion zone, which results in a variance reduction for the estimator. [Lynch and Gove \(2013\)](#) indicated that this type of estimator is unbiased as well as conditionally unbiased on the inclusion zone, satisfying the condition of [eq. 2](#). Thus, estimator 13 ([eq. 13](#)) and similar estimators discussed by [Lynch and Gove \(2013\)](#) will be unbiased when used with this generalized mirage method. A detailed proof is given in [Appendix A](#).

Estimated tree or CWD attribute independent of sample point location

There have been a number of unbiased individual tree volume estimators that are independent of sample tree or sample point location. Good examples of these would be the individual tree importance sampling volume estimators proposed by [Gregoire et al. \(1986\)](#) and [Van Deusen and Lynch \(1987\)](#). Upper-stem height measurements with these methods are obtained by generating a uniform random number independently for each sample tree (and not depending on distance to the sample point) and using the inverse transform method with a distribution function based on a proxy taper function for the tree. In contrast to SRIA, we will term these methods spatially independent individual tree attribute estimators (SIIA). When these estimators are used, the mirage estimator can be formulated as

$$(14) \quad \hat{T}^I = |A| \sum_{i=1}^N \frac{\hat{t}_{i,\delta}^I \delta_i + \hat{t}_{i,\omega}^I \omega_i}{|\Theta_i|}$$

where $\hat{t}_{i,\delta}^I$ is an estimate of t_i from the original sample point that is independent of spatial location; $\hat{t}_{i,\omega}^I$ is an estimate of t_i from the mirage point that is independent of spatial location; and $\hat{T}^I$ is an estimate of T , the sum of attributes t_i on the tract.

By taking expectations, we have

$$(15) \quad E(\hat{T}^I) = \frac{|A|}{|\Theta_i|} \sum_{i=1}^N \{E[\hat{t}_{i,\delta}^I \delta_i | \delta_i = 1]P(\delta_i = 1) + E[\hat{t}_{i,\omega}^I \omega_i | \omega_i = 1]P(\omega_i = 1)\}$$

Because $\hat{t}_{i,\delta}^I$ and $\hat{t}_{i,\omega}^I$ are design unbiased (e.g., [Gregoire et al. \(1986\)](#); [Van Deusen and Lynch \(1987\)](#)) and independent of spatial location, it is clear that $E[\hat{t}_{i,\delta}^I \delta_i | \delta_i = 1] = t_i$ and $E[\hat{t}_{i,\omega}^I \omega_i | \omega_i = 1] = t_i$. Therefore, we have

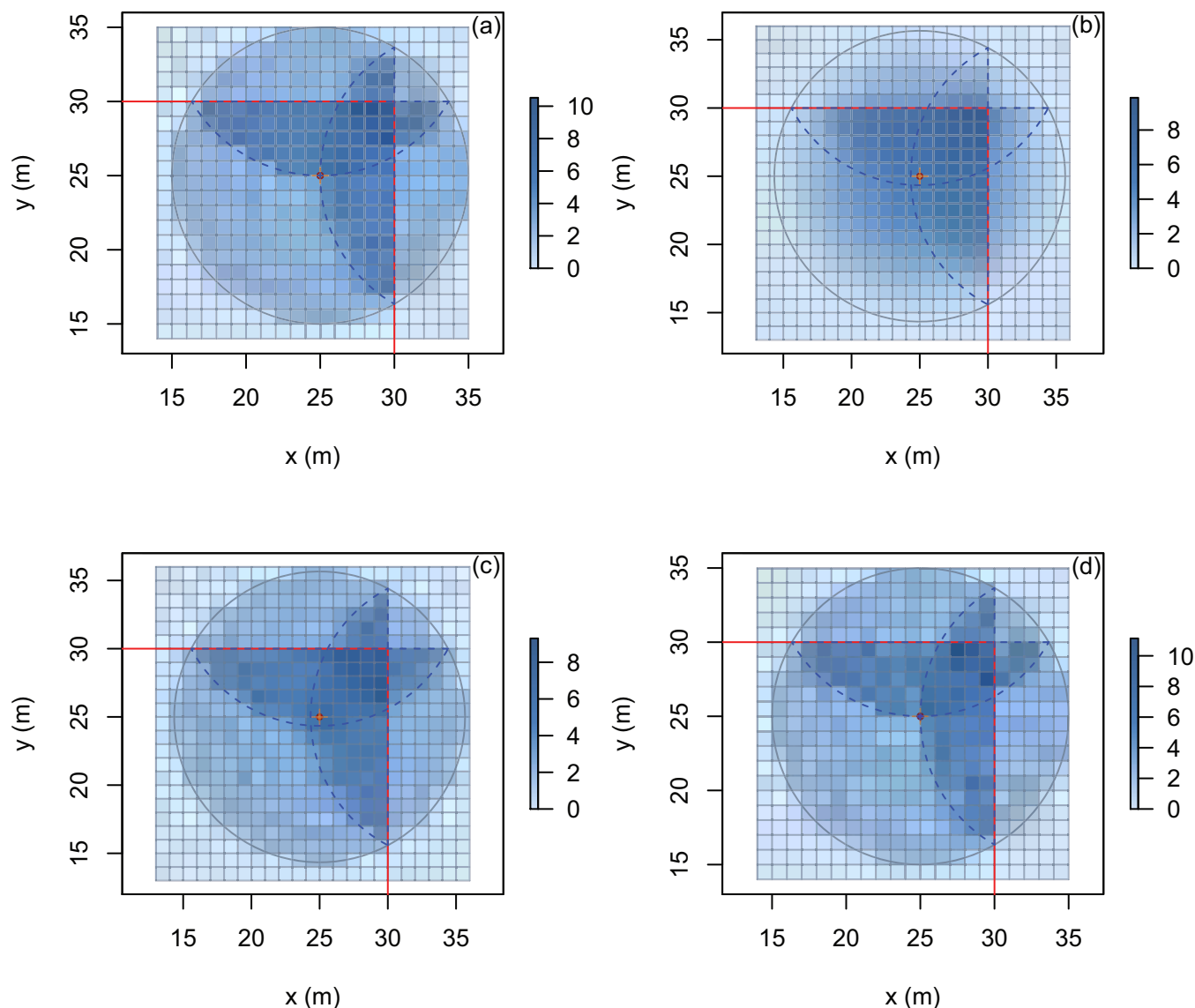
$$(16) \quad E(\hat{T}^I) = \frac{|A|}{|\Theta_i|} \sum_{i=1}^N \{t_i(P(\delta_i = 1) + P(\omega_i = 1))\} \\ = \frac{|A|}{|\Theta_i|} \sum_{i=1}^N \left\{ t_i \left(\frac{|\Theta'_i| + |\Psi_i|}{|A|} + \frac{|\Omega_i|}{|A|} \right) \right\} \\ = \frac{|A|}{|\Theta_i|} \frac{|\Theta_i|}{|A|} \sum_{i=1}^N t_i = T$$

which shows that the estimator is design unbiased. This is similar to [Gregoire's \(1982\)](#) proof that the mirage method is unbiased with tree volumes treated as if known without error. Note that where the estimate of tree attribute t_i is independent of spatial location, $\hat{t}_{i,\delta}^I$ and $\hat{t}_{i,\omega}^I$ could be the same estimates (i.e., one could count the estimate obtained on the original sample point twice for trees tallied on the mirage point and the resulting estimator would still be unbiased for T). Although both are unbiased, the two different approaches would have slightly different variances, with the double count variance expected to be smaller. An example would be HPS in which individual tree volumes were estimated by importance sampling (IS) with independent random numbers generated for each tree (not dependent on the distance between the sample tree and the sample point). We will refer to this method as HPS + IS in the following simulations.

Simulations

To illustrate the theory of the proposed generalized mirage method, a set of four small simulations was developed. The simulations consider HPS, CHS, ICHS, and HPS + IS. As noted earlier, boundary overlap under HPS can be handled with the normal mirage in which replicated estimates are summed at a given sample point ([Gregoire and Valentine 2008, pp. 227, 344](#)). The other three methods require the generalization proposed here. In the simulations, a small plot was tessellated into square grid cells of 1 m resolution where the center of each grid cell is a sample point. A tree with a diameter at breast height of 40 cm and a height of 15 m was positioned such that its inclusion zone overlapped the boundary on a corner of the tract ([Fig. 2](#)). Simulations were also conducted for a tree of the same size interior to the tract so that there was no overlap of the inclusion zone with the boundary. An estimate of the total volume was established for each grid cell within the tree's inclusion zone under each sampling method and the mirage method was used to correct the overlap at the

Fig. 2. Sampling surfaces for (a) HPS, (b) CHS, (c) ICHS, and (d) HPS + IS. The tract boundary (solid; red in digital version) and reflected inclusion zone polygons (dashed; blue in digital version) are also shown, as are the sample values for points lying outside the tract that get reflected back into these polygons. [See digital version for colour rendition.]



boundary. The R statistical language (R Core Team 2014) “samp-Surf” package (Gove 2012) was used for the simulations.

The results are shown in Fig. 2 and Table 1. Note that the two CHS-based methods used reference heights for the inclusion zone at the tree base, whereas the HPS-based methods had inclusion zones established at breast height as usual. This was done to ensure that all of the volume was correctly accounted for in the critical height estimators (e.g., Lynch and Gove 2013). As a consequence, the number of samples (m_e , the effective sample size) inside the tract within each inclusion zone differed for the two sets of methods (Table 1). The simulations demonstrate that the generalized mirage method is unbiased for all methods, regardless of spatial restrictions (i.e., SRIA versus SIIA). Because of the juxtaposition of the inclusion zone and the corner, sample points can have one, two, three, or four accumulated estimates depending on proximity to the corner as illustrated in Fig. 2.

Each of the methods illustrates how the variance (SD) increases over the sample points within the tract because of extra attribute density added from the external mirage points. HPS and HPS + IS are both SIIA methods; the former has a constant surface within

the inclusion zone when it lies entirely within the tract and the latter has a variable surface because of the randomly chosen importance heights and associated volume estimates at each sample point (Gove 2013; Gregoire and Valentine 2008, p. 111). This extra variability for HPS + IS is reflected in the larger maximum surface height and the slightly larger variance estimates for HPS + IS for inclusion zones lying within the tract and corrected by mirage (note that both the maximum value and the variance will be somewhat different under HPS + IS if a new simulation is run and will generally decrease if more importance subsamples are taken at each sample point because of the smoothing effect of averaging subsample estimates). The two SRIA methods based on CHS also showed differences in variance for both internal and mirage cases. Variance reduction is achieved in both cases under ICHS compared with CHS in accordance with theory (Lynch and Gove 2013).

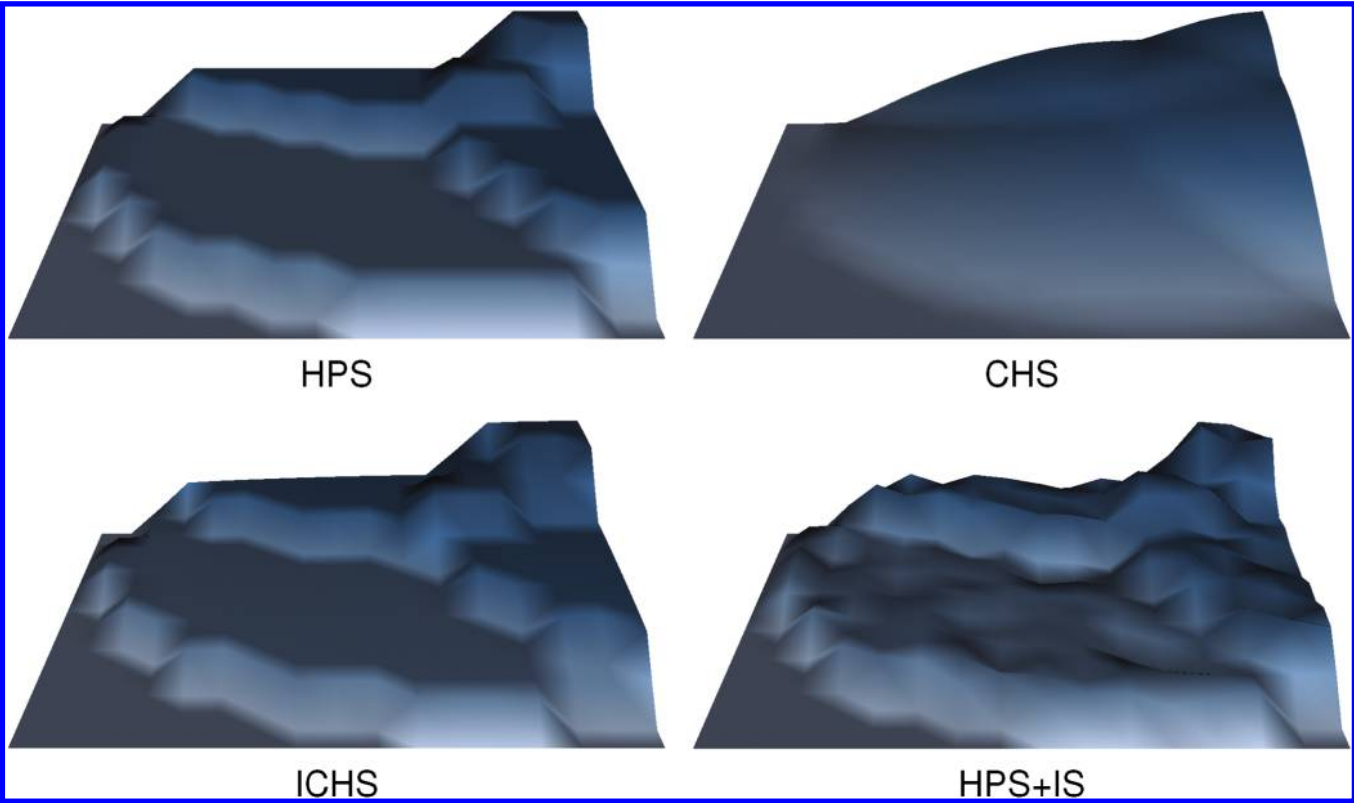
Figure 2 shows both the internal and external sample point estimates for comparison in a two-dimensional representation of the sampling surface where generalized mirage has been applied. Alternatively, Fig. 3 depicts the surface three dimensionally. In this representation, only the surface within the tract boundaries

Table 1. Simulation results for a tree with total volume of 0.918 m³ for inclusion zones that are fully internal to the tract and mirrored at the corner.

Method	Sampling surface		Bias (m ³)	Bias (%)	SD (m ³)	Maximum (m ³)	<i>m_e</i> ^a
	Estimate (m ³)						
HPS							
Inside	0.923	0.005	0.586	1.256	2.63	316	
Corner	0.923	0.005	0.586	1.983	10.52	200	
CHS							
Inside	0.917	−0.001	−0.082	1.461	5.308	360	
Corner	0.917	−0.001	−0.082	2.058	9.865	219	
ICHS							
Inside	0.925	0.007	0.721	1.137	2.678	360	
Corner	0.925	0.007	0.721	1.899	9.299	219	
HPS + IS							
Inside	0.923	0.008	0.832	1.278	3.068	316	
Corner	0.923	0.008	0.832	1.996	11.135	200	

^aNumber of sample points within inclusion zone.

Fig. 3. Sampling surfaces for each of the four methods shown in Fig. 2 showing only the mirage estimate surface internal to the tract boundary (refer to Fig. 2 or Table 1 for surface heights).



is shown. This presents the true picture of the surface at the corner of the tract with generalized mirage from a sampling perspective.

Discussion

A generalized version of the mirage method has been demonstrated to be unbiased when individual tree volume estimates are used that depend on spatial location of the sample point and the individual tree volume estimator is unbiased over the individual tree inclusion zone. Because the proof did not depend on a model such as a random spatial distribution of sampled objects, the modified mirage method is design unbiased. Generally, foresters have assumed that tree stems are symmetric objects and most individual tree volume estimators implicitly use this assumption. However, we should note that the proof of unbiasedness given

here for the modified mirage method does not depend on symmetry, so it is valid for nonsymmetrical objects or for nonsymmetrical individual tree volume estimators. This might be important for possible future applications to nonsymmetrical objects.

We also note that if the regions in Fig. 1 were not required to be portions of the circle, but allowed to take on other geometric shapes, then the proof would be valid for noncircular inclusion zones that may occur with HLS or sampling CWD. Because the proof uses integration over generally defined sets, it is valid for many alternative inclusion zone geometries. In the case of noncircular inclusion zones such as may occur with point relascope sampling (Gove et al. 1999) for downed and dead woody material, it is possible that material could be sampled from the mirage point that was not sampled from the original sample point. However, this proof would still hold with the inclusion and overlap

zones properly defined, and estimator 1 (eq. 1) would still be valid. Clearly, the proof is also valid for fixed-radius plot sampling or for fixed-sized plots that are square or rectangular, as well as various other shapes.

For trees located in tract corners selected using an HPS protocol, as many as three mirage points (for a total of four points including the original sample point) may need to be implemented (see Fig. 2), as indicated by Gregoire and Valentine (2008, p. 228, fig. 7.9). For trees that are selected from these mirage points, estimates of individual tree volume (e.g., critical heights for CHS) would need to be made and added to the tally from the original sample point. A proof of unbiasedness for this procedure would be very similar to that given previously, but would simply involve additional conditional expected values for regions of the inclusion zone falling outside the boundary with respect to corner orientation.

The methods described here preserve the individual estimates at sample points within the inclusion zone. When the sampling (response) surface (Williams 2001a, 2001b) is flat, then one simply double counts the attribute of interest. However, when the surface is variable, then it is important to use the correct estimate for the mirage point in the inclusion zone to preserve the variance of the surface. Under SRIA methods, this means using two spatially explicit estimates. Under independent variable surface methods, one may simply resample at the internal point within the tract to generate the second estimate because the spatial reference is not necessary. In this way, the variance of the two estimates is taken into account. In the independent case, one could simply reuse the original estimate at the original sample point. This would cause the variance to be slightly less (it may not make much difference in practical situations when the sample size is sufficiently large) and is akin to imagining that the surface within Ω_i is a mirror image of Ψ_i for a given stem. The magnitude of the decrease in variance depends on the size of $|\Psi_i|$ relative to $|\Theta_i|$. A small amount of overlap (stem further away from the boundary) would lead to less of a difference than when the stem was located just adjacent to the boundary. Of course, for a given distance from the boundary, trees with larger inclusion zones (e.g., large diameter trees in HPS) would have a greater effect on the variance of the estimate than trees having smaller inclusion zones.

Results from simulations in Table 1 confirm that the generalized mirage method is unbiased for examples including HPS, CHS, ICHS, and HPS + IS. As expected, ICHS was more precise than CHS for an interior tree and a tree located in a corner because of variance reduction provided by using IS to determine volumes for sample trees. These two methods cannot be directly compared with HPS and HPS + IS in this simulation because trees in CHS and ICHS were selected with gauge angles located at the base of the tree, whereas trees in HPS and HPS + IS were selected with gauge angles located at BH, resulting in differences in the sizes of the respective inclusion zones. It is interesting to note that the variance associated with HPS + IS is essentially the same as the variance for HPS even though HPS + IS uses estimated sample tree volumes, while HPS uses actual tree volumes (again note that the variance of HPS + IS will be somewhat different in different simulations unless the same random number sequence is used). Similar phenomena were noted in the simulations conducted by Lynch and Gove (2013).

As noted by Lynch and Gove (2013), the walkthrough method (Ducey et al. 2004) can also be used for boundary correction for CHS and related methods in which individual tree volume estimates depend on spatial location. To perform the walkthrough method, the portion of the inclusion zone overlapping the boundary is rotated by 180° to be coincident with a portion of the inclusion zone within the boundary. Trees are counted twice within this overlap zone. This does depend on radial symmetry, but that would work with most methods of individual tree volume estima-

tion. For the walkthrough method, there is no mirage point and the radial distance between the tree and the sample point is the same for the overlap and the original zone. Thus, for the walkthrough method, there is no need to re-estimate tree volume, one simply does a double count. When the walkthrough method is applied in the field, one walks a distance D_i between the subject tree and the sample point. Proceeding along the same azimuth, one walks to the boundary an additional distance, D_i . Should the boundary be met before this additional distance D_i , then the tree will be counted twice. The walkthrough method may have advantages over the mirage method because one does not need to cross a boundary to implement it and it does not require additional tree measurements from the mirage point. The walkthrough method works for many curved boundary configurations, whereas the mirage method as described here works only for straight boundaries and corners. However, the mirage method would be advantageous for application with nonsymmetric objects or estimators. The toss-back method (Iles 2003, p. 641; Gregoire and Valentine 2008, p. 344) can also be used for the situations described here and can be applied to curved boundaries readily.

Conclusions

A generalized version of the mirage method must be used to correct for edge effects when CHS or other SRIA methods in which sample tree or CWD volumes depend on spatial location. The typical double count mirage method is biased with CHS and other SRIA estimators because critical heights in the part of the inclusion zone outside the boundary cannot be sampled and are not generally equal to critical heights in the inclusion zone inside the boundary. The proof of unbiasedness presented here depends on the assumption that the individual tree volume or CWD estimator is unbiased over the domain indicated by the inclusion zone. This condition is satisfied by CHS and several other SRIA estimators described by Lynch and Gove (2013), as well as several CWD estimators. The typical mirage method needs to be modified so that it includes SRIA estimates made from the mirage point rather than simply double counting the estimate made on the original sample point. For future applications, the fact that this modified mirage method would work for nonsymmetric objects is a potential advantage.

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Appendix A

Conditional unbiasedness of CHS

From eq. 11, we have

$$\begin{aligned} E[\hat{t}_{i,c}(z, w) | \gamma_i = 1] &= \iint_{(z,w) \in \Theta_i} B_i h_{Ci} \frac{1}{|\Theta_i|} dz dw \\ &= \iint_{(z,w) \in \Theta_i} \frac{1}{\pi R_i^2} B_i h_{Ci} dz dw = \frac{B_i}{\pi R_i^2} \iint_{(z,w) \in \Theta_i} h_{Ci} dz dw \end{aligned}$$

Using the typical transformation to polar coordinates, we have $z = r \cos(\theta)$, $w = r \sin(\theta)$ when the sample point is located r metres from tree i and positioned at an angle of θ radians from the polar axis (Wikipedia Contributors 2013). Note that the limits of integration for the integral transformed to polar coordinates are $0 < r < R_i$, where R_i is the borderline distance from tree i and $0 < \theta < 2\pi$ and the Jacobian for the polar coordinate transformation is $J = r$. Transforming to polar coordinates yields

$$E[\hat{t}_{i,c}(z, w) | \gamma_i = 1] = \iint_{(z,w) \in \Theta_i} B_i h_{Ci} \frac{1}{|\Theta_i|} dz dw = \frac{B_i}{\pi R_i^2} \int_0^{2\pi} \int_0^{R_i} h_{Ci} r dr d\theta$$

An additional transformation of $r_i = \kappa 2r_{Ci}/100$, where r_{Ci} (m) is the stem radius at critical height and $\kappa = 1/(2^{1/2})$ is the plot radius factor (PRF) (Gregoire and Valentine 2008, p. 249) and letting d_i be basal stem diameter (cm) and R_{Ci} be basal stem radius (m) yields

$$\begin{aligned} E[\hat{t}_{i,c}(z, w) | \gamma_i = 1] &= \frac{\pi d_i^2}{40\,000 \times \pi \kappa^2 d_i^2} \int_0^{2\pi} \int_0^{R_{Ci}} h_{Ci} 40\,000 \kappa^2 r_{Ci} dr_{Ci} d\theta \\ &= \int_0^{R_{Ci}} 2\pi r_{Ci} h_{Ci} dr_{Ci} = t_i \end{aligned}$$

In this case, t_i is individual-tree cubic metre stem volume. The last step in eq. 12 is to recognize the cylindrical shells integral of stem volume, as has been noted previously by Lynch (1986) and Van Deusen and Meerschaert (1986). The integrand is a cylindrical shell with radius r_{Ci} , circumference $2\pi r_{Ci}$, and height h_{Ci} , which is integrated over all values of the stem radius, resulting in total stem volume. Because eq. 12 shows that the condition of eq. 2 is satisfied, then it follows from this proof that CHS is unbiased when the generalized mirage method is used.

Conditional unbiasedness of IS

For this IS estimator, the upper-stem measurements are obtained using the proxy taper function to obtain the following probability density:

$$(A1) \quad p_k(c) = \frac{h_k(c)}{v_k(B_i)}, \quad 0 \leq c \leq B_i$$

Estimator 13 was originally designed so that samples would be drawn from the distribution function of density (eq. A1) with a uniform random variate using the inverse transformation method. To do this for a SRIA tree volume estimator, we use a uniform random variate based on the distance r between the randomly located sample point and tree i :

$$u_i^* = \frac{\pi r^2}{\pi R_i^2}$$

and then we obtain a sample upper-stem cross-sectional area using the inverse transformation method from

$$c = F_k^{-1}(u_i^*) = F_k^{-1}\left(\frac{\pi r^2}{\pi R_i^2}\right)$$

where using density (eq. A1)

$$F_i(c) = \int_0^c p_k(x) dx$$

Now using estimator 13 in eq. 2 and noting that $du_i^* = 2\pi r / \pi R_i^2 dr$, we have

$$(A2) \quad E[\hat{t}_{i,e}(z, w) | \gamma_i = 1] = \int_0^1 \hat{t}_{i,e}(u_i^*) du_i^* = \int_0^{R_i} \hat{t}_{i,e}(r) \frac{2\pi r}{\pi R_i^2} dr$$

We now transform variables to upper-stem cross-sectional area to be consistent with the mathematical form of eq. 13. To do this, we work with the inverse transformation of the distribution function of density (eq. A1) as follows:

$$(A3) \quad \begin{aligned} c &= F_k^{-1}\left(\frac{\pi r^2}{\pi R_i^2}\right) \\ \Rightarrow F_k(c) &= \frac{\pi r^2}{\pi R_i^2} \\ \Rightarrow F'_k(c) &= p_k(c) dc = \frac{\pi 2r}{\pi R_i^2} dr \\ \Rightarrow dr &= p_k(c) \frac{\pi R_i^2}{\pi 2r} dc \end{aligned}$$

Substituting for dr in eq. A2 and noting the range of integration for upper-stem cross-sectional area $0 < c < B_i$, we have

$$\begin{aligned} E[\hat{t}_{i,e}(z, w) | \gamma_i = 1] &= \int_0^{R_i} \hat{t}_{i,e}(r) \frac{2\pi r}{\pi R_i^2} dr = \int_0^{B_i} \hat{t}_{i,e}(c) p_k(c) \frac{2\pi r}{\pi R_i^2} \frac{\pi R_i^2}{\pi 2r} dc \\ &= \int_0^{B_i} \hat{t}_{i,e}(c) p_k(c) dc \end{aligned}$$

By using eqs. 13 and A1, we have

$$(A4) \quad \int_0^{B_i} \frac{h(c) w_k(B_i)}{h_k(c)} \frac{h_k(c)}{v_k(B_i)} dc = \int_0^{B_i} h(c) dc = t_i$$

where the last step above recognizes as in Lynch et al. (1992) that the integral of upper-stem height with respect to corresponding upper-stem cross-sectional area is equal to individual tree volume, symbolized here as t_i . Because eq. A4 shows that condition of eq. 2 in the proof of unbiasedness for the generalized mirage method is satisfied, then the generalized mirage method is unbiased for this IS application.