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Forest Products Laboratory Contributions to the Use of Weibull Distribution in Wood Engineering

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Abstract

This paper discusses contributions developed at the Forest Products Laboratory that have advanced utilization of the Weibull distribution in wood engineering. The increased use of the Weibull distribution in wood engineering primarily occurred since 1975. The goals of this paper are to (1) unite, clarify, and simplify information published in several sources into a cohesive summary, (2) advance the use of this research through corrections of misinterpretations and errors of some earlier publications, and (3) introduce some potential new concerns about the use of the Weibull distribution in some ASTM standards that present an opportunity for further research.

Keywords: Weibull distribution, goodness-of-fit, proof-loading, reliability based design, bivariate distributions, In-Grade Program, estimation

Contents

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Forest Products Laboratory Contributions to the Use of Weibull Distribution in Wood Engineering

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1. Introduction

This paper discusses contributions developed at the USDA Forest Service (FS) Forest Products Laboratory (FPL) that have advanced utilization of the Weibull distribution in wood engineering. The increased use of the Weibull distribution in wood engineering primarily occurred around 1975 and continues today. As a national research center with the assignment to increase the effective utilization of wood, FPL plays a dominant role in wood utilization research, including the development of allowable properties values (design values) used to design wood structures. In collaboration with wood industry partners, academicians, and other interested parties, FPL scientists, including mathematical statisticians, have been and are actively involved in developing and advocating for improvements to consensus standards published by organizations such as the American Society for Testing Materials (ASTM) and in providing support to government agencies, such as the American Lumber Standards Committee's (ALSC) Board of Review (BOR) and the National Grading Rule Committee.

When a redwood cooling tower collapsed in 1978, which resulted in a person's death, it became a responsibility of the FPL to help understand the problem. This tragedy was followed by claims that the current design procedure led to overestimating design values. A consensus developed that a new testing program was needed, one that should be directed by a steering committee whose members were associated with national research laboratories, grading agencies, industry representatives, and university researchers working in wood property research from both the United States and Canada. The FPL staff and individuals from U.S. grading agencies formed a technical committee that would design and conduct tests in a new multistage program. After the first stage, Canadian representatives joined the technical committee. All procedures developed were to be incorporated in consensus standards. Using some other consensus standards as models to begin designing the study, it was decided that it would be appropriate to produce estimates using the normal distribution, using nonparametric estimates, and using estimates based on another parametric distribution. The normal distribution was a logical choice because it had been shown in several studies to be useful for wood stiffness measured by modulus of elasticity (MOE) values. A nonparametric approach was already coded in several standards. This left the need to determine which additional parametric distribution to use for strength properties, such as modulus of rupture (MOR). The concept of weakest link, often associated with the Weibull distribution, seemed to be a logical choice. However, wood strength design value properties are typically lower-tail estimates of the distribution, such as 5th percentile estimates or tolerance limits for 5th percentile estimates, and the Weibull distribution's use is typically not for lower-tail properties. It was not clear if these properties had been developed for the Weibull distribution or could easily be developed. In addition, all procedures needed to be easily explained and sold to the wood-based consensus, steering, and technical committees.

The research team needed to determine if the Weibull-based design value that the whole committee had identified as a candidate distribution would serve as well as the normal-distribution-based design value and the nonparametric-based design values. At this time, the technical committee had a statistician from the FPL who worked with Dr. Richard A. Johnson, a Ph.D. statistician from The University of Wisconsin. The results of this collaborative research were published in two statistics journals. This initiated a research program aimed at developing information about the Weibull that might be useful in wood engineering. This program has proceeded with several FPL mathematical statisticians and other researchers contributing to the effort. This paper discusses some of the accomplishments and explains the motivation that led to the research. The goals of this paper are to (1) unite information published in several sources into a cohesive summary, (2) correct misinterpretations of some earlier publications by further discussion, and (3) introduce some potential concerns about the use of the Weibull distribution in some ASTM standards.

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2. Weibull Distribution and U.S. In-Grade Program

Motivation for the Research

In 1978, the FPL, in cooperation with the lumber industry, initiated the In-Grade Testing Program (FPRS 1989). The objectives of this program were to evaluate the mechanical properties of nominal 2-in. (standard 38-mm) dimension lumber sold in the United States and to develop analytical models to predict the performance of light-frame structures constructed with this lumber. More than 70,000 specimens were tested in the In-Grade Program, the largest single research effort ever undertaken in forest products research. It was evident from the start of this 10-year, $7 million effort that critical statistical issues would need to be addressed for the program to succeed. Central among these were several issues dealing with the Weibull distribution. Current design practices for wood and related material, known as allowable stress design (ASD), are based on an estimate of the 5th percentile of the strength distribution. From the inception of the In-Grade Program, considerable interest was voiced in using the two- or three-parameter Weibull distribution to form the basis of these estimates. Three critical questions were identified at that time:

- If the population has a Weibull distribution, is the maximum likelihood estimate of the 5th percentile a good estimator of the population parameter for the desired sample size?
- If we want a more conservative estimate for design of wood structures, how do we calculate tolerance limits and confidence limits for percentile estimates based on the Weibull distribution?
- How do we know if the Weibull distribution fits the data we are going to collect?

Statistical research programs were initiated to answer each of these questions. During the In-Grade Program, the research team included at least one mathematical statistician as a member.

Calculation of Estimates and Confidence Percentiles

In one of its three-parameter forms, the Weibull family can be represented by the density function

$$f(x) = \frac{a}{b}\left(\frac{x_i - c}{b}\right)^{a-1} \exp\left\{-\left(\frac{x_i - c}{b}\right)^{a}\right\}$$

where a is the shape parameter, b the scale parameter, and c the location parameter with support for $x_i \geq c$. A more common representation of the Weibull distribution in the wood community is the use of the cumulative distribution function

$$F(x) = 1 - \exp\left\{-\left(\frac{x_i - 1}{b}\right)^{a}\right\}$$

with c set to zero, so that the family of two-parameter Weibull distributions follows (that is, when $c = 0$).

For the three-parameter family, if we let ξ_α denote the population 100αth percentile, the maximum likelihood estimate of ξ_α is

$$\hat{\xi}_\alpha = \hat{c} + \hat{b}\{-\ln(1-\alpha)\}^{1/\hat{a}}$$

where $(\hat{a}, \hat{b}, \hat{c})$ maximize the likelihood

$$L_n(a,b,c) = \prod_{i=1}^{n} \frac{a}{b}\left(\frac{x_i - c}{b}\right)^{a-1} \exp\left\{-\left(\frac{x_i - 1}{b}\right)^{a}\right\}$$

and our random sample is such that $x_i > c$ for $i = 1, \ldots, n$. If values of $a < 1$ are allowed, the likelihood will be unbounded. Johnson and Haskell (1983) showed that the estimated percentile converges to the population percentile with probability one. Harter and Moore (1967) claimed that for shape parameter $a > 2$,

$$\sqrt{n}(\hat{a} - a, \hat{b} - b, \hat{c} - c) \rightarrow \mathbf{N}\left(\mathbf{O}', \mathbf{I}^{-1}(a,b,c)\right)$$

in distribution, where $\mathbf{I}^{-1}(a,b,c)$ is the inverse of the information matrix

$$\mathbf{I}(a,b,c) = \left[-\mathrm{E}\frac{\partial^2 \ln\{f(x_i; a,b,c)\}}{\partial w_k \partial w_j}\right]$$

and $(w_1, w_2, w_3) = (a, b, c)$. Johnson and Haskell (1983) used a computer simulation to investigate the idea that the joint distribution of the maximum likelihood estimators for the shape, scale, and location parameters is normally distributed. They looked at shape parameter values from $a = 1.2$ to 4.0 and sample size $n = 70$ and 200. These researchers concluded that a sample size of 70 was still too small to obtain adequate estimation results using the large-sample normal approximation and that the maximum-likelihood 5th-percentile estimator had a strong tendency to overestimate the population percentile.

Their second conclusion generated a second question of interest. If the point estimate has a tendency to overestimate the population 5th percentile, how do we calculate a tolerance limit for the 5th percentile? Johnson and Haskell (1984) noted that if Harter and Moore's (1967) claim of asymptotic normality holds, then the earlier equation of $\hat{\xi}_\alpha$ is a differentiable function of $(\hat{a}, \hat{b}, \hat{c})$, meaning that $\sqrt{n}(\hat{\xi}_\alpha - \xi_\alpha)$ would also have a limiting normal distribution with mean zero and variance

$$\sigma^2 = \left(\frac{\partial \xi_\alpha}{\partial a}, \frac{\partial \xi_\alpha}{\partial b}, \frac{\partial \xi_\alpha}{\partial c}\right) \mathbf{I}^{-1} \begin{pmatrix} \frac{\partial \xi_\alpha}{\partial a} \\ \frac{\partial \xi_\alpha}{\partial b} \\ \frac{\partial \xi_\alpha}{\partial c} \end{pmatrix}$$

where

$$\left(\frac{\partial \xi_\alpha}{\partial a}, \frac{\partial \xi_\alpha}{\partial b}, \frac{\partial \xi_\alpha}{\partial c}\right) = \left(-\frac{b}{a^2}[-\ln(1-\alpha)]^{1/a} \ln[-\ln(1-\alpha)], [-\ln(1-\alpha)]^{1/a}, 1\right)$$

The information matrix can be estimated by

$$\hat{\mathrm{I}}_n = \left[-\frac{1}{n}\sum_{i=1}^{n} \frac{\partial^2}{\partial w_j \partial w_k} \ln f\left(x_i; \hat{a}, \hat{b}, \hat{c}\right)\right]_{j,k=1,2,3}$$

The second-order partial derivatives of the log-likelihood going into this estimated information matrix are

$$-\hat{\mathrm{I}}_{11} = -\frac{1}{a^2} - \frac{1}{n}\sum_{i=1}^{n}\left(\frac{x_i - c}{b}\right)^a \ln^2\left(\frac{x_i - c}{b}\right)$$

$$-\hat{\mathrm{I}}_{22} = \frac{a}{b^2} - \frac{a(a+1)}{b^2}\frac{1}{n}\sum_{i=1}^{n}\left(\frac{x_i - c}{b}\right)^a$$

$$-\hat{\mathrm{I}}_{33} = -(a-1)\frac{1}{n}\sum_{i=1}^{n}\frac{1}{(x_i - c)^2} - \frac{a(a-1)}{b^2}\frac{1}{n}\sum_{i=1}^{n}\left(\frac{x_i - c}{b}\right)^{a-2}$$

$$-\hat{\mathrm{I}}_{12} = -\frac{1}{b} + \frac{1}{b}\frac{1}{n}\sum_{i=1}^{n}\left(\frac{x_i - c}{b}\right)^a + \frac{a}{b}\frac{1}{n}\sum_{i=1}^{n}\left(\frac{x_i - c}{b}\right)^a \ln\left(\frac{x_i - c}{b}\right)$$

$$-\hat{\mathrm{I}}_{13} = -\frac{1}{n}\sum_{i=1}^{n}\frac{1}{(x_i - c)} + \frac{1}{b}\frac{1}{n}\sum_{i=1}^{n}\left(\frac{x_i - c}{b}\right)^{a-1} + \frac{a}{b}\frac{1}{n}\sum_{i=1}^{n}\left(\frac{x_i - c}{b}\right)^{a-1} \ln\left(\frac{x_i - c}{b}\right)$$

$$-\hat{\mathrm{I}}_{23} = -\frac{a^2}{b^2}\frac{1}{n}\sum_{i=1}^{n}\left(\frac{x_i - c}{b}\right)^{a-1}$$

Thus, we can estimate the variance by

$$\hat{\sigma}_\alpha^2 = \left(\frac{\partial \xi_\alpha}{\partial a}, \frac{\partial \xi_\alpha}{\partial b}, \frac{\partial \xi_\alpha}{\partial c}\right)\Bigg|_{(\hat{a},\hat{b},\hat{c})} \hat{\mathrm{I}}_{\mathrm{n}}^{-1} \begin{pmatrix} \frac{\partial \xi_\alpha}{\partial a} \\ \frac{\partial \xi_\alpha}{\partial b} \\ \frac{\partial \xi_\alpha}{\partial c} \end{pmatrix}\Bigg|_{(\hat{a},\hat{b},\hat{c})}$$

Johnson and Haskell (1984) proposed the following procedure to obtain a tolerance limit for the 100αth percentile with probability $1 - \gamma$:

1. If $\hat{a} > 1$, take $\hat{\xi}_\alpha - Z_\alpha \hat{\sigma}_\alpha / \sqrt{n}$

2. If $\hat{a} = 1$, take $\xi_\alpha = X_{(1)} + b[-\ln(1-\alpha) - (n-1)^{-1} (1 + \ln(1-\alpha))]$ and its estimated variance to get the approximate tolerance bound

$$\hat{\xi}_\alpha - Z_\alpha \hat{b}\sqrt{\left[-\ln(1-\alpha) - (n-1)^{-1}(1+\ln(1-\alpha))\right]^2 \left[\frac{1}{n} - \frac{1}{n^2}\right] + \frac{1}{n^2}}$$

where $X_{(1)}$ is the first-order statistic. The second case uses a modified maximum likelihood solution for the two-parameter exponential distribution. Johnson and Haskell (1984) evaluated this procedure through computer simulation and concluded that a sample size of 70 was not large enough in general for the asymptotic results to hold; results for $n = 200$ were much better, but still slightly low in coverage probability.

The results of this simulation also suggested that for sufficiently large sample sizes, confidence intervals for percentiles of the population can be obtained using the same asymptotic normality for sample percentile estimators. The simulation also showed that the assumption of asymptotic normality improved as percentiles approached the median.

In the In-Grade Program, 360 or more specimens were taken for every size and grade cell tested on major species. Based on the improvement in this tolerance limit from $n = 70$ to $n = 200$, the sample sizes were thought to be sufficient to allow use of this procedure to estimate tolerance limits of the 5th percentile.

In the In-grade Program, shape parameters for various size and grade cells covered an expanse of values. For Hem–Fir values for 13 grade–size combinations of specimens with sample sizes ranging from 20 to 428 at 15% moisture content, the shape parameter estimate for the three-parameter Weibull distribution ranged in values from 1.38 to 4.15 (Evans and Green 1988b). The smallest sample sizes were of some grades and sizes not in the basic sampling plan, which was devised to achieve 360 or more specimens.

Goodness-of-Fit Tests

The next question was how would we know if a two- or three-parameter Weibull distribution fit the data? The answer to this question was important to determining (1) if the tolerance limit should be used based on using the distribution to estimate allowable engineering properties and (2) if the distribution could be used in simulations of lumber performance in structures, such as floor and wall systems.

Goodness-of-fit tests for the two-parameter Weibull distribution have received considerable attention in statistical literature. Mann and others (1973), Smith and Bain (1976), Stephens (1977), Littell and others (1979), Chandra and others (1981), Tiku and Singh (1981), and Wozniak and Warren (1984) all discussed aspects of this problem. The three-parameter Weibull distribution has received considerably less attention, probably because critical values depend on the unknown shape parameter. Woodruff and others (1983) produced tables of critical values when the shape parameter is known. A study of goodness-of-fit tests for two- and three-parameter Weibull distributions was initiated on the basis of apparent inconsistencies in published critical values for some statistics for the two-parameter Weibull, the lack of two-parameter Weibull critical values for sample sizes of 80 to 400 expected in the U.S. In-Grade Program, and the almost complete absence of any goodness-of-fit tests for the three-parameter Weibull distribution. Results of this study are published in Evans and others (1989, 1997). We correct a minor error in Evans and others (1997) in the following section on critical value approximations for the three-parameter distribution.

Evans and others (1989) considered four goodness-of-fit statistics. The first two were Shapiro–Wilk-type correlation statistics with scores suggested by Filliben (1975). If we let $X_{(1)}, \ldots, X_{(n)}$ denote an ordered sample of size n from the population of interest, a test statistic considered is R_{w}^2, where

$$R_{\mathrm{w}} = \frac{\sum_{i=1}^{n}(X_{(i)} - \overline{X})m_{\mathrm{w},i}}{\left[\sum_{i=1}^{n}(X_{(i)} - \overline{X})^2 \sum_{i=1}^{n}\left(m_{\mathrm{w},i} - \overline{m}\right)^2\right]^{1/2}}$$

and

$$m_{\mathrm{w},i} = \{-\ln[1-(i-0.3175)/(n+0.365)]\}^{1/\hat{a}}$$

If a variable X has a two-parameter Weibull distribution, the variable $Y = \ln X$ has an extreme value distribution. Because the extreme value distribution is defined by location and scale, the critical values of the correlation statistic are not dependent on the true shape parameter. Thus, a second statistic is given by R_{we}^2, where

$$R_{\mathrm{we}} = \frac{\sum_{i=1}^{n}(\ln X_{(i)} - \overline{\ln X})\ln m_{\mathrm{w},i}}{\left[\sum_{i=1}^{n}(\ln X_{(i)} - \overline{\ln X})^2 \sum_{i=1}^{n}\left(\ln m_{\mathrm{w}.i} - \overline{\ln m}\right)^2\right]^{1/2}}$$

and

$$\ln m_{\mathrm{w},i} = \ln(-\ln[1-(i-0.3175)/(n+0.365)])^{1/\hat{a}}$$

Evans and others (1989) also considered a modified Kolmogorov–Smirnov D statistic given by

$$D = \sup\left|F(x;\hat{a},\hat{b},\hat{c}) - F_n(x)\right|$$

where $F_n(x)$ is the empirical distribution function of the sample and $F(x;\hat{a},\hat{b},\hat{c})$ is the fitted distribution. Finally, Evans and others (1989) considered a modified Anderson–Darling A^2 statistic given by

$$A^2 = -\frac{\sum_{i=1}^{n}(2i-1)\left[\ln U_{(i)} + \ln\left(1-U_{(n+1-i)}\right)\right]}{n} - n$$

where $U_{(i)} = F(X_{(i)}, a, b, c)$ and $X_{(i)}$ is the ith order statistic. Evans and others (1989) showed that the statistics D and A^2 are the same if calculated on the original scale or the log scale. In the two-parameter case, they showed that R_{we}, D, and A^2 do not depend on the shape and scale parameters. For the three-parameter case, they showed that R_{w}, D, and A^2 depend only on the shape parameter. Computer simulation was used to develop critical values for the test statistics. For the two-parameter Weibull distribution, Evans and others (1989) provided critical values for sample sizes of 10, 15, 20, 25, 30, 35, 40, 45, 50, 60, 70, 80, 90, 100, 120, 140, 160, 180, 200, 240, 280, 320, 360, and 400. For the three-parameter Weibull distribution where the shape parameter affects the critical values, shape parameters of 2.0, 2.8, 3.6, 4.4, and 5.2 (typical range for Weibull fits of structural lumber) were used and tables provided for n = 20, 40, 60, 80, 100, 120, 140, 160, 180, and 200. These critical values were then modeled to develop curves that can be used for any sample size, n, in this range.

Critical Value Approximations for Two-Parameter Distribution

For a sample of size n, with the Kolmogorov–Smirnov D statistics at the 0.10, 0.05, and 0.01 levels of significance, critical value approximations are given by

$$D_{0.10} = 0.82645983 - (0.199103/n^{1/2})$$

$$D_{0.05} = 0.89820336 - (0.221577/n^{1/2})$$

$$D_{0.01} = 1.04550210 - (0.282595/n^{1/2})$$

For the Anderson–Darling A^2 statistic, Stephens's (1977) modification of the asymptotic critical values is very good. This requires multiplying the value found by the test statistic by $(1 + 0.2/n^{1/2})$ and comparing the result to the asymptotic critical values of 0.637, 0.757, and 1.038 for significance levels of 0.10, 0.05, and 0.01, respectively.

For the correlation statistic R_{we}^2, approximate critical values come from

$$\mathrm{RWE}_{0.10} = 0.99550280 - (3.46422/n) + (61.17125245/n^2) - (706.629/n^3) + (3047.57446/n^4)$$

$$\mathrm{RWE}_{0.05} = 0.99373844 - (4.69737/n) + (91.36608058/n^2) - (1093.48/n^3) + (4804.52152/n^4)$$

$$\mathrm{RWE}_{0.01} = 0.98826584 - (8.82798/n) + (205.65876975/n^2) - (2548.8/n^3) + (11329.68065/n^4)$$

Critical Value Approximations for Three-Parameter Distribution

Because the critical values of each test statistic for the three-parameter Weibull distribution depend on the shape parameter, two versions of critical values were produced for each statistic. One version ignores the effect of the shape parameter and expresses just the general trend. A second model takes into account the slightly quadratic effect of the critical values for each sample size when plotted against the shape parameter.

For the Kolmogorov–Smirnov *D* statistics at the 0.10, 0.05, and 0.01 levels of significance, the first set of critical value models that do not incorporate an effect for the shape parameter are

$$D_{0.10} = 0.77946799 - (0.199103/\ n^{1/2})$$

$$D_{0.05} = 0.84427004 - (0.221577/\ n^{1/2})$$

$$D_{0.01} = 0.97673771 - (0.282595/\ n^{1/2})$$

and the second set that does incorporate an adjustment for a shape effect *a* are

$$D_{0.10} = 0.83249052 - (0.199103/\ n^{1/2}) - (0.02551\ a) + (0.00272591\ a^2)$$

$$D_{0.05} = 0.90395920 - (0.221577/\ n^{1/2}) - (0.028543\ a) + (0.0030243\ a^2)$$

$$D_{0.01} = 1.06649581 - (0.282595/n^{1/2}) - (0.0443486\ a) + (0.004908494\ a^2)$$

Notation in Evans and others (1989) caused some problems for some readers in the past and has been updated here. To include the effect of the shape parameter, we start with the first step critical values and adjust them by adding a quadratic adjustment that changes the intercept and adds a linear and quadratic term of the shape parameter. In practice, substitute an estimated shape, $\hat{a}$ for a in the equation.

For the Anderson–Darling A^2 statistic, multiply the value found by the test statistic by $(1 + 0.2/n^{1/2})$, but now compare the result to the following values when not taking an adjustment for shape,

$$AD_{0.10} = 0.55138780$$

$$\underline{AD}_{0.05} = 0.6513147$$

$$AD_{0.01} = 0.88711011$$

and the following when taking into account the shape effect,

$$AD_{0.10} = 0.66187219 - (0.0535362\ a) + (0.005775685\ a^2)$$

$$AD_{0.05} = 0.\ 77169857 - (0.0575924\ a) + (0.006105963\ a^2)$$

$$AD_{0.01} = 1.07062855 - (0.0880853\ a) + (0.009381216\ a^2).$$

For the correlation statistic R^2_{w}, approximate critical values are derived from

$$RW_{0.10} = 0.99885011 - (1.45389/n) + (7.25250378/n^2)$$

$$RW_{0.05} = 0.99877755 - (1.77495/n) + (8.67291150/n^2)$$

$$RW_{0.01} = 0.99910494 - (2.66292/n) + (12.86169089/n^2)$$

and

$$RW_{0.10} = 0.98340421 - (1.45389/n) + (7.25250378/n^2) + (0.008294643\ a) - (0.00101228\ a^2)$$

$$RW_{0.05} = 0.97726385 - (1.77495/n) + (8.67291150/n^2) + (0.01150321\ a) - (0.00139732\ a^2)$$

$$RW_{0.01} = 0.95877564 - (2.66292/n) + (12.86169089/n^2) + (0.02112679\ a) - (0.00250893\ a^2).$$

For the correlation statistic R^2_{we}, approximate critical values are derived from

$$RWE_{0.10} = 0.99411418 - (1.81407/n) + (12.38547217/n^2)$$

$$RWE_{0.05} = 0.99229032 - (2.24194/n) + (16.33414042/n^2)$$

$$RWE_{0.01} = 0.98757887 - (3.37283/n) + (26.99680370/n^2)$$

and

$$RWE_{0.10} = 0.98706289 - (1.81407/n) + (12.38547217/n^2) + (0.003971786\ \alpha) - (0.000508929\ \alpha^2)$$

$$RWE_{0.05} = 0.98677107 - (2.24194/n) + (16.33414042/n^2) + (0.00348\ \alpha - (0.000492187\ \alpha^2)$$

$$RWE_{0.01} = 0.989386299 - (3.37283/n) + (26.99680370/n^2) + (0.0006810714\ \alpha) - (0.000299107\ \alpha^2).$$

Note that the models for the critical values of the three-parameter Kolmogorov–Smirnov and Anderson–Darling tests are incorrect in Evans and others (1997).

Conclusions from Goodness-of-Fit Tests

After completing power studies on both the two- and three-parameter Weibull goodness-of-fit tests, Evans and others (1989, 1997) concluded the following:

- For the two-parameter Weibull, the Anderson–Darling A^2 statistic appeared best.
- For the three-parameter Weibull, R^2_{we} and R^2_{w} appear best.
- The approximating equations for critical values appear good; the equations using the shape parameters are only slightly better.
- The power of the tests is very dependent on the true distributional form of the data.
- Goodness-of-fit tests are often used to help decide distribution selection.

In-Grade Program Use

With the methodology available to fit the Weibull distribution to data sets, to evaluate the fit with goodness-of-fit tests, and to develop tolerance limits and confidence intervals for percentiles from a Weibull distribution, the Weibull distribution was ready to play a significant role in the U.S. In-Grade Program. Seventeen commercial species of wood commonly used in structural applications were tested in the In-Grade Program. For each species, data were collected for several sizes and grades, typically at least three sizes for each of two grades. Mechanical properties included MOE; MOR; the product of MOE and the moment of inertia (EI), often denoted as MOEEI; the product of MOR and section modulus (RZ), denoted as MORRZ; and for major species, ultimate tensile stress (UTS) and ultimate compressive stress (UCS). Sample sizes were at least 360 per size–grade cell for major species and 60 per size–grade cell for species that would later be combined into species groups. The data were adjusted to three different standard moisture content conditions. Two- and three-parameter Weibull fits to each data cell for each property and moisture content value were provided along with goodness-of-fit tests and estimates of 50th and 5th percentiles with 95% and 75% confidence limits and lower tolerance limits. The results of this effort are contained in nine volumes of tables (Green and Evans 1988a–d, 1989; Evans and Green 1988a–d). The information in these volumes has been extremely useful to researchers and others. The distributional fits allow simulations of wood-based structures under different moisture conditions.

The statistical methodology has also been the basis of computer programs now available on the Internet. A draft program that will fit two- and three-parameter Weibull distributions to data, evaluate goodness-of-fit for the distributions, and produce point estimates, confidence limits, and tolerance limits for percentiles of the population is available from the Forest Products Laboratory at http://www1.fpl.fs.fed.us. An improved version of this program is under development, and a user manual for the program is expected shortly.

3. Use of Bivariate Distributions to Model Lumber Properties

Motivation for the Research

The problem began with the question of estimating the correlation between two measures of wood strength, such as bending and tension strength, that each must be measured with destructive testing; or, how do we break the same board twice? This problem was solved using a bivariate normal distribution by Green and Evans (1983), Evans and others (1984), and de Amorim and Johnson (1986). This encouraged researchers to examine more complicated problems, such as using other bivariate distributions to model lumber properties.

Use of Bivariate Weibull Distribution to Model Lumber Properties

The U.S. In-Grade Program provided useful two- and three-parameter Weibull fits to structural lumber data. However, to realistically determine reliability-based safety factors for wood structures, it is necessary to accurately model the joint stochastic nature of two or more strength properties of single members. Johnson and others (1999) evaluated some bivariate distributions for modeling the strength properties of lumber. Chief among these were bivariate Weibull distributions. Several types of these distributions were investigated for possible use in modeling lumber properties. The most promising (see Gumbel (1958) and Lu and Bhattacharyya (1990)) has marginal distributions that are two-parameter Weibull distributions, and its survival function is given by

$$\begin{aligned}\overline{F}(x,y) &= [X > x,\ Y > y]\\ &= \exp\left\{-\left[\left(\frac{x}{\theta_1}\right)^{\beta_1/\delta}+\left(\frac{y}{\theta_2}\right)^{\beta_2/\delta}\right]^{\delta}\right\}, 0<\delta\le 1\end{aligned}$$

The probability density function (pdf) for this bivariate Weibull is

$$\begin{aligned}f(x,y) = &\left\{\frac{\beta_1}{\theta_1}\left(\frac{x}{\theta_1}\right)^{\frac{\beta_1}{\delta}-1}\right\}\left\{\frac{\beta_2}{\theta_2}\left(\frac{y}{\theta_2}\right)^{\frac{\beta_2}{\delta}-1}\right\}\\ &\times\left\{\left(\frac{x}{\theta_1}\right)^{\frac{\beta_1}{\delta}}+\left(\frac{y}{\theta_2}\right)^{\frac{\beta_2}{\delta}}\right\}^{\delta-2}\\ &\times\left\{\left[\left(\frac{x}{\theta_1}\right)^{\frac{\beta_1}{\delta}}+\left(\frac{y}{\theta_2}\right)^{\frac{\beta_2}{\delta}}\right]^{\delta}+\frac{1}{\delta}-1\right\}\\ &\times\exp\left\{-\left[\left(\frac{x}{\theta_1}\right)^{\frac{\beta_1}{\delta}}+\left(\frac{y}{\theta_2}\right)^{\frac{\beta_2}{\delta}}\right]^{\delta}\right\}\end{aligned}$$

Thus, the log of the likelihood is

$$\begin{aligned}\ln L = {} & n\ln\left(\frac{\beta_1}{\theta_1}\right) + n\ln\left(\frac{\beta_2}{\theta_2}\right) + \left(\frac{\beta_1}{\delta}-1\right)\sum_{i=1}^{n}\ln\frac{x_i}{\theta_1} \\ & + \left(\frac{\beta_2}{\delta}-1\right)\sum_{i=1}^{n}\ln\frac{y_i}{\theta_2} + (\delta-2)\sum_{i=1}^{n}\ln\left[\left(\frac{x_i}{\theta_1}\right)^{\frac{\beta_1}{\delta}} + \left(\frac{y_i}{\theta_2}\right)^{\frac{\beta_2}{\delta}}\right] \\ & + \sum_{i=1}^{n}\ln\left\{\left[\left(\frac{x_i}{\theta_1}\right)^{\frac{\beta_1}{\delta}} + \left(\frac{y_i}{\theta_2}\right)^{\frac{\beta_2}{\delta}}\right]^{\delta} + \frac{1}{\delta} - 1\right\} \\ & - \sum_{i=1}^{n}\left[\left(\frac{x_i}{\theta_1}\right)^{\frac{\beta_1}{\delta}} + \left(\frac{y_i}{\theta_2}\right)^{\frac{\beta_2}{\delta}}\right]^{\delta}\end{aligned}$$

$$\begin{aligned}\ln L = {} & n\ln\left(\frac{\beta_1}{\theta_1}\right) + n\ln\left(\frac{\beta_2}{\theta_2}\right) + \left(\frac{\beta_1}{\delta}-1\right) \\ & \times\sum_{i=1}^{n}\ln\frac{x_i}{\theta_1} + \left(\frac{\beta_2}{\delta}-1\right)\sum_{i=1}^{n}\ln\frac{y_i}{\theta_2} \\ & + (\delta-2)\sum_{i=1}^{n}\ln\left[\left(\frac{x_i}{\theta_1}\right)^{\frac{\beta_1}{\delta}} + \left(\frac{y_i}{\theta_2}\right)^{\frac{\beta_2}{\delta}}\right] \\ & + \sum_{i=1}^{n}\ln\left\{\left[\left(\frac{x_i}{\theta_1}\right)^{\frac{\beta_1}{\delta}} + \left(\frac{y_i}{\theta_2}\right)^{\frac{\beta_2}{\delta}}\right]^{\delta} + \frac{1}{\delta} - 1\right\} \\ & - \sum_{i=1}^{n}\left[\left(\frac{x_i}{\theta_1}\right)^{\frac{\beta_1}{\delta}} + \left(\frac{y_i}{\theta_2}\right)^{\frac{\beta_2}{\delta}}\right]^{\delta}\end{aligned}$$

Johnson and others (1999) developed a computer program to provide maximum likelihood estimates of parameters for the bivariate Weibull distribution. Looking at sample sizes of 20 and 100 for combinations of parameters believed to be representative of the range of values one might encounter for the strength properties of lumber, these researchers showed that all the parameters could generally be estimated with reasonable accuracy when given a sample size of 100. However, a sample size of 20 resulted in far less satisfactory estimates. Using a cell of In-Grade data consisting of 412 specimens of visually graded No. 2 Southern Pine 2 by 10 lumber, Johnson and others (1999) showed that a bivariate Weibull distribution fit the lower tail of the joint MOR and MOE data better than did a bivariate normal distribution. For reliability calculations, this suggests that a bivariate Weibull distribution may give more realistic results in wood engineering simulations than a bivariate normal distribution.

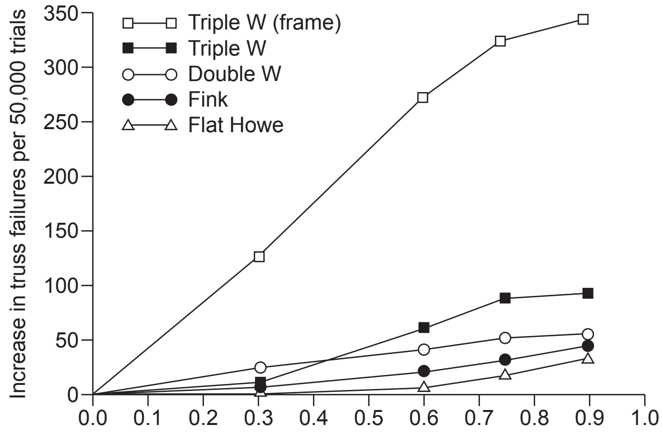

Figure 1. Effect of residual correlation between bending and tensile strength on the increase in truss failures.

In a study to illustrate this, Hamon and others (1985) simulated roof truss performance using two regression equations to predict MOR and UTS strength properties from MOE values generated from a three-parameter Weibull distribution and regression residuals generated from a bivariate normal distribution. The simulation showed that roof trusses with a high ratio of snow to dead load, such as large agricultural trusses, may be quite sensitive to the correlation between lumber strength properties (Fig. 1). Use of a bivariate Weibull distribution to estimate the bivariate nature of MOR and UTS should provide a more realistic distributional assumption that could better assess this potential risk.

Johnson and others (1999) recognized that measuring strength properties often requires a destructive test. It is therefore often impossible to measure two different strength properties on the same specimen. Following an initial paper (Galligan and others 1980), Evans and others (1984), Green and others (1984), and Green and Evans (1983) wrote a series of papers in which they developed methods of estimating the parameters of a bivariate normal distribution using a proof-loading scheme (see also de Amorim and Johnson 1986). The scheme involves loading specimens in strength mode 1 up to a specified load L. If a specimen fails, its mode-1 strength is recorded. Otherwise, the specimen is loaded in strength mode 2 until failure. Estimation of all five parameters in a bivariate normal distribution worked best (Green and Evans 1983) when using a symmetric version of the procedure in which the specimens were randomly split in half. Half the specimens were first loaded in strength mode 1, and the survivors were loaded to failure in strength mode 2. The other half were loaded first in strength mode 2, and the survivors were loaded to failure in strength mode 1.

Johnson and others (1999) showed that this form of the bivariate Weibull distribution is amenable to estimation under a proof-load scheme. They showed that if $R = \{i{:}\ X_i < L\}$ denotes the index set of specimens that failed the proof-load at some load L on strength property X and if the survivors were loaded to failure on strength property Y, then the likelihood is

$$\prod_{i\in R}\frac{\beta_1}{\theta_1}\left(\frac{x_i}{\theta_i}\right)^{\beta_1-1}e^{-\left(\frac{x_i}{\theta_1}\right)^{\beta_1}}$$

$$\times\prod_{i\notin R}\overline{F}(L,y)\left\{\left(\frac{L}{\theta_1}\right)^{\beta_1/\delta}+\left(\frac{y}{\theta_2}\right)^{\beta_2/\delta}\right\}^{\delta-1}\left(\frac{\beta_2}{\theta_2}\right)\left(\frac{y}{\theta_2}\right)^{\frac{\beta_2}{\delta}-1}$$

Thus, the likelihood for the proof-load case can be expressed simply. Estimation of parameters should be possible with sufficient sample sizes. A symmetric approach is also easy to express through its likelihood and again offers the opportunity to simultaneously estimate all parameters. Further evaluation is in progress.

The bivariate Weibull distribution offers opportunities in wood engineering to simulate the performance of wood members in structures more effectively and hence to better understand the reliability of those structures. For example, at the beginning stages of using more sophisticated distributions to model lumber strength properties, one might expect the bivariate Weibull distribution to play an increasingly important role. Again, a draft computer program is available on the FPL web site for researchers who wish to estimate the parameters of a bivariate Weibull distribution from complete data sets.

Use of a Bivariate Gaussian–Weibull Distribution to Model Lumber Properties

Development of new ways to analyze and understand wood engineering data to model lumber properties has continued long after the end of the formal In-Grade Program. Work on bivariate distributions has covered more than just Weibull-related distributions as described above, such as the paper by Pearson (1980), "Potential of the S_B and S_{BB} distributions for describing mechanical properties of lumber." Although the bivariate Weibull distribution had been used successfully to model two related lumber strength properties, other bivariate distributions can be considered, especially those that may have marginal distributions that may capture separate wood property behaviors. For example, knowing a Weibull distribution with a shape parameter of 3.6 is essentially a normal distribution is not intuitive to many people in wood engineering. Recent research has focused on a bivariate distribution that matches a normal distribution for one property to a Weibull distribution for a second property.

Two important properties of wood are stiffness (MOE) and bending strength (MOR). Historically, wood engineers have used a normal (Gaussian) distribution to model MOE and a lognormal or Weibull distribution to model MOR. So using a bivariate distribution to model MOE and MOR suggests that we might want to consider a bivariate Gaussian–Weibull distribution as a possible bivariate distribution to model wood properties. Whereas the bivariate Weibull distribution has been investigated by many researchers, including Johnson and others (1999), the Gaussian–Weibull distribution is less well explored. Verrill and others (2012a,b) derives a Gaussian–Weibull distribution following methods in Johnson and Kotz (1972). This is not the only Gaussian–Weibull distribution possible, but it appears to be a useful step in the construction and evaluation of bivariate distributions because wood engineers often assume that MOE is normally distributed and MOR has a Weibull distribution.

Summary

The bivariate Weibull distribution presented earlier offered opportunities in wood engineering to simulate the performance of wood members in structures more effectively and hence to better understand the reliability of those structures. The development of a bivariate Gaussian–Weibull distribution by Verrill and others (2012a,b) is a new step in using more sophisticated distributions to model lumber strength properties. The bivariate Gaussian–Weibull distribution will be judged on its ability to play an increasingly important role and its incorporation into consensus standards. Again, a draft computer program is available on the FPL web site for researchers who wish to estimate the parameters of a bivariate Gaussian–Weibull distribution from complete data sets.

4. Reliability-Based Design in Wood Engineering

Motivation for the Research

The U.S. In-Grade Program was begun over concerns for the reliability of design values that were being used for dimension lumber. During the time that the U.S. In-Grade Program was occurring, a probability-based procedure called reliability-based design was being introduced to the wood industry as an improvement to the load and resistance factor approach currently being used. As a possible reliability-based procedure was being developed, the Weibull distribution would play an important role. Unfortunately, the way the proposed reliability-based design procedure was written creates potential problems as this statistical review discusses.

Use of the Weibull Distribution in Reliability-Based Design in Wood Engineering

In 1984, the American Society of Civil Engineers (ASCE) Committee on Wood formed a Task Committee on Load and Resistance Factor Design (LRFD) for Engineered Wood Construction. The intent was to develop a reliability-based approach to safety and to put wood practices in line with those being used for concrete and steel. This new LRFD approach is described in AF&PA/ASCE 16-95 (AF&PA/

ASCE 1996). Calculation methods for the resistance part of the LRFD approach are described in ASTM D5457 (ASTM 2009). This ASTM standard applies only to individual wood elements, not to assemblies. The two-parameter Weibull distribution is the mandated basis of ASTM D5457 calculations. The standard does not use full reliability methods but is loosely based on the approach presented by Thoft-Christensen and Baker (1982). This design procedure starts with fitting a two-parameter Weibull distribution to either a complete data set or a set of data representing the lower tail of the distribution. The fit can be made using either maximum likelihood methods or a regression-based estimation procedure. A distribution percentile estimate R_p is calculated from the fitted two-parameter Weibull distribution. For many properties like strength properties of wood, this will be a 5th percentile estimate. A coefficient of variation (CV) is calculated from the fitted shape parameter α using the formula

$$\mathrm{CV} \cong \alpha^{-0.92}$$

From this value and the sample size, a data confidence factor Ω is calculated from a table in the standard. Generally this factor (≤1) goes up as sample size increases and down as the CV increases. Finally, a reliability normalization factor K_R is also calculated from a table in the standard based on the CV and the mechanical property being considered. In general, this factor reaches its largest value at a CV of 12% and decreases as CV rises above 12%. The final LRFD resistance factor is a product of the three values:

$$R_n = \left(R_p\right)\left(\Omega\right)\left(K_R\right)$$

The resulting value is used as the upper limit of the 50-year maximum load to which the element can be subjected. Fifty-year maximum loads are based on fitting various distributions to combinations of dead loads and wind, rain, snow, earthquake, and occupancy live loads. These distributions are codified in ASCE 7-88 (ASCE 1990).

Because of the sensitivity of reliability calculations to distributional form and method of fit, the wood engineering community has been interested in evaluating the effect of using a three- versus a two-parameter Weibull distribution in estimation method and use of censored data sets versus full data sets. Simulations (such as Durrans and others (1998)) addressed some of these issues. Further, the standard states that “Estimates of the distribution and its parameters give the most accurate reliability estimates when they represent a tail portion of the distribution rather than the full distribution” (ASTM D5457 (ASTM 2009), appendix X1.1.2). Whereas most strength properties for lumber are based on lower tail properties, the use of the standard is not limited to lower tail properties.

Using a Weibull distribution to represent various properties of wood is not a new concept. Verrill, and others (2012a, 2013, 2014) suggested theoretically and empirically that MOR values of lumber grades are not distributed as a two-parameter Weibull distribution. This may be due in some cases to a grading process where a grade may have a high grade removed in some part from it, or a lower bound on the size of defects allowed in the grade, or multiple sources producing the grade in different ways. However, in practice, the Weibull distribution is often used as a possibly conservative estimate of lower-tail percentile estimates. Data sets of randomly selected specimens drawn from a population to examine other properties, such as MOE, have commonly exhibited goodness-of-fit of the sample distribution for a normally shaped distribution. What is being proposed in the reliability-based design section in ASTM D5457 is slightly different than the usual procedure. Lumber properties can change over time due to changes in the population. For example, changes in the population can occur if the proportion of juvenile wood changes. With increased fertilization and periodic thinning of trees to allow more space between trees to promote faster growth, the result will often be an increase in the proportion of juvenile wood. Trees grown in this way are often referred to as plantation wood. Such changes in the population have led to mandatory monitoring of lumber properties. Under the current procedure, a random sample of the population is taken in the size and grade most likely to show a change earliest. Usually this is a 2 by 4 No. 2 size and grade. This grade cell is one with defects. With the size being small, any increase in juvenile wood would not be masked as much as in larger sized specimens with a larger cross section that may contain more mature wood.

The current monitoring procedure for allowable properties for visually graded dimension lumber in ASTM D1990–07 (ASTM 2009) starts with a random sample of approximately 360 specimens that is representative of the population. This is done by dividing the growth region into sub-regions, which in past studies have been shown to be somewhat homogenous in wood properties. Because economic conditions have eliminated all commercial production of 2 by 4 No. 2 data in some sub-regions, the total number available has been greatly reduced for some species. Each remaining sub-region is sampled in proportion to production. Mills in a sub-region are chosen at random, sampled with a limit of 20 specimens coming from any mill. The properties of the resulting sample can then be calculated and compared to the claimed properties to see if there is evidence of a change in properties, which would result in further sampling to better evaluate a potential change. ASTM D5457 is aimed at providing reliability estimates rather than allowable property values, like ASTM D1990. However, it is not unreasonable to use the same sample to evaluate the properties under each of these standards.

Because allowable design values for strength properties such as MOR in ASTM D1990 are lower tail properties, it has been suggested that methods can be used to reduce the number of specimens to be tested, thus saving money and resources. Recalling that the ASTM D5457 standard suggests using a tail portion of the distribution rather than the full distribution leads to some ideas on how to single out the lower tail specimens. Because very few situations arise where someone knows in advance which specimens constitute the lower tail, researchers must employ some procedure to break the lower tail pieces without breaking all the specimens. The most common way of doing this has become known as the Warren–Glick method (Johnson 1980). In this procedure, if you wanted the bottom 10% of a sample of 600 specimens, you would break the first 60 pieces. You would load the next piece to the highest load level of the original 60 pieces tested. If it does not fail, the piece being tested is considered to have passed and loaded no more. The test sample and test load remain the same. If the specimen fails, it becomes part of a new test sample. The piece in the test sample with the highest load is removed from the test sample. The highest load remaining becomes the new test load. This continues until the test sample contains the 60 pieces that failed at the lowest load.

Other methods have been suggested to further lower the cost of a Warren–Glick sample. If there is a variable that is highly correlated with the test variable of interest, it can be used to estimate the lowest 60 pieces. With a high correlation, the number of specimens broken beyond the initial 60 pieces can be reduced to save money.

At this point under ASTM D5457, the random sample data can be used to create numbers that wood engineers call a reliability-based design procedure. The reliability-based design procedure does not use full reliability procedures but uses a procedure from Thoft-Christensen and Baker (1982) that is loosely based on it. The ASTM D5457 procedure starts with fitting a two-parameter Weibull distribution to either a complete data set or a set of data representing the lower tail of the distribution. The fit of the two-parameter Weibull can come from maximum likelihood estimates or regression-based estimates. A predicted percentile R_{p} is calculated. An estimated coefficient of variability (COV) is calculated from the estimated shape parameter using the equation above. Finally, a reliability normalization factor is calculated using ASTM D5457. The three factors can then be used to calculate reliability-based percentile estimate R_n based on the second equation above. For wood, the reliability-based percentile is the design value used for strength-based wood properties like MOR.

The random sample data and the procedures discussed have produced a design value using a reliability-based percentile. It would be nice at this point to conclude that the reliability-based design procedure is working well for wood engineering. Unfortunately, we have not created a unique design value. The standard allows the formation of multiple design values. Any design system that can create multiple design values is open to someone creating multiple values and then picking the one most favorable to them.

This problem of creating multiple design values begins with the problem that there are multiple sample data sets that someone could argue should be used. For example, the standard states that "estimates of the distribution and its parameters give the most accurate reliability estimates when they represent a tail portion of the distribution rather than the full distribution" (ASTM 2009). So a data set that represents the tail portion of the distribution is a potential data set that could be used. It is easy to test every specimen and get a complete data set, so that a complete data set could be used. If a procedure like the Warren–Glick procedure is used to get a lower-tail distribution around the 5th percentile and any other specimens broken in the Warren–Glick added, we would be following a procedure recommended by many statistics books that teach reliability. This hybrid data set would appear to be another reasonable choice.

With each data set created, the procedure allows two ways to estimate the Weibull distribution parameters. The parameters could use maximum likelihood estimates of the two parameters. Statistical studies claim that estimates from censored data sets should be the same as for the whole data set if maximum likelihood estimates are used. Regression-based estimates provide no such assurance that there will be similar estimates for the different data sets. In addition, the regression-based estimates can be very different from maximum-likelihood estimates for censored data sets, as shown by Genschel and Meeker (2010) who compared maximum-likelihood (ML) estimation of Weibull parameters and median-rank regression (MMR) methods through simulations with censored data sets. They point out that the two methods often give very different estimates and conclude that "under any reasonable criterion for comparing estimators, ML methods are better in almost all situations" (Genschel and Meeker 2010). With two or more allowed sample sizes (a complete sample, a fixed percentage of the lower tail specimens, and the bottom 60 specimens) as well as more than one regression method available for estimation, it is very possible that each regression-based estimate and maximum-likelihood estimate could produce a different design value.

In a series of recent articles, Gromala and others (2017a–d) look at reliability-based design and ASTM D5457. The editorial at the beginning of *Wood Design Focus*, vol. 27, no. 1, states the following:

> The four papers in this issue of *Wood Design Focus* examine the past, present, and potential future of reliability-based wood design. The first paper explores the relationship between early LRFD terms and discusses how today's differing terminology interpretations can lead to confusion. The second paper discusses several assumptions underlying all reliability analyses, and clarifies how these assumptions lead to end-use simplification (rather than complication). The third paper provides

step-by-step calculations for users to follow when developing input parameters for reliability analysis. Finally, the fourth paper traces the history of closed-form reliability equations from the late 1960s to today's ASCE 7 recommendations.

These are excellent papers to help begin the process of reevaluating ASTM D5457. The implication of the papers is that ASTM D5457 needs to be reevaluated. However, the issue of how you get the estimates of the Weibull shape and scale parameters is not discussed. This could be an issue. Any procedure that allows so many choices for design procedures is too subjective to be of any real use. Until the standard reduces its dependency on choices and settles on a specified procedure, reliability-based design may not really exist in wood engineering.

5. Ratios of Percentiles from Two Distributions

Motivation for the Research

When dealing with physical properties that can be modified by treatments, conditions of use, conditions of manufacturing, or other factors, differences in properties are often described in terms of ratios. When dealing with two properties that are both normally distributed or log-normally distributed, it is easy to talk about ratios of percentiles of the properties and confidence limits for these ratios. When it was decided that the Weibull distribution would be considered for properties, it became important to study ways of putting confidence limits on ratios of percentiles from Weibull distribution.

Confidence Intervals on Ratios of Percentiles from Two Distributions

The ratio of two property estimates is commonly used in wood engineering design codes to establish allowable properties. For example, "dry–green" ratios are given in ASTM D2555 (ASTM 2009) for a number of timber species and may be used to calculate allowable properties. Also, an "E/G ratio" of 16 is assumed in ASTM D2915 (ASTM 2009) for adjusting the flexural modulus MOE based on any span-to-depth ratio and several loading modes. As a third example, ASTM D245 specifies that "strength ratios in tension parallel to grain are 55% of the corresponding bending strength ratios" (ASTM 2009).

Ratios often are used when it is not practical or perhaps possible to conduct tests for all combinations of factors that may affect allowable properties, such as grades, sizes, test modes, and environmental conditions. The usual practice has been to conduct extensive tests for one combination of factors and to develop ratios for adjusting allowable properties for a different set of factors. For example, by 1935 extensive tests had been conducted on wood specimens with "green" moisture contents (Markwardt and Wilson 1935). To reduce sampling and other costs, only a limited amount of information was available on "dry" wood. By careful evaluation of the results where both dry and green data were obtained from the same trees, dry–green ratios were developed, which could be used to adjust allowable green properties to 12% moisture content. Current values of green properties for clear wood and appropriate dry–green ratios for several properties are given in ASTM D2555 (ASTM 2009).

Most of these ratios were originally established by analysis of mean trends. However, the ratios are being applied without regard to position in the distribution. Recent studies have begun to focus on the ratio of properties at other percentile levels, especially the 5th percentile. These include studies of the effect of loading rate on tensile strength (Gerhards and others 1984), effect of moisture content on flexural strength (Aplin and others 1986, McLain and others 1984), and effect of redrying on strength of CCA-treated lumber (Barnes and Mitchell 1983, 1984). However, fitting confidence intervals on ratios of percentiles, particularly when the ratio of percentile estimates was based on two different three-parameter Weibull distributions, was problematic.

Johnson and others (2003) suggest four possible solutions to the problem of confidence intervals for ratios. Two solutions, which we examine here, consider ratios of percentile estimates from three-parameter Weibull distributions. One solution investigates a confidence interval for the log of the ratio, and one is a nonparametric solution that does not assume a Weibull distribution. The first solution began as an extension of the problem of estimating confidence intervals for percentiles from Weibull distributions, as discussed earlier in this report. Recall that the Johnson and Haskell (1984) procedure for confidence intervals for the $100(1-\alpha)$ percentile with probability $1-\gamma$ would be as follows:

1. If $\hat{a} > 1$, then

$$\hat{\xi}_\alpha - Z_{\gamma/2}\left[\hat{\sigma}_\alpha / \sqrt{n}\right] \le \xi_\alpha \le \hat{\xi}_\alpha + Z_{\gamma/2}\left[\hat{\sigma}_\alpha / \sqrt{n}\right]$$

2. If $\hat{a} = 1$, then take

$$\hat{\xi}_\alpha = X_{(1)} + \hat{b}[-\ln(1-\alpha) - (n-1)^{-1}(1+\ln(1-\alpha))]$$

and its estimated variance to obtain

$$\hat{\xi}_\alpha - Z_{\gamma/2}\hat{b}\sqrt{\left[-\ln(1-\alpha) - (n-1)^{-1}(1+\ln(1-\alpha))\right]^2\left[\frac{1}{n} - \frac{1}{n^2}\right] + \frac{1}{n^2}}$$

$$\le \xi_\alpha \le \hat{\xi}_\alpha + Z_{\gamma/2}\hat{b}$$

$$\times\sqrt{\left[-\ln(1-\alpha) - (n-1)^{-1}(1+\ln(1-\alpha))\right]^2\left[\frac{1}{n} - \frac{1}{n^2}\right] + \frac{1}{n^2}}$$

If we choose confidence levels for each of two percentile estimates, denoted by $(1-\gamma_1)$ and $(1-\gamma_2)$, such that $(1-\gamma_1)(1-\gamma_2) = (1-\gamma)$, then by independence

$$P\left[\frac{\hat{\xi}_{1\alpha} - Z_{\gamma_1/2}\hat{\sigma}_{1\alpha}/\sqrt{n_1}}{\hat{\xi}_{2\alpha} + Z_{\gamma_2/2}\hat{\sigma}_{2\alpha}/\sqrt{n_2}} \le \frac{\xi_{1\alpha}}{\xi_{2\alpha}} \le \frac{\hat{\xi}_{1\alpha} + Z_{\gamma_1/2}\hat{\sigma}_{1\alpha}\Big/\sqrt{n_1}}{\hat{\xi}_{2\alpha} - Z_{\gamma_2/2}\hat{\sigma}_{2\alpha}\Big/\sqrt{n_2}}\right]$$

when neither shape parameter estimate equals 1. If either or both shape parameter estimates equal 1, then the modified maximum likelihood estimate and confidence interval based on the two-parameter exponential distribution can be used. One problem Johnson and others (2003) noted with this solution is how to choose γ_1 and γ_2. Because any combinations of γ_1 and γ_2 that meet our restriction can be chosen, this procedure can lead to a variety of confidence intervals for the same two sets of data. We could search for the values of γ_1 and γ_2 that produce the "shortest" confidence interval, but we may not achieve the desired coverage with the shortest interval. Also, with the coverage dependent on estimates of $\xi_{1\alpha}$, $\xi_{2\alpha}$, $\sigma_{1\alpha}$, and $\sigma_{2\alpha}$, as well as γ_1 and γ_2, this procedure may not lead to a simple confidence interval for our ratio.

Another approach proposed by Johnson and others (2003) employs maximum likelihood estimators to obtain a large sample approximation to the distribution of $\hat{\xi}_{1\alpha}/\hat{\xi}_{2\alpha}$.

From a Taylor expansion, we see that

$$\frac{\hat{\xi}_{1\alpha}}{\hat{\xi}_{2\alpha}} = \frac{\hat{c}_1 + \hat{b}_1\left[-\ln(1-\alpha)\right]^{1/\hat{\alpha}_1}}{\hat{c}_2 + \hat{b}_2\left[-\ln(1-\alpha)\right]^{1/\hat{\alpha}_2}}$$

$$= \frac{\xi_{1\alpha}}{\xi_{2\alpha}} + \frac{1}{\xi_{2\alpha}}\left(\hat{\xi}_{1\alpha} - \xi_{1\alpha}\right) - \frac{\xi_{1\alpha}}{\xi_{2\alpha}}\left(\hat{\xi}_{2\alpha} - \xi_{2\alpha}\right)$$

+ higher order terms

This suggests the large sample approximation

$$\hat{Var}\left(\frac{\hat{\xi}_{1\alpha}}{\hat{\xi}_{2\alpha}}\right) = \frac{1}{\hat{\xi}^2_{2\alpha}}\hat{Var}\left(\hat{\xi}_{1\alpha}\right) + \frac{\hat{\xi}^2_{2\alpha}}{\hat{\xi}^4_{2\alpha}}\hat{Var}\left(\hat{\xi}_{2\alpha}\right)$$

Then, a large sample approximate $(1-\gamma)$ 100% confidence interval for $\hat{\xi}_{1\alpha}/\hat{\xi}_{2\alpha}$ is given by

$$\frac{\hat{\xi}_{1\alpha}}{\hat{\xi}_{2\alpha}} \pm Z_{\gamma/2}\sqrt{\frac{1}{\hat{\xi}^2_{2\alpha}}\left(\frac{\hat{\sigma}^2_{1\alpha}}{n_1}\right) + \frac{\hat{\xi}^2_{1\alpha}}{\hat{\xi}^4_{2\alpha}}\left(\frac{\hat{\sigma}^2_{2\alpha}}{n_2}\right)}$$

where the estimates of $\sigma^2_{i\alpha}$ come from our earlier estimate

$$\hat{\sigma}^2_\alpha = \left(\frac{\partial\xi_\alpha}{\partial a}, \frac{\partial\xi_\alpha}{\partial b}, \frac{\partial\xi_\alpha}{\partial c}\right)\Bigg|_{(\hat{a},\hat{b},\hat{c})} \left.\begin{pmatrix}\dfrac{\partial\xi_\alpha}{\partial a}\\ \dfrac{\partial\xi_\alpha}{\partial b}\\ \dfrac{\partial\xi_\alpha}{\partial c}\end{pmatrix}\right|_{(\hat{a},\hat{b},\hat{c})}$$

Using a computer simulation, Johnson and others (2003) evaluated this approach for a variety of three-parameter Weibull distributions with parameter values typical of lumber industry applications. A sample size of 100 for each distribution was chosen, and 250 replications of each combination of true parameter values were simulated. Both 90% and 95% confidence intervals were evaluated for ratios of nine percentiles: 2%, 5%, 10%, 30%, 50%, 70%, 90%, 95%, and 98%. For the 95% confidence intervals, coverage for the ratios of 10th through 98th percentiles was generally within 2% of the claimed confidence level, with average coverage very near the claimed confidence level. The 5th percentile ratio confidence intervals had an average coverage of 92.8%. Even the 2nd percentile ratio confidence interval had an average coverage of 91.1%. Results for the 90% confidence intervals were similar. Although the number of replications was small, the intervals are good for sample sizes considerably smaller than Johnson and Haskell (1984) found necessary.

6. Calculation of Volumetric Effects

Motivation for the Research

It is quite appropriate to be discussing volumetric effects last because in a way it brings into full circle FPL's role in the use of the Weibull distribution in wood engineering. Volume effects were needed in the In-Grade Program to estimate properties from samples of combinations of 2×4, 2×6 2×8, 2×10 and possibly 2×12 data into one number that designers of wood structures could use from the design books for wood species by grade of 2×8, 144 in. in length, and 15% moisture content. The theoretical model for calculation of volumetric effects that Bohannan (1966) derived still forms a basis for volumetric effects. Work done during the In-Grade Program has modified the overall volumetric effect into width, length, and thickness portions that maintain the overall effects. The circular nature of this work is also quite interesting. Bohannan (1966) acknowledges in his work two statisticians at FPL "for their assistance in the statistical formation and analysis" in the paper. When the In-Grade Program was begun, Bohannan was the Assistant Director with the responsibility for directing FPL efforts in solving the problem. He placed a Mathematical Statistician as key member of FPL's research team.

Calculation of Volumetric Effects

Testing all combinations of factors that may affect allowable properties for wood members is often not possible. The Weibull distribution has played a major role in handling size differences of wood members. It has long been recognized that as a wood bending member is increased in size, its bending strength apparently decreases. Newlin and Trayer (1924) investigated this in 1924 and related the decrease to the depth of the bending member (see Tucker (1941)). Bohannan (1966) used the statistical strength theory of

Weibull (1939) to develop a theoretical model for strength loss related to increasing size of a wood member. The basis of the theory was that increasing size raises the probability of a region of low strength, and this region of low strength is assumed to cause the failure of the member. Using this "weakest link theory" of Weibull and the resulting two-parameter Weibull distribution, Bohannan derived a model for predicting strength change resulting from a change in depth of the wood member. Specifically, the ratio of the strength of a beam having depth d_1 to a beam having depth d_2 is given by $(d_2/d_1)^{2/a}$, where a is the shape parameter of the Weibull distribution. For simply supported beams tested in bending with a center-point load, Bohannan used test data to estimate $a = 18$. Thus for bending, the size adjustment was $(d_2/d_1)^{1/9}$.

Adjustments for dimensional changes of this form are now found in several ASTM standards related to wood engineering. ASTM D245, which is used to establish properties of visually graded solid sawn structural lumber starting with property values from clear wood, uses the exact model of Bohannan for bending.

ASTM D1990, which is used to establish properties of visually graded dimension lumber from In-Grade tests of full-size lumber, has the same theoretical basis behind adjustments for width, length, and thickness, but it uses different data to develop the Weibull shape parameter estimates. In this standard, property test values F_1 can be adjusted using

$$F_2 = F_1\left(\frac{W_1}{W_2}\right)^w \left(\frac{L_1}{L_2}\right)^l \left(\frac{T_1}{T_2}\right)^t$$

where

F_1 is property value at volume 1, lb/in^2,
F_2 property value at volume 2, lb/in^2,
W_1 width at F_1, in.,
W_2 width at F_2, in.,
L_1 length at F_1, in.,
L_2 length at F_2, in.,
T_1 thickness at F_1, in.,
T_2 thickness at F_2, in.,
w 0.29 for MOR and UTS,
0.13 for UCS parallel to grain,
0 for MOE,
l 0.14 for MOR and UTS,
0 for UCS parallel to grain and MOE, and
t 0 for MOR, UTS, UCS parallel to grain, and MOE.

ASTM D5456 (ASTM 2009), which is used for structural composite lumber intended as an engineering material, uses the same form of a volume adjustment. The multiplication factor for bending is given as

$$K_\mathrm{d} = \left(\frac{d_1}{d_2}\right)^{1/m} \left(\frac{L_1}{L_2}\right)^{1/m}$$

where

K_d is factor applied to design stress of the member of unit volume,
d_1, L_1 are depth and length of unit volume members,
d, L are depth and length of application member, and
m is parameter determined in accordance with annex A1 of the standard.

In ASTM D5456, the multiplication factor for axial tension is

$$K_L = \left(\frac{L_1}{L}\right)^{1/m}$$

where

K_L is adjustment factor,
L_1 base length between grips,
L end-use length, and
m parameter determined in accordance with annex A1 of the standard.

Annex A1 of ASTM D5456 for bending and axial tension offers two ways of estimating m. If single-size (or length in the case of axial tension) testing is done, then

$$m = \mathrm{CV}^{-1.08}$$

where CV is the coefficient of variation of the data. If CV < 0.15, then set $m = 8$. For multiple-size (or length) testing, which requires at least four depths (or lengths), including a base depth (or length), the annex shows how to compute an "empirical" value and a "theoretical" value of m. The theoretical value requires fitting a two-parameter Weibull distribution to a data set generated through testing using either the full data set or the bottom-tail data if 75 or more data points are used and if the points include at least 10% of the distribution. Through references to other standards, it is implied that the Weibull fit should be done using a regression fit to data transformed to make it linear. The annex then shows how to reconcile the two estimates.

Finally, annex A1 of ASTM D5456 shows how to adjust flexure stress based on different types of loading conditions (center-point, third-point, and uniform loading). Adjustment factors given in table A1.1 of the standard are said to be developed from weak-link theory and to account for variations in stress distribution along the length of the member (ASTM 2009).

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