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Probability Distributions as Models for Mortality

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ABSTRACT. The necessary attributes for a mortality model for an even-aged forest stand are stated. The Weibull distribution, the gamma distribution, and the negative binomial distribution are proposed based on their previous use in failure research and as mortality models. A distribution derived from the Richards generalization of the von Bertalanffy growth equation is proposed. The four functions are examined mathematically and empirically using data from a loblolly pine spacing study to determine their usefulness as mortality models. The negative binomial distribution and its continuous analog, the gamma distribution, show instability under right-censoring and are computationally difficult. The Weibull distribution shows extreme instability under right-censoring due to constraints on the location of the inflection points of its probability density function, limiting its value as a mortality model. The distribution derived from the Richards generalization of the von Bertalanffy function is stable under right-censoring, shows no constraints on assumable shapes, and is computationally simple. FOREST SCI. 31:331–341.

ADDITIONAL KEY WORDS. Negative binomial distribution, gamma distribution, Weibull distribution, Richards function.

DECISION PROCESSES in forest management depend on reliable estimates of stand growth and yield. Stand values depend upon the volume of living material, the distribution of stem sizes, and the number of living stems. One of the most important components of a growth and yield prediction system is the mortality, or survival, component. It is imperative that we be able to model accurately the progress of mortality in a forest plantation, and to predict accurately the number of living stems at any point in a stand's life, given its history.

After stand establishment, tree growth is relatively uninhibited until crown closure occurs. Intertree competition for space and light then begins and competition-related mortality commences. Mortality occurs at an increasing rate up to some maximum rate, after which it occurs at a continually decreasing rate (Harms and Langdon 1976). Many of the models developed for loblolly pine mortality are linear or polynomial in nature (Coile and Schumacher 1964, Lenhart 1972, Smalley and Bailey 1974, Feduccia and others 1979). While these functions may predict well over the range of data used in their development, their functional forms dictate behavior outside this range which is inconsistent with the biological behavior of the system.

To be useful as a model for cumulative mortality in an even-aged stand, a function must be monotonically increasing towards an asymptote, have one in-

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flection point, and have a unimodal first derivative or rate curve. A large class of functions that displays these traits is that class of functions referred to as cumulative distribution functions. Using such functions, we develop the following definitions:

$$\text{Mortality}_t{}^{1} = N\cdot F(t) = \begin{cases} N\displaystyle\int_0^t f(x)\,dx & \text{continuous case} \qquad (1a) \\ N\displaystyle\sum_{x=1}^{t} f(x) & \text{discrete case} \qquad (1b) \end{cases}$$

$$\text{Mortality rate}_t = N\cdot f(t) \tag{2}$$

$$\text{Survival}_t = N\cdot[1 - F(t)] \tag{3}$$

where:

N = number of trees living at time of stand establishment
$F(t)$ = cumulative distribution function (cdf)
$f(t)$ = probability density function (pdf).

As a class, cumulative distribution functions display behavior corresponding to that of mortality in plantations. Moreover, they provide a sound mathematical basis for studying the biological process of mortality at the stand level.

Cunia (1974) suggested the negative binomial distribution as a mortality model for a forest attacked yearly by a given insect. Somers and others (1980) used the Weibull distribution to predict mortality in natural loblolly pine stands thinned at age 3 to five residual densities. Hafley and others (1982) used the negative binomial distribution as the mortality model in a yield model for loblolly pine in plantations. The literature shows a limited recognition and use of cumulative distribution functions for mortality models. No investigation has produced a mathematical or empirical framework for evaluating the suitability of different distribution functions for mortality models.

This paper considers the behavior and constraints of the negative binomial distribution, its continuous analog the gamma distribution (Bartko 1961), and the Weibull distribution. In addition, a distribution derived from the Richards (1959) generalization of the von Bertalanffy growth function is proposed and its attributes and behavior as a model for competition-induced mortality are evaluated.

Data

Two sets of data were used in this study. The first set is unpublished from a 48-year-old loblolly pine spacing study on a pocosin site on the Hofmann forest in southeastern North Carolina. The spacing study consists of one quarter-acre plot each of 4 × 4, 6 × 6, and 8 × 8 foot spacings, representing 2,722, 1,210, and 680 stems per acre (spa), respectively. Number of stems surviving was recorded for each plot at ages 3, 10, 13, 16, 20, 24, 28, 32, 37, 44, and 48. Except for the first and last ages, all diameters at breast height (dbh) were recorded. All heights were recorded at ages 37 and 44, and a subsample of heights was taken at the other ages.

The second set of data consists of yearly measurements from age 3 to age 22 on four sets of 0.016-acre plots of naturally seeded loblolly pine on the Santee

[1] For the remainder of this paper, the term mortality refers to cumulative mortality.

Experimental Forest in the lower coastal plain of South Carolina.[2] As described by Harms and Langdon (1976), the plots were thinned at age 3 to 1,000, 2,000, 4,000, 8,000, or 16,000 stems per acre. Number of stems surviving was averaged across the four plots representing each of the five initial densities to give one survival sequence for each of the five densities. Observations made at ages 3, 4, and 5 were excluded because the exceedingly high mortality in those years was considered to be the result of thinning damage rather than competition-induced mortality (Harms 1983). The stems removed in the thinning and those that succumbed to thinning damage were not considered to be part of the established stand. Consequently, the initial densities of the plots at age 6 were taken as 922, 1,844, 3,735, 7,578, and 14,781 stems per acre rather than 1,000, 2,000, 4,000, 8,000, and 16,000 stems per acre, respectively.

Methods and Discussion

Functional Forms

The form of the Weibull cumulative distribution function used was

$$F(x) = 1 - e^{-\left(\frac{x}{b}\right)^c} \qquad x > 0,\ b,\ c > 0.$$

The form of the gamma probability density function used was

$$f(x) = \frac{1}{\beta^\alpha \Gamma(\alpha)} x^{\alpha-1} e^{-\left(\frac{x}{\beta}\right)} \qquad x > 0,\ \alpha,\ \beta > 0,$$

since there is no closed form of the cumulative gamma distribution. The form of the negative binomial cumulative distribution function used was

$$F(x) = \sum_{x=1}^{n} \frac{\Gamma(k + x)}{x!\ \Gamma(k)} \left(1 + \frac{m}{k}\right)^{-k} \left(\frac{m}{m + k}\right)^{x} \qquad x = 1, 2, 3, \ldots \quad m, k > 0.$$

A commonly used form of the Richards generalization of the von Bertalanffy growth equation (Pienaar and Turnbull 1973) is

$$Y = A\,(1 - e^{-bt})^c \qquad t > 0, \qquad A, b, c > 0.$$

It can be shown that after dividing both sides by A, we are left with a cumulative distribution function of the form used in this study:

$$F(x) = (1 - e^{-bx})^c \qquad x > 0, \qquad b, c > 0.$$

Fits of the Distributions to the Data

The Weibull, gamma, Richards, and the negative binomial functions were fit to the Hofmann and South Carolina plot data using dominant height as a predictor variable. Adams and Chapman (1942), Beekhuis (1966), and Assmann (1970) discussed the relationship between dominant height and stand development, including the development of surviving stems. Dominant height reflects the relationship of age and site quality in a single variable, and if we recall that mortality is related to crown closure and competition (both functions of density, age, and site quality) then dominant height is a logical predictor variable.

[2] Permission to use the data from age 15 to age 22, not previously published, was given by Dr. William R. Harms of the Southeastern Forest Experiment Station, USDA Forest Service.

TABLE 1. Parameters of the four fitted distributions for cumulative mortality.[a]

Plot	Distribution			
	Weibull	Richards	Gamma	Negative binomial
Hofmann				
680 spa	$b =$ 83.4392	$b =$ 0.03849	$b =$ 7.0367	$m =$ 76.4201
	$c =$ 3.1809	$c =$ 11.1656	$c =$ 10.9498	$k =$ 7.7127
	$\chi^2 =$ 31.3795	$\chi^2 =$ 33.4898	$\chi^2 =$ 31.6156	$\chi^2 =$ 31.62
1,210 spa	$b =$ 69.9278	$b =$ 0.04477	$b =$ 6.04178	$m =$ 62.9768
	$c =$ 2.7505	$c =$ 9.6032	$c =$ 10.5204	$k =$ 6.5942
	$\chi^2 =$ 154.216	$\chi^2 =$ 129.992	$\chi^2 =$ 140.81	$\chi^2 =$ 139.695
2,722 spa	$b =$ 52.7265	$b =$ 0.05211	$b =$ 10.9017	$m =$ 47.0091
	$c =$ 2.1891	$c =$ 6.3786	$c =$ 4.3643	$k =$ 4.6813
	$\chi^2 =$ 175.878	$\chi^2 =$ 139.464	$\chi^2 =$ 148.195	$\chi^2 =$ 150.02
South Carolina				
922 spa	$b =$ 80.1330	$b =$ 0.03412	$b =$ 12.9512	$m =$ 75.3587
	$c =$ 3.1267	$c =$ 7.8067	$c =$ 5.8855	$k =$ 6.5169
	$\chi^2 =$ 32.044	$\chi^2 =$ 28.254	$\chi^2 =$ 29.1413	$\chi^2 =$ 29.4033
1,844 spa	$b =$ 69.6267	$b =$ 0.04672	$b =$ 8.1132	$m =$ 65.0576
	$c =$ 3.6432	$c =$ 12.8796	$c =$ 8.1007	$k =$ 9.2794
	$\chi^2 =$ 10.14975	$\chi^2 =$ 8.1531	$\chi^2 =$ 8.5439	$\chi^2 =$ 8.4819
3,735 spa	$b =$ 55.6964	$b =$ 0.05851	$b =$ 7.3403	$m =$ 50.7950
	$c =$ 3.0683	$c =$ 11.5239	$c =$ 6.9844	$k =$ 8.0500
	$\chi^2 =$ 24.4667	$\chi^2 =$ 23.0119	$\chi^2 =$ 22.7554	$\chi^2 =$ 22.9034
7,578 spa	$b =$ 46.5718	$b =$ 0.07042	$b =$ 6.5952	$m =$ 41.8857
	$c =$ 2.7828	$c =$ 11.11548	$c =$ 6.4252	$k =$ 7.3056
	$\chi^2 =$ 16.2298	$\chi^2 =$ 12.4327	$\chi^2 =$ 13.2549	$\chi^2 =$ 13.215
14,781 spa	$b =$ 35.6098	$b =$ 0.08634	$b =$ 6.0106	$m =$ 31.6196
	$c =$ 2.4562	$c =$ 8.8367	$c =$ 5.3557	$k =$ 6.1749
	$\chi^2 =$ 19.0818	$\chi^2 =$ 13.4463	$\chi^2 =$ 15.2060	$\chi^2 =$ 15.1823

[a] The χ^2 values are from the fits to the plot data rather than from the plot data transformed to a per-acre basis.

In this paper, stand establishment is taken as that point, after the time of early non-competition-induced mortality, when the trees begin to develop and capture the site, rather than time of planting. The trees remaining at this point comprise the established stand. These trees are the established stand and developed from time of planting and it is appropriate to model the mortality of the established stand starting at the origin (height = 0).

Permanent plot data, with observations made at intervals, like those of this study are not composed of spatially or temporally independent ungrouped random samples. The usual types of point estimators such as moment estimators or maximum likelihood estimators require an ungrouped random sample of independently and identically distributed observations (Hogg and Craig 1978). One of the goals of this study was to compare the behavior of the functions under various stages of right-censoring. Therefore, it was necessary to use the same type of estimation procedure for the parameters of all of the functions to avoid confounding behavior with type of estimator. The same type of estimator does not exist for the parameters of all the functions, particularly the Richards function. For these reasons, a fitting procedure which minimized the sum of squared deviations between the observed and predicted cumulative number of trees dead was used.

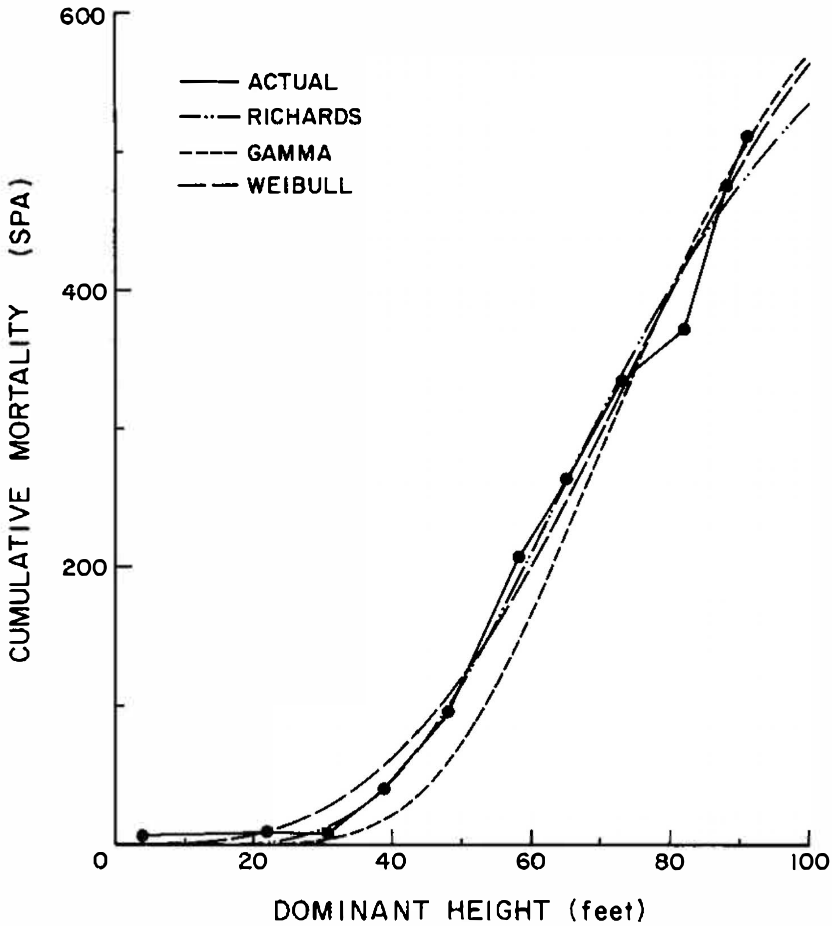

FIGURE 1. Actual and predicted cumulative mortality for the Hofmann 680 spa plot.

The particular procedure used was the nonlinear fitting procedure of Release SAS79.6 of the Statistical Analysis System computing package (SAS Institute 1979). Examination of the residuals from the fits using all the data as recommended by Draper and Smith (1981) showed no trend that would lead to rejection of the fitting procedure.

Table 1 gives the parameter values and the χ^2 statistic on a per plot basis for each of the fits. The values of the χ^2 statistic are presented as a relative measure of goodness of fit, with a smaller χ^2 indicating better fit to the given data. A χ^2 test of goodness of fit would not be correct here as the data are not a random sample of independently and identically distributed lifetimes, but result from repeated observations of the same plots (Elandt-Johnson and Johnson 1980). Table 1 shows that the Richards function gives the smallest χ^2 value in six of the eight instances, although there is little difference among the χ^2 values for a given plot. The large χ^2 values for the 1,210 spa and 2,722 spa Hofmann plots arise from one cell in each containing catastrophic mortality resulting from Hurricane Hazel in 1954. The 680 spa plot appeared essentially unaffected by the hurricane.

Figures 1 and 2 show the fitted cumulative mortality and the observed mortality for two of the plots and are representative of the results of the fitting for all of the plots. As was expected, the gamma distribution and the negative binomial

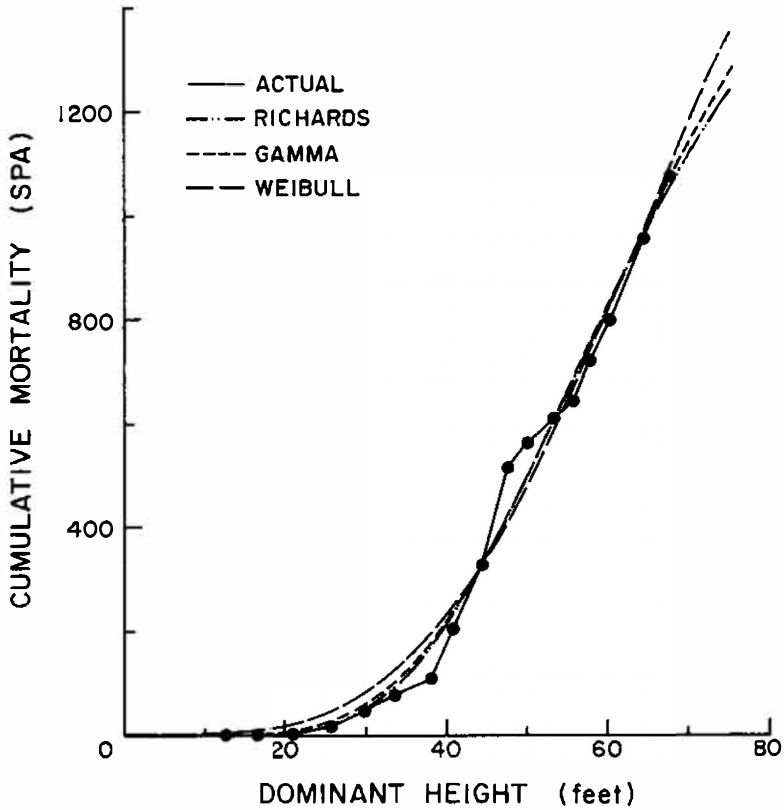

FIGURE 2. Actual and predicted cumulative mortality for the South Carolina 1,844 spa plot.

distribution gave essentially the same fit to every set of data. Their behavior proved to be identical during all phases of this study, so the plotted gamma represents both itself and the negative binomial. In Figures 1 and 2, the gamma and the Richards curves lie very close to each other and parallel the data very well. The Weibull consistently shows too shallow a curvature at the early heights, with predicted mortality consistently greater than that observed. In Table 1, the χ^2 statistic for the Weibull is consistently larger than for the other distributions: a result of the overprediction at the early heights.

The inflection points of the probability density functions were examined for constraints on their behavior since their placement controls the shapes that the distributions may assume. Table 2 summarizes the distribution functions and the functions defining the inflection points are quadratic functions of two of the parameters. This is a constraint in that, given one inflection point, the other is implicitly identified. However, only with the Weibull is this a problem. The right inflection point of the Weibull attains a maximum at $x = b \cdot (1.2454)$; i.e., when $c = 2.4562$ (corrected from Lehman 1963).[3] This implies that for any value of c, there is some maximum that the right inflection point can attain. Table 3 gives the maximum and actual right inflection points for the fitted Weibull distributions

[3] The value for c that maximizes the right inflection point is 2.4625 in Lehman's 1963 paper. However, work on this study revealed a probable transcription error in Lehman's paper. The actual value of c that maximizes the right inflection point is 2.4562. In Lehman's paper, the maximum occurs at $x = b \cdot (1.2451)$. Using the correct c, the actual maximum occurs at $x = b \cdot (1.2454)$.

TABLE 2. Inflection points of the Weibull, gamma, and Richards probability density functions.

Distribution functions	Inflection points of pdf
Weibull	
$F(x) = 1 - e^{-\left(\frac{x}{b}\right)^c}$	$x = b\left[\frac{3(c-1) \pm \sqrt{(5c-1)(c-1)}}{2c}\right]^{1/c}$
Gamma	
$F(x) = \int_0^x \frac{1}{b^c\Gamma(c)} t^{c-1} e^{-\left(\frac{t}{b}\right)} dt$	$x = b[(c-1) \pm \sqrt{c-1}]$
Richards	
$F(x) = (1 - e^{-bx})^c$	$x = \frac{1}{b}\log\left[\frac{(3c-1) \pm \sqrt{(5c-1)(c-1)}}{2}\right]$

in Figures 1 and 2. The inflection point of the Weibull is almost at the maximum for 7 out of the 8 plots and is at the maximum on the eighth plot, implying that the location constraint on the right inflection point of the Weibull pdf is a real constraint in fitting the mortality data presented here. Examination of the functions defining the inflection points of the gamma and Richards pdfs showed no

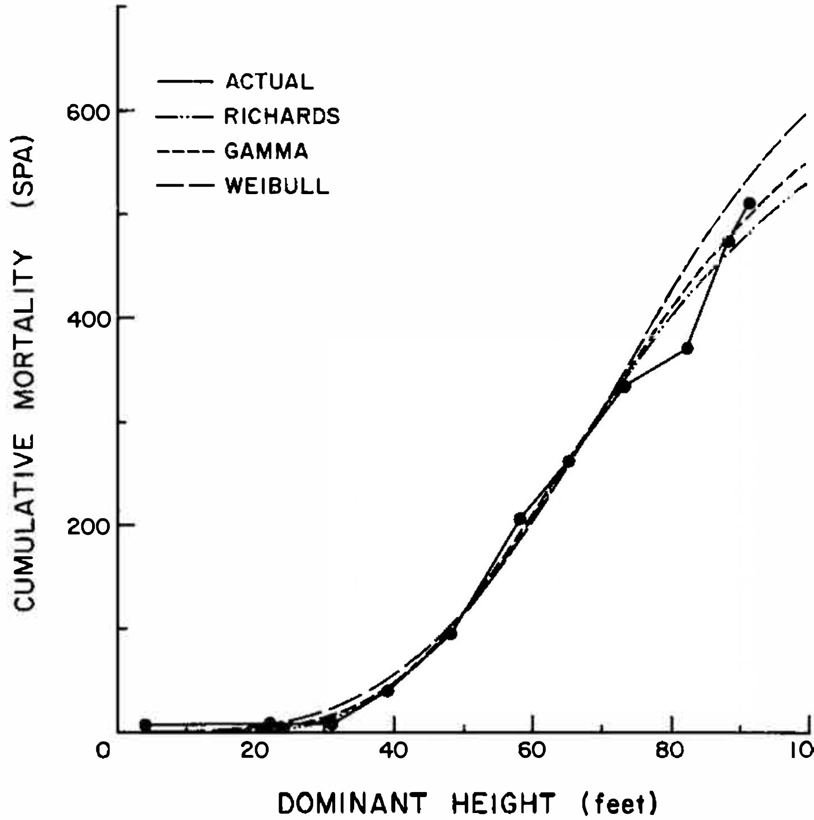

FIGURE 3. Actual and predicted cumulative mortality of Hofmann 680 spa plot using parameters of distributions fit to the eight observations made before age 35.

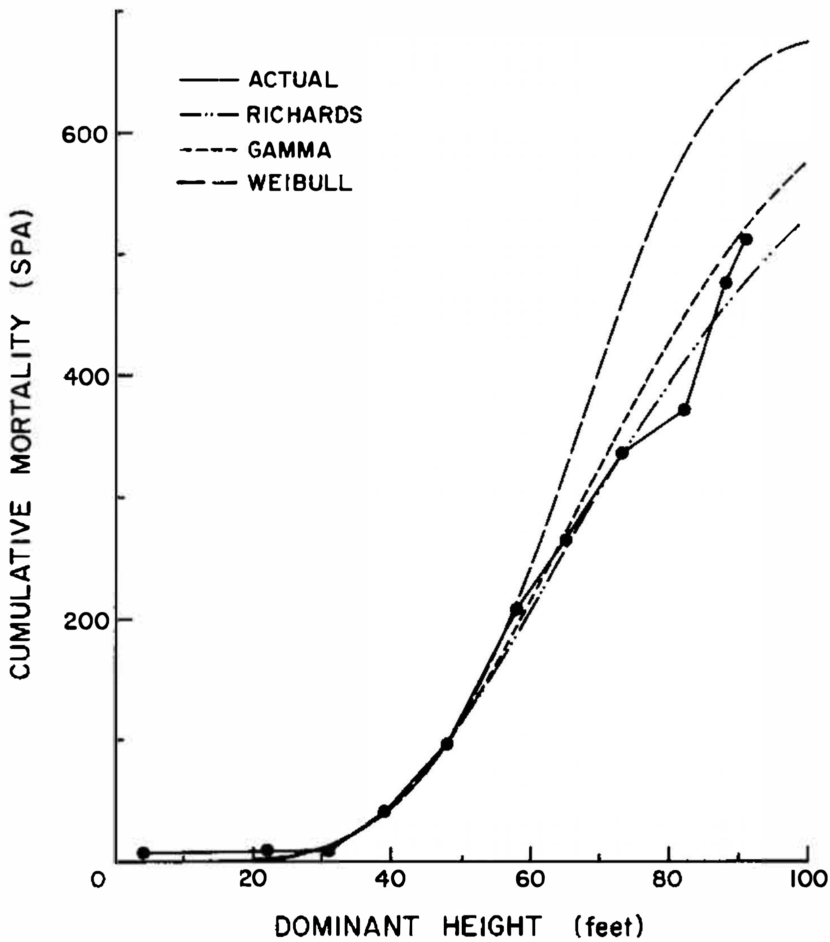

FIGURE 4. Actual and predicted cumulative mortality on Hofmann 680 spa plot using parameters of distributions fit to the five observations made before age 22.

such constraints, thus allowing more flexibility in their shapes than the shape of the Weibull.

EFFECT OF CENSORED DATA ON BEHAVIOR OF THE FUNCTIONS

In forest stand mortality problems, we often deal with both right- and left-censored data. That is, we may not have observations before a given age or after a given age, but we are interested in mortality during those periods. For plantations, left-censoring is not usually a serious problem since we may have a reliable estimate of the number of stems alive at stand establishment. Right-censoring is usually a serious problem since we often want to predict future stand development from a few early observations. We would like a model that gives stable and consistent predictions under varying degrees of right-censoring. For example, it is desirable that predictions based on observations made to age 15 be accurate and also consistent with predictions made after inclusion of later observations. Given this, it was necessary to examine the stability of the four distributions under right-censoring. Stability here refers to the ability of a function to predict mortality

TABLE 3. Maximum and actual right inflection points for the fitted Weibull distributions.

Plot		Maximum	Actual (fitted)
Hofmann	680	103.9159	102.5088
	1,210	87.0887	86.8244
	2,722	65.6660	65.3589
South Carolina	1,000	99.7983	98.5972
	2,000	86.7137	84.3800
	4,000	69.3647	68.6398
	8,000	58.0009	57.7890
	16,000	44.3487	44.3487

accurately over the life of the stand given that only a small number of early observations exist. This characteristic was demonstrated using the Hofmann data.

The Weibull, gamma, negative binomial, and Richards functions were fitted to the Hofmann data after removing the right-most observation, simulating differing degrees of right-censoring. The process of sequentially removing the right-most remaining observation and refitting the data was continued until only four ob-

TABLE 4. Predicted survival (spa) at 80 feet from the Weibull, gamma, and Richards functions fit to the sequentially right-censored Hofmann plot data.

	Predicted survival (spa)		
Age of censoring	Weibull	Gamma	Richards
Density: 2,722 spa[a]			
18	16	120	200
22	96	196	260
25	76	196	208
30	136	192	240
35	160	200	256
40	220	240	272
45	220	240	260
Density: 1,210 spa[b]			
18	644	684	716
22	348	440	488
25	124	224	272
30	164	248	272
35	228	264	300
40	288	300	308
45	288	288	288
Density: 680 spa[c]			
18	8	172	244
22	112	248	280
25	88	200	232
30	208	244	272
35	248	264	280
40	296	296	296
45	280	280	280

[a] Observed survival was approximately 380 spa.
[b] Observed survival was approximately 348 spa.
[c] Observed survival was approximately 290 spa.

TABLE 5. χ^2 values on a plot basis for the Weibull, gamma, and Richards fits to the sequentially right-censored Hofmann plot data.

Age of censoring	χ^2		
	Weibull	Richards	Gamma
Density: 2,722 spa			
18	3,003.250	174.409	257.683
22	304.415	144.344	165.829
25	344.138	162.702	185.193
30	227.370	149.663	167.353
35	196.127	144.684	158.590
40	184.771	144.043	154.868
45	183.439	145.379	177.066
Density: 1,210 spa			
18	131.567	128.270	130.484
22	220.768	162.850	177.479
25	326.437	252.296	271.465
30	299.902	253.324	270.654
35	269.900	240.832	253.056
40	244.673	233.280	237.070
45	247.028	241.139	242.165
Density: 680 spa			
18	61.7909	31.6432	38.439
22	53.3812	31.0304	34.3952
25	52.3757	31.9309	36.476
30	44.0864	30.9831	34.0449
35	40.9172	30.9156	33.4596
40	40.3725	31.8243	34.1212
45	40.1448	31.0762	33.5624

servations remained on each of the plots. The parameters from these fits were then used to predict the cumulative mortality on the plots over the entire range of the data. Figures 3 and 4 show some results for the Hofmann 680 spa plot. The cumulative gamma and the cumulative negative binomial are equivalent here also. The behavior on the other plots paralleled that presented in Figures 3 and 4. The Richards function is quite stable under right-censoring, the gamma distribution is somewhat less stable, and the Weibull is quite unstable.

Table 4 shows the predicted survival in stems per acre at 80 feet for the Hofmann plots from the Weibull, gamma, and Richards functions under varied right-censoring. A height of 80 feet was chosen as it corresponds to age 35, or approximately rotation age. From Table 4, the Richards curve gives the most stable and accurate prediction in all cases. Table 5 gives the χ^2 values for the fits corresponding to the predictions given in Table 4. Using χ^2 solely as a ranking tool, the Richards function consistently has the lowest value, indicating the best prediction of future behavior from limited early observations. Although none of the functions give exact prediction of survival, it is important to note the relative stability of the estimates in each case. The predicted survival at 80 feet is more stable from the Richards function than from the other functions.

Conclusions

Cumulative probability distributions are useful models for mortality in even-aged loblolly pine stands. They display behavior appropriate to describe the development of stem numbers over time. The Weibull distribution, gamma distribution, negative binomial distribution, and a distribution derived from the Richards function were shown to give comparably good fits to cumulative mortality data from loblolly pine spacing studies in North Carolina and South Carolina.

The Weibull distribution has constraints on the right inflection point of its probability density function that significantly impact prediction of mortality, particularly in the case of right-censored data common in forestry. The gamma, negative binomial, and Richards functions have no such restrictions upon shapes they may assume. As a result, all three functions consistently gave a better fit than the Weibull to the data used. The Richards function consistently gave the best fit when ranked by the χ^2 statistic, and was shown to be more stable than the other functions under sequential right-censoring, making it a useful model for predicting mortality in loblolly pine plantations.

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