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A Bivariate Model for Growth and Yield Prediction

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ABSTRACT. An application of Johnson's S_{BB} distribution to growth and yield prediction is presented. The mathematical techniques of describing stand structure and projecting yield for thinned and unthinned stands of loblolly pine in plantations are detailed. In addition, the assumptions employed in applying the techniques are discussed. The advantage of the bivariate approach to modeling thinning decisions is that height enters the removal decisions for low thinning which is more reflective of the decision process involved in an operational thinning. FOREST SCI. 31: 237–247.

ADDITIONAL KEY WORDS. Johnson's S_{BB} distribution, thinning.

IN A RECENT PUBLICATION, Hafley and others (1982) presented yield tables for loblolly pine (*Pinus taeda* L.) in unthinned plantations. That publication contained a summary of the validation efforts for their model of unthinned plantations. The model introduced in that publication has since been expanded to incorporate thinning operations. The validity of the expanded model has been discussed by Smith and Hafley.[1] The purpose of this paper is to present the mathematical techniques used to model the structure and dynamics of an unthinned stand and the application of a thinning logic to that stand model by rigorous mathematical treatment of the distribution describing the thinned stand. Readers interested in a detailed discussion of the validation exercises with the thinned and unthinned models and the data used in model development are referred to Hafley and others (1982) and Smith and Hafley.[1]

The plantation is envisioned as a bivariate population of diameters at breast height, *D*, and total heights, *H*. Characteristics of that population are estimated over time. Utilizing the suggestion of Schreuder and Hafley (1977), Johnson's S_{BB} distribution (Johnson 1949b) is employed to generate the bivariate distribution of diameters and heights for stand description.

Necessary input to the model is site index (base age 25), number of stems per acre planted, fraction of stems surviving one year after planting, desired thinning type, and ages at which reports are to be produced. The model is designed so that any site index, or height over age, equation can easily be substituted for the existing equation.

[1] Smith, W. D., and W. L. Hafley. 1986. Evaluation of a loblolly pine thinning model. (Accepted for publication by South J Appl For.)

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It is intuitively practical to describe the impact of management activities by describing changes in stand characteristics rather than by describing changes in the parameters of the mathematical model. Therefore, we identified those stand characteristics that are most appropriate for estimating parameters of the S_{BB} distribution and developed procedures for projecting those characteristics over the life of a plantation.

In the following sections we will identify the parameters of the S_{BB} distribution and the stand characteristics used to estimate them; the equations used to predict those stand characteristics and their coefficients; the assumptions and mathematics employed to describe the unthinned stand; the impact of thinning operations on the stand; and, finally, the assumptions made to project the stand forward in time after thinning.[2]

Distribution Specifics

Hafley and Schreuder (1977) demonstrated the applicability of Johnson's univariate S_B distribution for describing marginal distributions of *D* and *H*, and Schreuder and Hafley (1977) demonstrated the use of its bivariate analog, the S_{BB}, to describe the joint distribution of *D* and *H*.

If we suppose that the unit variables

$$z_D = \gamma_D + \delta_D \log_e\left(\frac{x_D - \xi_D}{\lambda_D + \xi_D - x_D}\right)$$

and

$$z_H = \gamma_H + \delta_H \log_e\left(\frac{x_H - \xi_H}{\lambda_H + \xi_H - x_H}\right)$$

have the joint bivariate normal distribution

$$f(z_D, z_H) = [2\pi(1 - \rho^2)^{1/2}]^{-1} \exp[-1/2(1 - \rho^2)^{-1} (z_D^2 - 2\rho z_D z_H + z_H^2)], \quad (1)$$

the joint distribution of x_D and x_H is determined. This means that there are nine parameters necessary for describing the joint distribution of *D* and *H*, viz., γ_D, γ_H, δ_D, δ_H, ξ_D, ξ_H, λ_D, λ_H, and ρ(the correlation between z_D and z_H).

Now let us consider these parameters in terms of the variables x_i, $i = D$ or H. The parameter ξ_i represents the smallest value of x_i in the population and the parameter λ_i represents the range of the x_i at any point in stand development. The parameters γ and δ (we shall drop the subscript unless required for clarity) are not so easily related to stand characteristics. Johnson (1949a) demonstrates that the mode of the S_B distribution occurs at the value of x which satisfies the relationship

$$\frac{2(x - \xi)}{\lambda} - 1 = \delta\left[\gamma + \delta \log_e\left(\frac{x - \xi}{\lambda + \xi - x}\right)\right] \quad (2)$$

except at $x = \xi$ or $x = \xi + \lambda$. For certain values of γ and δ the distribution is bimodal. Fortunately, these are not in the range of the parameters appropriate to describing loblolly pine stands.

Johnson (1949a) presents the maximum likelihood estimator of δ as

[2] The authors gratefully acknowledge the help of Mr. W. D. Smith in developing the thinning logic and assumptions regarding stand growth after thinning.

$$\hat{\delta} = \frac{1}{s_f},$$

where

$$s_f = \left(\frac{\Sigma(f_j - \bar{f})^2}{n}\right)^{1/2} \text{ and } f_j = \log_e\left(\frac{x_j - \xi}{\xi + \lambda - x_j}\right), j = 1, \ldots, n.$$

For the general case, taking a Taylor's series expansion of f about the point $a = 1/2$ and eliminating terms of third order and higher yields the approximation

$$f \simeq 4\left(\frac{x - \xi}{\lambda}\right) - 2,$$

the variance of which is

$$V(f) \simeq \frac{16\sigma_x^2}{\lambda^2},$$

giving the approximation used in the model

$$\delta \simeq \frac{\lambda}{4\sigma_x}. \tag{3}$$

Utilizing (3) and substituting the result into (2) we estimate γ from

$$\gamma = \frac{2x_m - \xi - \lambda}{\lambda\delta} - \delta \log_e\left(\frac{x_m - \xi}{\lambda + \xi - x_m}\right) \tag{4}$$

where x_m is the mode of x.

Thus, the problem is reduced to one of estimating the largest and smallest possible value of x, the modal value of x, and the standard deviation of x associated with those points in time for which a stand description is desired. The correlation between z_D and z_H will be discussed in the next section.

Stand Characteristics

Having identified the stand characteristics of interest it is necessary to obtain estimates of those characteristics. The data available to the authors for which adequate, long-term observations of both diameter and height of all trees existed consisted of data from a 44-year-old spacing study belonging to the School of Forest Resources, North Carolina State University, measurements, at age fifteen, of a spacing study on the Sumter National Forest in South Carolina (Harms and Lloyd 1981); and a limited number of measurements from a 30-year-old spacing study of the School of Forest Resources, Louisiana State University (Sprinz and others 1979). A detailed description of these data are given in Hafley and others (1982).

To obtain prediction equations for the stand characteristics of interest, the S_{BB} distribution was fitted to all sets of observations available from the studies identified above. The characteristics to be estimated were then extracted from each of those fits to form a data set from which the prediction equations were developed.

The equations developed for the nine characteristics necessary for estimating the parameters of the S_{BB} distribution are presented in Table 1. The coefficients of these equations were obtained from nonlinear least squares.

The independent variables used in the prediction equations are age, dominant height, and a measure of average initial growing space, S. The average initial

TABLE 1a. Equation forms used to estimate the nine stand characteristics employed by the yield prediction model.

1. $SI_a = a_1(b_1 + c_1 SI)$
2. $H_m = H_u(a_2 + b_2 A)$
3. $H_1 = H_u a_3[1 - \exp(-b_3 A)]$
4. $s_H = a_4[1 - \exp(-b_4 A)]^{c_4}$
5. $D_u = a_5[1 - \exp(-b_5 H_D)]^{c_5}$
6. $D_m = D_u a_6[1 - \exp(-b_6 H_D)]$
7. $D_1 = D_u a_7[1 - \exp(-b_7 H_D)]^{c_7}$
8. $s_D = a_8[1 - \exp(1 - b_8 H_D)]^{c_8}$
9. $\rho(z_D, z_H) = a_9 + b_9 \cos(c_9 H_D) + d_9 H_D$

where

H_u = the height obtained from the chosen height over age curve as a function of SI_a and age,
D_u = largest diameter,
H_m and D_m = the modal height and diameter, respectively,
H_1 and D_1 = the smallest height and diameter, respectively,
s_H and s_D = the standard deviation of height and diameter, respectively,
$\rho(z_D, z_H)$ = the correlation between z_D and z_H, $-1 \le \rho \le 1$,
H_D = dominant height,
SI = site index base age 25,
SI_a = adjusted site index,
A = stand age.

TABLE 1b. Coefficients for equations of Table 1a.

	a_i	b_i	c_i
1.	$1 - \exp(-0.693 \times S)$	-1.1	1.12
2.	$0.86165 - 2.2805/S^2$	$0.000973 + 0.01718/S^2$	
3.	$0.584 + 0.004 \times S$	0.0385	
4.	$23.164 - 0.59367 \times S - 0.0106 \times S^2$	$0.01198 + 0.0002435 \times S$	0.88
5.	$5.025 + 2.1456 \times S - 0.0398 \times S^2$	$0.0245 - 0.00205 \times S + 0.00007856 \times S^2$	$1.1615 - 0.04506 \times S + 0.00241 \times S^2$
6.	$0.8439 - 1.063/S$	$0.0467 + 0.31867/S$	
7.	$0.08 - 1.715/S$	$0.0226 + 0.0563/S$	$5.568 - 15.96/S$
8.	2.03	$0.02396 + 0.000487 \times S$	$2.164 - 0.05138 \times S$
9.	$0.91 \exp(-0.01061 \times S)$	$0.0725 \times S^{0.41587}$	$0.0445 + 0.0803/S$

$d_9 = 0.000708 + 0.001278/S$
where $S = (43{,}560/\text{initial surviving trees per acre})^{1/2}$

growing space is the square root of the ratio of the square foot area of an acre to the surviving number of stems per acre after planting, N_0; that is

$$S = (43{,}560/N_0)^{1/2}.$$

Using dominant height as a predictor variable for the diameter characteristics accounts for the site-age effect.

Within the model dominant height is defined as the average height of the tallest 100 trees per acre. If there are fewer than 100 trees per acre surviving, dominant height is equal to average height. The predicted maximum height, $\lambda_H + \xi_H$, is obtained by entering the height over age curve, selected by the user, with an adjusted input site index, SI_a. The use of SI_a results in a dominant height estimate at index age that is equal to the input site index, SI.

The prediction of the correlation between z_D and z_H is based on the following interpretation of the available data. Making the assumption that $\rho(z_D,z_H)$ is essentially one at stand establishment, the correlation declines as the stems grow and differential rates of height and diameter growth are induced by genetic and microsite differences. As crown closure begins, and competition for light and growing space increases, the correlation declines at a faster rate, generally paralleling in time the increase in mortality rate. Recalling that the correlation is related to the ratio of the axes of the ellipse defined by the scatter diagram of height-diameter pairs, and that mortality due to competition essentially occurs at one end of the ellipse, the scatter of remaining heights and diameters appears more circular and the correlation decreases. As the mortality rate decreases, the differential height and diameter growth rates result in a more pronounced elliptical shape of the scatter of height diameter pairs, and the correlation begins to increase.

MORTALITY

An important component of this yield model is the mortality function. To be useful as a model for cumulative mortality in an even-aged forest stand, a function must be monotonically increasing towards an asymptote, have one inflection point, and have a unimodal mortality rate curve. This type of behavior has been noted in loblolly pine stands by Harms and Langdon (1976). A class of functions which display these traits is that of cumulative distribution functions. Using the concept of cumulative distribution functions, we develop the following definitions:

$$\text{Cumulative mortality}_t = N_0F(t) \begin{cases} N_0 \int_0^t f(x)\,dx & \text{continuous case} \\ N_0 \sum_{x=1}^{t} f(x) & \text{discrete case} \end{cases}$$

$$\text{Instantaneous mortality rate}_t = N_0 f(t)$$

$$\text{Survival} = N_0[1 - F(t)]$$

where

N_0 = number of living trees following stand establishment,
$F(t)$ = cumulative distribution function (cdf),
$f(t)$ = probability density function (pdf).

This concept is illustrated in Figure 1.

A discrete distribution, the negative binomial, was chosen to model the continuous process of mortality because it was highly flexible and its two parameters, m and K, have direct biological meaning. The negative binomial probability density function can be written:

$$p(x) = \frac{\Gamma(K + x)}{x\Gamma(K)} \left(\frac{m}{m + K}\right)^x \left(\frac{K}{m + K}\right)^K.$$

K controls the mortality rate and m is the expected value of x, in this case, height. The parameters were estimated from the available data by the usual moment estimators (Johnson and Kotz 1969):

$$m = \bar{x}$$

and

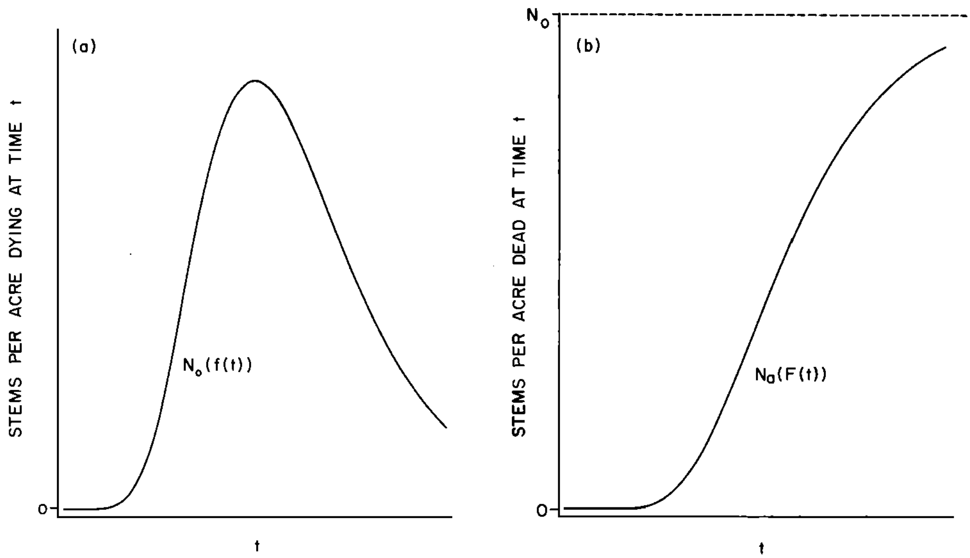

FIGURE 1. Instantaneous mortality rate (a) and mortality (b) from hypothetical loblolly pine plantation.

$$K = \frac{m^2}{s_x^2 - m}$$

where s_x^2 is the usual sample variance.

Equations and coefficients for predicting m and s_x under varied stand conditions were obtained from the data sets previously described. The equations are

$$m = 33.8N_0^{-(0.02359+0.004745\mathrm{SI})}\exp(0.04582\mathrm{SI})$$

and

$$s_x = -5.3 + 0.3\mathrm{SI} + S(0.3992 \times 10^{-6}\mathrm{SI}^{3.615})$$

where N_0 and S are as previously defined.

Using the algorithm given by Bliss and Fisher (1953), the number of trees dying in any growth interval is determined as a function of height. Height is rounded to the nearest foot to obtain the predicted cumulative number dead from the discrete negative binomial distribution. The number of surviving trees is obtained by subtraction.

ESTIMATING YIELD

The yield of the stand at any desired point in time is determined from the S_{BB} distribution whose parameters are defined by the stand characteristics, and the number of surviving trees per acre at that point. The quadratic mean diameter, basal area, average height, dominant height, and cubic foot volume are all obtained from moments of the respective marginal S_B distributions and the S_{BB} distribution using Gauss quadrature, a numerical integration technique. Also, integrating the bivariate distribution over the two-way cell boundaries gives two-way stand tables of frequency by diameter and height class.

Since the variable $y = (x - \xi)/\lambda$ follows the S_B distribution, the basal area per tree is obtained from

$$E[BA] = E[K_D^2] = KE[\lambda_D y_D + \xi_D)^2],$$

giving

$$BA = K(\lambda_D^2\mu_{2D}' + 2\xi_D\lambda_D\mu_{1D}' + \xi_D^2) \tag{9}$$

where $E[\cdot]$ is the expectation operator, μ_{1D}' and μ_{2D}' are the first and second noncentral moments of the marginal S_B distribution for diameter, and $K = 0.005454$. Volume per tree is based on a combined variable equation and the average cubic foot volume per tree is obtained from

$$E[V] = a + bE[D^2H] = a + bE[(\lambda_D y_D + \xi_D)(\lambda_D y_D + \xi_D)(\lambda_H y_H + \xi_H)],$$

giving

$$V = a + b(\lambda_D^2\lambda_H\mu_{2D1H}' + 2\lambda_D\lambda_H\xi_D\mu_{1D1H}' + \lambda_H\xi_D^2\mu_{1H}' + \lambda_D^2\xi_H\mu_{2D}' + 2\lambda_D\xi_D\xi_H\mu_{1D}' + \xi_D^2\xi_H) \tag{10}$$

where μ_{1D}' and μ_{2D}' are as above, μ_{1H}' is the first noncentral moment of the marginal distribution of height and μ_{1D1H}' and μ_{2D1H}' are joint noncentral moments of the S_{BB} distribution.

THINNING

One of the most sought after tools for the forest manager is a growth and yield model for estimating the impact of alternative management strategies. The fundamental goal of this modeling activity was to develop such a model. Thinning from below and row thinning, both individually and in combination, have been incorporated into the model at this writing. There are two aspects to introducing thinning into a growth and yield model, viz., simulation of the thinning itself and projection of growth following the thinning. This section will deal with the thinning simulation, or more appropriately the recovery of the stand conditions after thinning. The following section will deal with projection.

Part of the rationale for developing the bivariate stand description came from the belief that height information would be important for simulating thinning. Approaches to simulating thinning from below using diameter distributions have followed the procedure of truncating the diameter distribution, or of progressively reducing the proportion of trees removed in successive diameter classes (Cao and others 1982, Matney and Sullivan 1982). However, thinning from below means removing the shortest trees which are not always the trees with the smallest diameters. Hence, height is an important component of the thinning decision process.

To simulate thinning from below, the bivariate S_{BB} distribution is doubly truncated from below. The approach adopted for determining the truncation point is to search for the coordinates of the truncation point along the line in the S_{BB} variable space defined by the relationship

$$\frac{x_D - \xi_D}{\lambda_D} = \frac{x_H - \xi_H}{\lambda_H}; \text{ i.e., } y_D = y_H,$$

starting at the coordinate (ξ_D, ξ_H), and integrating at each iteration to determine the residual basal area or number of stems until a truncation coordinate giving the desired residual basal area or residual number of stems is found. This decision criterion results in the following: when the correlation between z_D and z_H is high, thinning decisions are primarily made on the basis of diameter; when this correlation is low, height plays a bigger role in the decision process.

The resulting doubly truncated S_{BB} distribution defines the residual stand. The quadratic mean diameter, average height, basal area, and volumes of the residual stand are obtained using the moments of the truncated S_{BB} distribution in equations (9) and (10). Following Rosenbaum (1961), the moments of the doubly truncated bivariate normal distribution implied by the doubly truncated S_{BB} are used to determine the correlation between z_D and z_H and the standard deviations of diameter and height. The correlation is obtained from

$$\text{RHO} = \frac{\mu_{1D1H}{}^t}{\left(\mu_{2D}{}^t \mu_{2H}{}^t\right)^{1/2}}, \tag{11}$$

where $\mu_{1D1H}{}^t$, $\mu_{2D}{}^t$ and $\mu_{2H}{}^t$ are the joint first moment about the mean of the doubly truncated bivariate normal distribution, and the second moments about the mean of the marginal distributions of the doubly truncated bivariate normal distribution, respectively. The standard deviations s_D and s_H are obtained from

$$s_i = \frac{4\left(\mu_{2i}{}^t\right)^{1/2}}{\lambda_i}\ ;\ i = D \text{ or } H. \tag{12}$$

The modal values of the diameter and height distributions require satisfying a relationship similar to (2) derived from the marginal distribution of the doubly truncated S_{BB} distribution. The relationship to be satisfied is

$$\rho\delta_i \exp\left(-z^2{}_{mi}\right) = \delta_i\left[\pi\left(1 - \rho^2\right)/2\right]^{1/2}\left[\gamma_i + \delta_i \log_e\left(\frac{y_{mi}}{1 - y_{mi}}\right) + 1 - 2y_{mi}\right]\psi(z_{mi});\ i = D \text{ or } H \tag{13}$$

where

$$y_{mi} = \frac{x_{mi} - \xi_i}{\lambda_i},$$

$$z_{mi} = \gamma_i + \delta_i \log_e\left(\frac{y_{mi}}{1 - y_{mi}}\right);\quad i = D \text{ or } H,$$

$$\psi(z_{mi}) = \frac{1}{\sqrt{2\pi}} \int_{z_{mi}}^{\infty} e^{-u^2/2}\, du\ .$$

γ_i, δ_i, ξ_i, λ_i, and ρ are the parameters of the S_{BB} distribution prior to truncation, and x_{mi} is the desired mode.

After truncation the nine stand characteristics of the residual stand are defined. The smallest values are the truncation points, the largest values are unchanged, the modal values are the result of a search procedure to satisfy (13), the standard deviations result from (12), and the correlation between z_D and z_H is obtained from (11). A tacit assumption of the model is that following the next growing period the residual stand may be approximated by the S_{BB} distribution. This assumption is supported by the available data.

For row thinning, the nine characteristics of interest in the residual stand are assumed to be unchanged. The only change is a reduction in surviving number of stems.

Growth After Thinning

Estimating projected yields after thinning necessitates incorporation of certain assumptions into the model. The fundamental assumption applied to modeling thinning from below is that the growth rates of the characteristics of the residual stand are equal to the growth rates of the characteristics of an unthinned stand of the same height, but having had an initial survival density equal to the residual density of the thinned stand.

The growth assumptions following row thinning are that the trees adjoining the removed rows are afforded some releases while the trees within the leave rows are unaffected. To simulate this differential response the lower bounds of height and diameter are assumed to grow as they would have without the thinning, the growth rate of the upper bound and standard deviation of diameter are assumed to respond in the same manner as for thinning from below, and the growth rates of the remaining characteristics are assumed to be equal to an unthinned stand with an initial survival density of the thinned stand reduced by the same fraction as imposed by the row thinning. The result of this approach is that in future periods the average stand diameter is little different from what would have occurred if the stand had not been thinned while the largest diameters show an increase.

The result of these assumptions is that the value of the average growing space, S, is redetermined after thinning by

$$S = (43{,}560 N_t^{-1})^{1/2},$$

for thinning from below, or

$$S = \{43{,}560[N_0(1 - p)]^{-1}\}^{1/2},$$

for row thinning, where N_t is the residual number of trees per acre, N_0 is as before, and p is the fraction of the stand removed by row thinning. The coefficients resulting from the equations of Table 1b are then recalculated using the new value of S. Growth is determined by differencing, using the equations of Table 1a.

The basal area increment of the thinned stand is assumed to be equivalent to that of an unthinned stand of the same height, but having had an initial survival equivalent to the residual density of the thinned stand. Hence, the basal area growth of the thinned stand is determined from the diameter growth of an unthinned stand of the same height and density, and the diameter of the thinned stand is obtained from the incremented basal area.

We assume that stand characteristics relating to height are impacted only by the change in density. The same is true for the characteristics relating to diameter with the exception of the standard deviation of diameter. While it is reasonable to assume that the differential rate of height growth of the residual stand will not be impacted by the thinning, this is not a reasonable assumption for diameter growth. The residual stems will have variable crown sizes and, therefore, a variable potential for crown expansion. This variation in crown expansion can be expected to relate to variation in diameter growth rate. Remembering that the diameter characteristics are functions of dominant height, the "effective" dominant height for use in calculating the future change in the standard deviation of diameter is determined as that dominant height that would have produced a standard deviation equivalent to the standard deviation of the residual stand. This is a common technique used in modeling which amounts to picking up the standard deviation of diameter in biologic time. Biologic time means that the size of the organism is used to describe its age rather than chronologic time. For a detailed discussion of the concept of biologic time we refer the reader to one of the following: Spinner and others (1975), Sharpe and others (1977), or Curry and others (1978).

The concept of biologic time is also used for the correlation between z_D and z_H. Since the value for this correlation reaches a minimum in an unthinned stand and then begins to increase, there are four states that may exist for this value in the residual stand. The value of the correlation may be greater than or less than the expected minimum in an unthinned stand, and it may occur at a dominant height greater than or less than the dominant height at which the minimum occurs in an unthinned stand. If the dominant height of the residual stand is greater than the dominant height at which the minimum occurs in an unthinned stand, the effective dominant height for computation of the correlation is the current dominant height. If the dominant height of the residual stand is smaller than the dominant height at which the minimum occurs in an unthinned stand, the effective dominant height for computation of the correlation is obtained as follows: If the value of the residual correlation is less than the minimum, the effective dominant height is set at the dominant height at which the minimum occurs in an unthinned stand. If the value of the residual correlation is greater than the minimum, the effective dominant height is set equal to the dominant height that would have produced that correlation in an unthinned stand. The result of these assumptions is that, except for very light thinnings, the value of the correlation between z_D and z_H is increased following thinning.

Since none of the stand characteristics are assumed to be altered by row thinning, the biologic time for the standard deviation of diameter and the correlation is unchanged for row thinning.

For estimating survival following thinning from below, the parameters of the negative binomial are recalculated from (7) and (8) and mortality rates for future growth periods are based on those parameters. For row thinning, the negative binomial parameters are not recalculated; the probability of death in any given growth interval remains the same as for an unthinned stand, but it is multiplied by N_0p where N_0 and p are as defined above. In the computer version of the model, the option of performing a thinning from below on the remaining rows is provided. If that option is selected the characteristics of the residual stand and growth after thinning are as for thinning from below.

Conclusion

The mathematical basis for a bivariate distribution model for growth and yield of loblolly pine in plantations is given. As described, the model gives a unified approach to predicting yields for unthinned and thinned plantations. Yield following thinning is predicted by applying a thinning logic in a rigorous mathematical treatment of the S_{BB} distribution describing the unthinned stand.

The stand is described at any instance using Johnson's S_{BB} distribution on the assumption that the stand is a bivariate population of diameters and heights. The model predicts changes over time in nine stand characteristics and utilizes those characteristics to derive the nine parameters of the S_{BB} distribution. The thinning logic is applied to the stand characteristics and then translated to the parameters of the distribution based on the mathematics of truncation.

The bivariate stand model described here is more biologically appealing than the univariate stand model for simulating thinning and projecting growth after thinning since it uses both height and diameter in the decision process. Except for pure row thinning, most thinning applications are removals of trees in the lower canopy, thinning from below. These are decisions that are often based on height rather than on diameter. The logic and the mathematics utilized in the model presented here reflect that process.

Literature Cited

Bliss, C. I., and R. A. Fisher. 1953. Fitting the negative binomial distribution to biological data. Biometrics 9:176–199.

Cao, O. V., H. E. Burkhart, and R. C. Lemin. 1982. Diameter distributions and yields of thinned loblolly pine plantations. Sch For & Wildlife Resour, VPI & SU, Blacksburg, Va, Publ FWS-1-82, 62 p.

Curry, G. L., R. L. Feldman, and K. C. Smith. 1978. A stochastic model for a temperature dependent population. Theor Popul Biol 13:197–213.

HAFLEY, W. L., and H. T. SCHREUDER. 1977. Statistical distributions for fitting diameter and height data in even-aged stands. Can J Forest Res 7:481–487.

HAFLEY, W. L., W. D. SMITH, and M. A. BUFORD. 1982. A new yield prediction model for unthinned loblolly pine in plantations. South For Res Cent, N C State Univ, Raleigh, N C, Tech Rep 1, 63 p.

HARMS, W. R., and O. G. LANGDON. 1976. Development of loblolly pine in dense stands. Forest Sci 22:331–337.

HARMS, W. R., and LLOYD, F. T. 1981. Stand structure and yield relationships in a 20-year-old loblolly pine spacing study. South J Appl For 5:162–166.

JOHNSON, N. L. 1949a. Systems of frequency curves generated by methods of translation. Biometrika 36:149–176.

JOHNSON, N. L. 1949b. Bivariate distributions based on simple translation systems. Biometrika 36: 297–304.

JOHNSON, N. L., and S. KOTZ. 1969. Distributions in statistics: discrete distributions. John Wiley and Sons, New York.

MATNEY, T. G., and A. D. SULLIVAN. 1982. Compatible stand and stock tables for thinned and unthinned loblolly pine stands. Forest Sci 28:161–171.

ROSENBAUM, S. 1961. Moments of a truncated bivariate normal distribution. J Roy Stat Soc, Ser B 23:405–408.

SCHREUDER, H. T., and W. L. HAFLEY. 1977. A useful bivariate distribution for describing stand structure of tree heights and diameters. Biometrics 33:471–477.

SHARPE, P. J. H., G. L. CURRY, D. W. DEMICHELE, and C. L. COLE. 1977. Distribution model of organism development times. J Theor Biol 66:21–38.

SPINNER, R. E., G. D. BUTLER, J. S. BACHELOR, and C. TUTTLE. 1975. Simulation of temperature dependent development in population dynamics models. Can J Entomol 107:1167–1174.

SPRINZ, P., T. CLASON, and D. BOWER. 1979. Spacing and thinning effects on the growth and development of a loblolly pine plantation. North La Hill Farm Exp Stn, Homer, La, For Res Rep, p 1–42.