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Flux-based estimation of parameters in *S*-systems

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Abstract

This paper proposes a technique for parameter estimation in *S*-systems which explicitly takes into account the structure of the *S*-system dynamic equations as the differences between two fluxes, each of which is a power-law function of state and external variables. If observed values of a flux are available for a number of sets of the variables which influence that flux, the technique of flux-based parameter estimation provides a simple method for estimating the parameters in the flux. Flux-based estimation applies multiple linear regression to the logarithms of the fluxes in terms of the logarithms of the variables and only requires that the number of sets of observations exceed the number of unknown parameters in the flux term. The technique is especially valuable for obtaining initial estimates for an application of a traditional nonlinear least-squares technique for fitting observed data to a dynamic model. Since the *S*-system fluxes are power-laws, the conclusions drawn here also apply to the estimation of power-laws. The straightforward application of flux-based estimation to data in which the state and external variables are subject to an allometric relationship can give estimates which are unrealistic. It is shown that under these conditions the estimated kinetic orders are linearly related to the logarithms of the rate constants, and that these relationships provide a powerful guide to selecting appropriate, meaningful estimates for the parameters. Different scenarios are illustrated with data obtained from an *S*-system model of forest growth. The effect of observational error is illustrated by using this model of forest growth to generate observations with varying amounts of observational error.

Keywords: *S*-systems; Power-laws; Parameter estimation; Forest growth model

1. Introduction

Mathematical models of natural systems are characterized by a set of parameters whose values remain constant during any application of the model. Parameter estimation is the process in which these parameters are given values such that the output of the model, in some optimal way, reproduces an observed behaviour of the system being modelled. Most techniques for estimating the parameters of a model involve making systematic changes to the parameter values until output matches observation within a preassigned measure of fidelity.

The measure of fidelity can range from a purely subjective 'eye-ball' fit to the objective criterion of 'least-squares' in which fidelity is measured by the sum of squares of the differences between the model

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output and the corresponding observation. The manner in which the parameters are changed can range from simple trial and error to linear regression (Draper and Smith, 1981, Ch. 1) or various techniques of nonlinear regression (Draper and Smith, 1981, Ch. 10). Parameter estimation is a computationally intensive process. It requires an efficient algorithm for solving the model (i.e., determining the outputs corresponding to a given set of inputs) and fairly good initial parameter values.

In the case of analytical models, i.e., models whose output is an explicit analytical function of its inputs, it suffices to write computationally efficient expressions for the outputs and, optionally, their first derivatives with respect to the parameters of the model. Software for statistical data analysis provide efficient procedures for estimating parameters in static linear and nonlinear models, e.g., GENSTAT (Payne et al., 1988). However, output of many models can not be expressed as explicit functions of inputs. For example, the model may be a system of differential equations with no analytical solution, or it may include logical switches. Such models include process-based models of a range of phenomena in the natural sciences, such as crop growth, population dynamics, biochemical systems. Solving the model may itself be computationally intensive, requiring the numerical solution of coupled differential equations. This adds greatly to the computational complexity of parameter estimation. Software which both solves the differential equations of the model and estimates its parameters do exist, e.g. BMDP (Ralston et al., 1988), Scientist (MicroMath, 1995) and ModelMaker (SB Technology, 1995), but is often only successful if the number of parameters is small.

Canonical models are classes of models characterized by a well-defined structure, such as the models of linear systems theory (Ogata, 1990), and *S*-system models (Savageau, 1976; Voit, 1991). Advantages of working with canonical models include existence of a well-defined prescription for determining the equations of the model and a systematic and efficient method for solving these equations which exploits their particular structure. For example, linear systems can be solved using the method of Laplace transforms (Ogata, 1990), and *S*-systems using recursive polynomial determinations based on Taylor-series expansion (Irvine and Savageau, 1990).

This paper concerns how the canonical structure of an *S*-system may facilitate parameter estimation. Savageau (1969) showed the canonical structure of an *S*-system leads to a simple method for estimation based on biochemical steady state data. Ecological systems are rarely in steady state, so our focus is on estimation in *S*-systems not in steady state. We show that the canonical structure of an *S*-system model does facilitate the estimation of its parameters: if an influx or efflux has been observed for sets of variables which influence this flux, multiple linear regression based on log-transformed data produces estimates of the parameters in the flux. We call this process flux-based parameter estimation. Although it is straightforward and systematic, problems are often encountered with particular combinations of model and data. We consider examples where these can be traced back to the existence of allometric relationships between observed variables.

The analyses of this paper focus on the estimation of the parameters in *S*-system fluxes. Since these fluxes are power-laws, the process of flux-based estimation and the conclusions drawn in this paper also apply to the estimation of power-law relationships in general.

The ideas presented here are based on our experience with a simple *S*-system model of forest growth (Voit and Sands, 1995a,b). They are not a definitive statistical treatment as the focus is on the practical application of a tool useful in developing *S*-System models. For the sake of completeness a brief overview of the forest growth model follows.

2. Canonical *S*-system models

2.1. Generic form of S-system models

Let X_i $(i = 1, 2, \ldots, n)$ be the n state variables of the system, and X_i $(i = n+1, n+2, \ldots, n+m)$ be the m external variables affecting the system. Then the rate of change $\dot{X}_i$ of each state variable is written as the difference between two terms, an influx F_i^+ and an efflux F_i^-, so

$$\dot{X}_i = F_i^+ - F_i^- \tag{1}$$

for $i = 1, 2, \ldots, n$. It is assumed all fluxes can be

approximated by power-law functions in the state and external variables X_i. Thus

$$F_i^+ = \alpha_i \prod_{j=1}^{n+m} X_j^{g_{ij}},\ F_i^- = \beta_i \prod_{j=1}^{n+m} X_j^{h_{ij}} \qquad (2)$$

The α_i and β_i are positive constants or zero and characterize the magnitude of the fluxes. The g_{ij} and h_{ij} characterize the influence of the variables on the fluxes. The effect of X_j on the influx F_i^+ is given by g_{ij}, which is positive if X_j is an enhancer (i.e., if an increase in X_j increases F_i^+), or zero if X_j has no effect on F_i^+, or negative if X_j is a repressor (i.e., if an increase in X_j decreases F_i^+). The interpretation of the h_{ij} in terms of the effect of X_j on the efflux F_i^- is similar. Experience with a wide range of *S*-systems suggests kinetic orders are usually in the range -2 to 2.

Eqs. 1–2 are the generic equations of *S*-system models. Their theoretical basis, the accuracy of the power-law approximations and typical applications of *S*-systems are discussed elsewhere (e.g., Savageau, 1976; Voit, 1991; and references therein). In the remainder of this paper we refer to the X_i ($i = 1, 2, \ldots, n+m$) as variables and the F_i^+ and F_i^- as fluxes. Note that a flux need not describe a biochemical reaction nor represent a physical flow but in general represents those processes which lead to an increase or decrease in a state variable. We also follow convention based on biochemical analogy by referring to the α_i and β_i as rate constants, and the g_{ij} and h_{ij} as kinetic orders.

An *S*-system model is a set of coupled first-order differential equations of the form of Eqs. 1–2. These can be solved using standard methods for solving differential equations, but the canonical structure of the *S*-system equations supports a more efficient method based on high-order Taylor-expansions of the X_i with the coefficients determined using recurrence relations (Irvine and Savageau, 1990). This method was implemented in ESSYNS, a software package specifically for working with *S*-systems (Voit et al., 1990).

2.2. An *S*-system model of forest growth

The structure of our forest growth model is based on a qualitative understanding of the processes underlying tree growth and biomass partitioning as influenced by nutritional status (Voit and Sands, 1995a, Voit and Sands, 1995b). The state variables are internal nitrogen X_1, foliage biomass X_2, stem and branch biomass X_3, and root biomass X_4, where all units are t ha^{-1}. The only external variable is X_5, a scalar measure of nutrient treatment. Time t is in years. The model was applied to an analysis of the

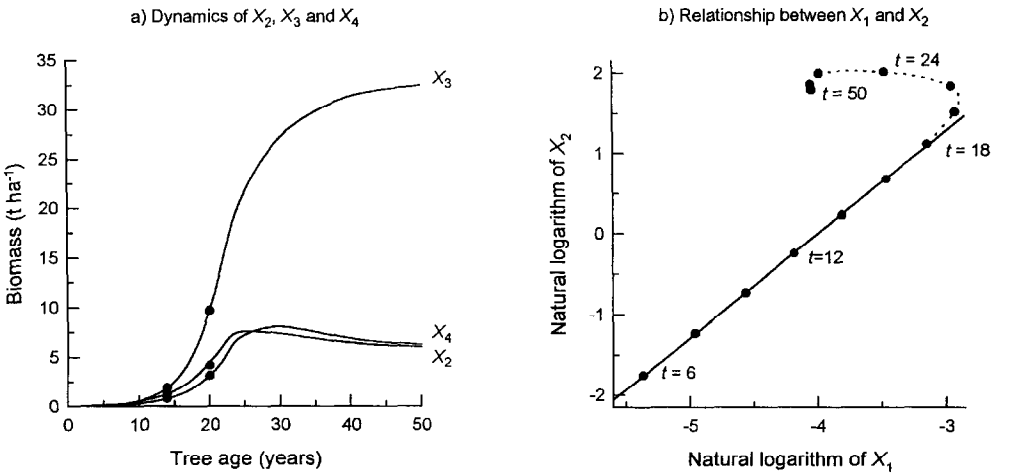

Fig. 1. (a) Dynamics of foliage (X_2), stem (X_3) and root (X_4) biomass compartments with age: (————), model output; (●), observed data. (b) Relationship between X_1 and X_2: (●), data from Table 2; (————), allometric relationship between X_1 and X_2 for ages $\leq$ 18 years; (- - -), nonallometric relationship between X_1 and X_2 for ages $>$ 18 years.

growth of fertilized and unfertilized stands of *Pinus sylvestris* growing in Sweden. The equations of the *S*-system model with parameter values for the *Pinus sylvestris* data are:

$$\dot{X}_1 = 9.5 \times 10^{-5} X_1^{-1} X_2^{0.72} X_4^{0.72} X_5 - 6.5 \times 10^{-5} X_2 X_3 X_4$$

$$\dot{X}_2 = 3.2 X_1^{0.5} X_2^{0.57} - 0.2 X_2$$

$$\dot{X}_3 = 5 X_1^{0.5} X_2^{0.7} - 0.2 X_3^{0.7}$$

$$\dot{X}_4 = 0.36 X_1^{-0.4} X_2 - 0.75 X_1^{-0.3} X_4^{0.8} \quad (3)$$

Initial values at time $t = 0$ are $X_1 = 0.0015$, $X_2 = 0.03$, $X_3 = 0.01$ and $X_4 = 0.01$, for all treatments X_5. Output from the model is illustrated by Fig. 1. Fig. 1a shows the dynamics of the biomass pools, and Fig. 1b shows the relationship between foliage biomass and internal nitrogen. This data set is used later to illustrate flux-based parameter estimation.

3. Parameter estimation in *S*-system models

In principle, any least-squares technique can be applied to estimate the parameters of an *S*-system not at steady state, e.g. the Gauss–Newton and the Levenberg–Marquardt methods (Draper and Smith, 1981). Let x_{ik} be the observed values of the X_i at times t_k ($k = 1, 2, \ldots, s$), and let $X_{ik} = X_i(t_k)$ be the corresponding values of X_i predicted by the *S*-system. The parameters are then adjusted so as to minimize the residual sum of squares $\phi = \sum_{i=1}^{n} \sum_{k=1}^{s} (x_{ik} - X_{ik})^2$. This assures the predicted trajectory of the *S*-system in the space spanned by the X_i passes close to the observed points x_{ik}. Estimation can also be based on observed values of the rates of change $\dot{X}_i$ of the state variables by including in ϕ the residuals between observed and predicted $\dot{X}_i$. This assures that the predicted trajectory of the *S*-system passes close to the observed points x_{ik} and that its slope near these points is close to the observed slopes.

These techniques have been used successfully with *S*-systems containing a single variable (Voit, 1992). However, experience with more complicated *S*-systems (Voit and Savageau, 1982; Torsella and Razali, 1991) has shown that the estimation process can be sensitive to the initial estimates of the parameters, and to the error criteria used both when solving the differential equations and when establishing convergence of the estimation process. Our experience with various systems also confirmed that correlations between observed variables, correlations between estimated parameters, and slow or no convergence of the estimation process often go hand in hand.

3.1. Ad hoc parameter estimation in the forest growth model

The data upon which the parameters in our forest growth model were estimated (Voit and Sands, 1995b) comprised only three sets of observations during the period of active growth but also included observed biomass fluxes into and out of major compartments. Two sets were at ages $t = 14$ and 20 years from a control treatment ($X_5 = 1$), and one at age $t = 20$ years from a fertilized and irrigated treatment ($X_5 = 9$). The treatment was applied from age 14 years on. This was implemented in the model by setting $X_5 = 1$ for all t in the control treatment and for $t \leq 14$ in the fertilized treatment, and $X_5 = 9$ for $t > 14$ in the fertilized treatment.

With the exception of the fluxes F_1^+ and F_1^- for internal nitrogen the observed data were just sufficient to estimate all parameters: there were three parameters in each flux term and three sets of observations for each flux. Parameters in F_1^+ and F_1^- were assigned by trial and error (Voit and Sands, 1995b). A traditional method for nonlinear parameter estimation was not used to estimate the remaining parameters because experience suggested parameter estimation in *S*-systems of this size can be quite difficult due to a largely unexplained lack of convergence. We also had to optimize simultaneously the fit of observed fluxes to static functions and dynamic output of the model to observed state variables.

Because of these difficulties, a simple ad hoc process was used to estimate the parameters in the model. Consider the flux F_4^+. This is a power-law in two variables (X_1 and X_2) and involves three parameters (α_4, g_{41} and g_{42}): $F_4^+ = \alpha_4 X_1^{g_{41}} X_2^{g_{42}}$. The three observations (one at age 14 and two at age 20) are given in Table 1. These determined unique values for the three parameters since a log-transforma-

Table 1
Subset of observed data for the forest growth model

age	Treatment	X_1	X_2	F_4^+
14	Control ($X_5 = 1$)	0.02	1.2	2.13
20	Control ($X_5 = 1$)	0.06	4.2	6.86
20	Fertilized ($X_5 = 9$)	0.12	10.6	6.92

Observed values of the compartment sizes X_1 and X_2 and of the flux F_4^+ used for parameter estimation in the forest growth model. Data are given for a control treatment at ages 14 and 20 years, and for a fertilized treatment at age 20 years.

tion of the equation for F_4^+ gave three linear equations for $\ln\alpha_4$, g_{41} and g_{42} which were solved exactly. However, this process led to biologically unreasonable values: $\ln\alpha_4 = 29.934$, $g_{41} = 7.208$, $g_{42} = -5.387$. Although these reproduce the three observed values for F_4^+, they give unacceptable values for other reasonable combinations of X_1 and X_2.

This problem was addressed by recognizing that observations are subject to observational error and parameters calculated using this simple approach are sensitive to this error. Synthetic data sets were obtained by perturbing the observed data with a random relative error to simulate observational error. If X was the observed value of a variable, synthetic observed values X' were obtained from

$$X' = (1 + N(0,\epsilon))\,X \qquad (4)$$

where $N(\mu,\epsilon)$ is a normal distribution with mean μ and standard deviation ϵ. When ϵ is small the $\ln X'$ have a normal distribution with mean $\ln X$ and standard deviation ϵ. This form was chosen so the error $(X' - X)$ scales with the observed value and matches the error distribution frequently found in ecological data and assumed for our forest growth model. Observed data from Table 1 were perturbed by applying Eq. 4 with $\epsilon = 0.1$ (i.e. 10% relative error) and new parameter values calculated as above. This was repeated for 50 distinct and random sets of perturbations. It was found that the estimates of $\ln\alpha_4$, g_{41} and g_{42} varied widely (in our case $-410 < \ln\alpha_4 < 68$, $-103 < g_{41} < 17$ and $-12 < g_{42} < 78$); data in the range $-60 < \ln\alpha_4 < 60$ are shown in Fig. 2a (4 extreme points are excluded). Fig. 2a shows there are strong linear relationships between g_{41} and g_{42} and $\ln\alpha_4$ ($r^2 > 0.99$, $n = 50$). Thus, when a value is assigned to one of the three parameters, ranges for the other two are determined.

The biology of the processes subsumed in F_4^+ suggests leaf biomass X_2 and root growth F_4^+ are positively correlated. Thus $g_{42} > 0$, and probably close to 1. We assumed $g_{42} = 1$, and the linear regressions between g_{41}, g_{42} and $\ln\alpha_4$ then suggested $\ln\alpha_4 = -1.07 \pm 0.6$, $g_{41} = -0.4 \pm 0.2$. The errors were obtained by taking account of the stan-

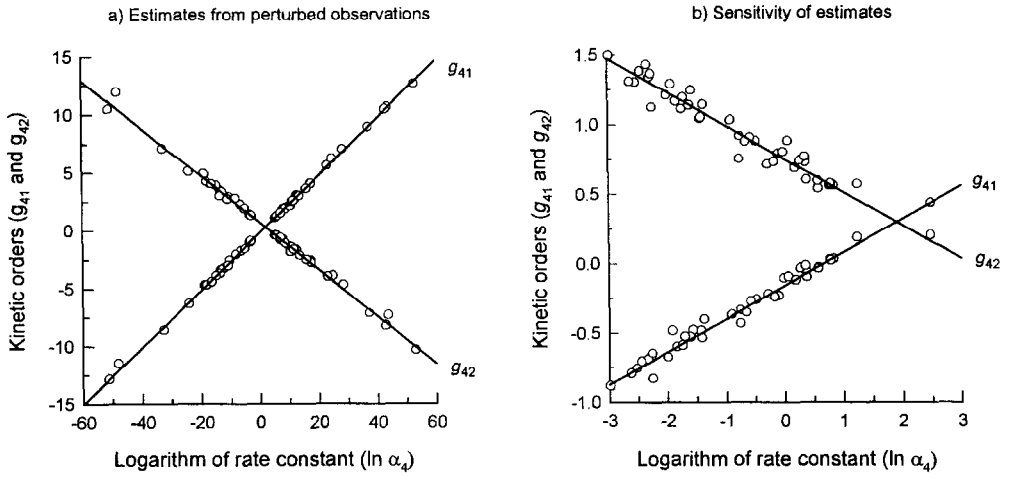

Fig. 2. (a) Flux-based estimates for g_{41}, g_{42} and $\ln\alpha_4$ obtained from the observed data in Table 1: (○), basic estimates from 50 sets of observations obtained by applying 10% random relative error to data in Table 1; (————), linear relationships obtained by applying Eq. 8 to observed data. (b) Sensitivity analysis of the estimates: (○), basic estimates from 50 synthetic sets of observations corresponding to Table 1 generated by applying 10% random relative error to model output; (————), linear relationships obtained by applying Eq. 8 to unperturbed model output.

dard errors from the regressions of g_{41} and g_{42} as functions of $\ln \alpha_4$.

Similar relationships were found for all fluxes in the model and this process was used to assign ranges to the parameters in each flux. The dynamic behaviour of the model was checked for consistency with observed state variables and fluxes. Agreement was improved using the freedom implied by presumed observational error to change the parameter values. Fine-tuning generally required varying the rate constants only. In the end we obtained parameter values for which output from the model was consistent with the observations.

4. Flux-based parameter estimation

4.1. Definition of flux-based estimation

The above ad hoc estimation process was systematic, tedious and somewhat subjective, but it led to a number of insights into the process of parameter estimation in *S*-systems. These have been formulated in a process which makes explicit use of the canonical form of *S*-system models to estimate parameters from observed fluxes. We define an observation to be a set of simultaneous measurements of a flux and of all variables which affect that flux. Given a sufficient number of observations, flux-based estimation can be used to estimate directly parameters characterising that flux. The minimum number of observations required is one more than the number of variables affecting the flux.

Since most fluxes in our forest growth model depend on only two variables our discussion is based on two-variable fluxes, but generalizations to many variables is straightforward. A typical influx or efflux then has the form

$$F = \alpha X_a^{g_a} X_b^{g_b} \tag{5}$$

where X_a and X_b are the variables, α the rate constant and g_a and g_b the kinetic orders. Taking logarithms of both sides gives

$$\ln F = \ln \alpha + g_a \ln X_a + g_b \ln X_b \tag{6}$$

so $\ln F$ is a linear function of $\ln \alpha$, g_a and g_b. Accordingly, given 3 or more sets of observed values of F, X_a and X_b the parameters $\ln \alpha$, g_a and g_b can be estimated by multiple linear regression using $\ln F$ as dependent variable and X_a and X_b as independent variables. This is *flux-based parameter estimation* and the parameter estimates obtained by linear regression based on Eq. 6 are called the *basic estimates*.

4.2. Observational and measurement error

Each observed variable and flux is subject to a direct error in measurement and a sampling error due to natural variation. We refer to these collectively as observational error. In the case of our forest growth model the statistical nature of the observational error was unknown. It is often assumed in ecological contexts that observational errors in pools and fluxes are proportional to the means and that errors in log-transformed data are normally distributed. If fluxes are calculated as the difference between pool sizes, errors in fluxes are larger than errors in the corresponding pools.

Linear regression based on ordinary least-squares estimation assumes all independent variables are error free and the dependent variable has normally distributed error. This assumption is generally not satisfied in the context of Eq. 6 as F and the X_j are all observed data. If all variables are subject to error, estimates based on ordinary least squares will be biased (e.g. Madansky, 1959). However, consistent unbiased estimates are obtained from the corrected least-squares estimator (Ketellapper, 1983).

If errors in $\ln F$ are normally distributed and larger than the errors in $\ln X_j$ the assumptions necessary for ordinary least-squares regression are approximately satisfied and flux-based estimation may give reliable estimates of the parameters. In general, errors in the $\ln X_j$ can not be neglected so flux-based estimates will not be unbiased. However, flux-based estimation using ordinary least-squares nonetheless gives useful estimates which subsequently can be used as initial values of the parameters for conventional nonlinear least-squares parameter estimation.

4.3. The effects of allometry

Multiple linear regression requires the inversion of a square matrix (Draper and Smith, 1981, Ch. 2.6)

whose determinant Δ is a measure of the collinearity of the independent variables: the higher the degree of collinearity the smaller Δ. If Δ is zero the regression cannot be performed, and if Δ is small the estimates have large standard errors and are sensitive to small changes in the independent variables. In flux-based estimation the independent variables are the $\ln X_j$ and linear relations between the $\ln X_j$ are equivalent to allometric (power-law) relations between the X_j. Thus, allometry between the variables leads to poor estimation. However, if variables in a flux are allometrically related a linear relationship between parameter estimates can be predicted theoretically. Thus, allometry effectively hides the underlying power-law.

Let a flux F determined by variables X_a and X_b be given by Eq. 6 and assume that X_a and X_b are allometrically related, i.e.

$$X_b = \gamma X_a^k \tag{7}$$

where γ and k are constants. Let $F' = \alpha' X_a^{g'_a} X_b^{g'_b}$ be another representation for the same flux, but with distinct parameters α', g'_a and g'_b. The requirement that F and F' are identical for all X_a and X_b for which Eq. 7 holds leads to the relations

$$\begin{aligned} g_a &= (g'_a - k\ln\alpha'/\ln\gamma) + (k/\ln\gamma)\ln\alpha \\ g_b &= (g'_b + \ln\alpha'/\ln\gamma) - (1/\ln\gamma)\ln\alpha \end{aligned} \tag{8}$$

Thus, g_a and g_b are linearly related to $\ln\alpha$. The slopes are given directly by the parameters of the relationship (7) and if a set of values for α', g'_a and g'_b is known for which F, X_a and X_b are related by Eq. 5 the intercepts can also be calculated.

Under these conditions if a value of one parameter is known a priori then Eq. 8 can be used to assign values for the other parameters.

4.4. Sensitivity of estimates

The sensitivity of parameter estimates can in principle be determined by error propagation from a knowledge of the statistics of the distribution of errors in each variable. However, in practice neither the observations nor the parameters themselves are independent and hence it is necessary to include in the error propagation formulae terms representing the correlations. Accordingly, we use a Monte-Carlo simulation technique to study the sensitivity of parameter estimates to observational errors.

If flux-based estimation is used to estimate parameters in an S-system model, the model can also be used to determine a measure of the sensitivity of the estimates to observational error. First, the model is used to generate output corresponding to the observed data used when estimating the parameters. Then, sufficiently many sets of synthetic observations with random observational error are generated from this output by repeated application of Eq. 4 with a desired value of ϵ. Basic estimates are determined from each of these sets as described in Section 4.1. These estimates constitute an approximation to the distribution of estimates consistent with the assumed observational error.

If there is little or no correlation between the parameters estimated from these synthetic data sets, the standard deviation of these estimates provides a measure of the sensitivity of the estimates. If the estimates are correlated (e.g. as in Fig. 2a) and it is possible to assign an a priori value to one of the parameters, plots analogous to Fig. 2a can be used to assign a range within which the estimates of the other parameters lie.

5. Flux-based estimation applied to a minimum of observed data

Our ad hoc estimation of parameters in the forest growth model based on only three sets of observations is now examined in light of the above findings. Three observations are the minimum for flux-based estimation when fluxes are determined by two variables. Again consider the flux F_4^+ as an example. Observed values of F_4^+, X_1 and X_2 are given in Table 1 and as X_1 and X_2 are allometrically related, flux-based estimation may give estimates whose distributions have large standard errors. Indeed, the basic estimates based on Table 1 (i.e., $g_{41} = 7.2$, $g_{42} = -5.4$, $\ln\alpha_4 = 29.9$) were sensitive to small errors in the observations (Fig. 2a).

Since the observed X_1 and X_2 are allometric the 'true' values of g_{41}, g_{42} and $\ln\alpha_4$ must satisfy Eq. 8. Using Eq. 7 to fit X_2 to X_1 for the data in Table 1 gives $\ln\gamma = 4.891 \pm 0.2$ and $k = 1.209 \pm 0.05$ (r^2

$= 0.998$, $n = 3$) and substitution of these values and the basic estimates for g_{41}, g_{42} and $\ln\alpha_4$ (as $g'_{41} = 7.208$, $g'_{42} = -5.387$ and $\ln\alpha'_4 = 29.934$) into Eq. 8 gives

$$g_{41} = 0.247\ln\alpha_4 - 0.192$$

$$g_{42} = -0.204\ln\alpha_4 + 0.733 \quad (9)$$

These lines are shown in Fig. 2a and clearly fit the various estimates of g_{41}, g_{42} and $\ln\alpha_4$ based on random 10% relative error imposed on the data of Table 1. As before, if $g_{42} = 1$ Eq. 9 gives $g_{41} = -0.51$ and $\ln\alpha_4 = -1.31$.

The model was used to obtain a measure of the sensitivity of the estimates to observational error. Model output corresponding to the observed data in Table 1 was generated. Fifty synthetic data sets corresponding to Table 1 were then generated by applying Eq. 4 with a 10% relative error to this output. Basic estimates of g_{41}, g_{42} and $\ln\alpha_4$ generated from these 50 data sets are shown in Fig. 2b, truncated to the range $-3 < \ln\alpha_4 < 3$. The lines are given by Eq. 9. By reading off Fig. 2b the ranges of values of g_{41} and $\ln\alpha_4$ which are consistent with $g_{42} = 1$ it was shown that the range of $\ln\alpha_4$ corresponding to $g_{42} = 1$ is $-1.5 < \ln\alpha_4 < -0.4$ and the corresponding range for g_{41} is then $-0.7 < g_{41} < -0.3$. These ranges encompass the values ultimately used in the model ($g_{42} = 1$, $g_{41} = -0.4$, $\ln\alpha_4 = -1.02$) and those estimated from Eq. 9.

None of these various estimates are exactly the values finally used in the model. We used a manual process to adjust our flux-based estimates to ensure consistency between the dynamic behaviour of the model and observed state variables and fluxes. In general, a nonlinear least-squares technique with flux-based estimates as initial values would be used.

6. Flux-based estimation applied to time-series data

We now consider an example where there is more than the minimum data sufficient for an application of flux-based estimation. The forest growth model of Eq. 3 was used to generate time-series for the flux F_4^+ and for the state variables X_1 and X_2 which affect this flux. These data are given to 4 significant digits in Table 2. The application of flux-based estimation to time-series data is illustrated using synthetic observations derived from these time-series.

Table 2
Predicted time series of data from forest growth model

t	X_1	X_2	F_4^+
6	0.004666	0.1739	0.5357
8	0.007014	0.2920	0.7644
10	0.01043	0.4824	1.077
12	0.01533	0.7851	1.503
14	0.02222	1.258	2.077
16	0.03155	1.985	2.847
18	0.04310	3.066	3.882
20	0.05364	4.570	5.301
24	0.03108	7.484	10.80
50	0.01763	6.030	10.92

Time series for the compartment sizes X_1, X_2 and flux F_4^+, obtained from the forest growth model of Eq. (3). The external variable has the value $X_5 = 1$. All values were rounded to 4 significant digits.

Eq. 4 with a 1% random relative error ($\epsilon = 0.01$) was applied to the data in Table 2 for ages 6, 10, 12, 16, 18 and 24 years to generate a synthetic observed time-series for X_1, X_2 and F_4^+. Flux-based estimation was used to estimate the parameters in F_4^+ from this particular synthetic time-series, i.e. by linear regression of $\ln F_4^+$ as dependent variable with $\ln X_1$ and $\ln X_2$ as independent variables. The basic estimates for this particular synthetic time-series were $\ln\alpha_4 = -1.12 \pm 0.07$, $g_{41} = -0.42 \pm 0.02$ and $g_{42} = 1.01 \pm 0.01$ ($n = 6$, standard errors from linear regression). This process was repeated a total of 50 times using the same 6 sets of output from the model but with distinct streams of random numbers in Eq. 4. In this way 50 distinct basic estimates of $\ln\alpha_4$, g_{41} and g_{42} were generated. Their means and standard deviations were $\ln\alpha_4 = -1.04 \pm 0.08$, $g_{41} = -0.40 \pm 0.02$ and $g_{42} = 1.00 \pm 0.01$.

The process described in the previous paragraph was repeated using 6 distinct relative errors for each of 3 distinct sets of 6 ages selected from Table 2 as the basis for generating the time-series for X_1, X_2 and F_4^+. The 6 relative errors ϵ were 0.0005, 0.005, 0.01, 0.05, 0.1 and 0.2. The 3 sets of 6 ages were 14, 16, 18, 20, 24 and 50 years, 6, 10, 12, 16, 18 and 24 years, and 6, 8, 12, 14, 16 and 20 years. In each case 50 sets of estimates of $\ln\alpha_4$, g_{41} and g_{42} were

generated and Table 3 gives the mean and standard deviation of each set of 50 estimates.

It is clear from Table 3 that for low ϵ the estimates agree closely with the parameter values used in the model (i.e., $\ln \alpha_4 = -1.022$, $g_{41} = -0.4$ and $g_{42} = 1$) when generating the original time-series. Also, the standard deviation of the estimates increases with increasing ϵ. Finally, the reliability of the estimates for high ϵ depends markedly on the choice of time series. This last point is a further illustration of the effect of allometry between variables. Fig. 1b shows that X_1 and X_2 are allometrically related for $t < 18$ but not for larger values of t. The results in Table 3a were derived using time series with no allometric relationship between X_1 and X_2 ($r^2 = 0.01$), for Table 3b there was considerable allometry ($r^2 = 0.86$), while for Table 3c there is an excellent allometric relationship between X_1 and X_2 ($r^2 = 0.99$) and the estimated parameters are subject to significant error.

7. Replication and flux-based estimates

Additional data may also be obtained by replication, e.g., through repeated measurements at a few observational times. The effect of replication on flux-based estimation and its interaction with allometry is considered below.

7.1. Replication and sensitivity

Three random synthetic replicates of the observed data in Table 1 were generated by using Eq. 4 to apply a 10% random relative error to the corresponding output from the model. These synthetic observations were then used in place of the observed data of Table 1 in the ad hoc estimation (Section 3.1) and in our application to a minimum of observed data (Section 5). Direct comparison with the results of those earlier estimations are then suggestive of effects due to replicated observations.

First consider the application of our ad hoc estimation (Section 3.1) to the synthetic replicated observations. The basic estimates ($g_{41} = -0.19 \pm 0.2$, $g_{42} = 0.86 \pm 0.2$, $\ln \alpha_4 = -0.19 \pm 0.9$, $n = 9$, standard errors from linear regression) are biologically reasonable but have large standard errors and differ markedly from the values used to generate the data (i.e., the values in the model). 50 random perturbations of these synthetic observations were performed and the corresponding flux-based estimates are shown

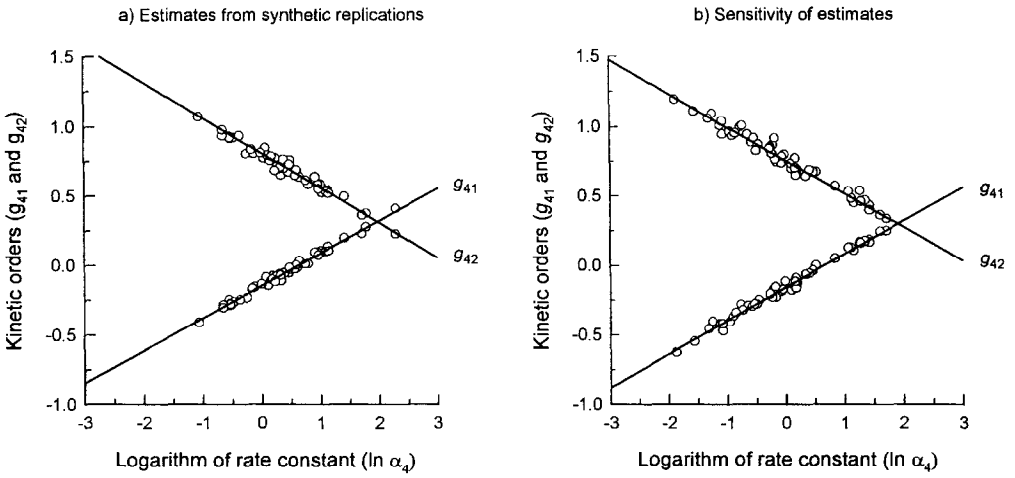

Fig. 3. (a) Flux-based estimates for g_{41}, g_{42} and ln α_4 based on three replicates of the data in Table 1, generated by applying 10% random relative error to model output: (○), basic estimates from 50 sets of observations obtained by applying 10% random relative error to the synthetic replicated observations: (————), linear relationships obtained by applying Eq. 8 to the synthetic replicated observations. (b) Sensitivity analysis of estimates based on replicated observations: (○), basic estimates from 50 sets of 3 synthetic replicated observations obtained by applying 10% random relative error to model output; (————), linear relationships obtained by applying Eq. 8 to model output.

Table 3
Flux-based estimates of parameters in F_4^+

ϵ	$\ln\alpha_4$	g_{41}	g_{42}
(a) Estimates derived from the time series: $t = 14, 16, 18, 20, 24, 50$			
0.0005	-1.023 ± 0.003	-0.400 ± 0.001	1.000 ± 0.000
0.005	-1.026 ± 0.03	-0.401 ± 0.009	1.000 ± 0.005
0.01	-1.029 ± 0.06	-0.402 ± 0.02	0.999 ± 0.009
0.05	-1.051 ± 0.3	-0.407 ± 0.08	0.993 ± 0.05
0.1	-1.075 ± 0.6	-0.414 ± 0.2	0.983 ± 0.1
0.2	-1.092 ± 1.2	-0.422 ± 0.3	0.949 ± 0.2
(b) Estimates derived from the time series: $t = 6, 10, 12, 16, 18, 24$			
0.0005	-1.023 ± 0.004	-0.400 ± 0.001	1.000 ± 0.001
0.005	-1.031 ± 0.04	-0.402 ± 0.01	1.001 ± 0.006
0.01	-1.040 ± 0.08	-0.404 ± 0.02	1.002 ± 0.01
0.05	-1.101 ± 0.4	-0.416 ± 0.1	1.008 ± 0.06
0.1	-1.160 ± 0.8	-0.426 ± 0.2	1.011 ± 0.1
0.2	-0.987 ± 1.4	-0.377 ± 0.4	0.971 ± 0.2
(c) Estimates derived from the time series: $t = 6, 8, 12, 14, 16, 20$			
0.0005	-1.019 ± 0.03	-0.398 ± 0.008	1.000 ± 0.006
0.005	-1.067 ± 0.3	-0.411 ± 0.09	1.008 ± 0.07
0.01	-1.101 ± 0.7	-0.419 ± 0.2	1.014 ± 0.1
0.05	0.295 ± 2.0	-0.069 ± 0.5	0.746 ± 0.4
0.1	0.884 ± 2.1	0.081 ± 0.5	0.631 ± 0.4
0.2	1.064 ± 2.2	0.135 ± 0.5	0.581 ± 0.4

Means and standard deviations of 50 sets of flux-based estimates of parameters in F_4^+, obtained from each of various synthetic time-series data based on output from the forest growth model and perturbed with a relative error ϵ using Eq. (4).

in Fig. 3a. This figure corresponds directly to Fig. 2a and suggests the range of estimates ($-1.1 < \ln\alpha_4 < 2.3$, $-0.4 < g_{41} < 0.4$ and $0.2 < g_{42} < 1.2$) based on the synthetic replicated data is considerably reduced from the range obtained using the original observed data. Linear regressions of the data in Fig. 3a gave estimates corresponding to $g_{42} = 1$: $\ln\alpha_4 = -0.81 \pm 0.02$, $g_{41} = -0.34 \pm 0.01$. Comparison with results in Section 3.1 suggests replication has greatly reduced standard errors.

Now consider the analyses of Section 5 applied to the replicated data. As there is a high degree of allometry between these (synthetic) replicated observations, estimation was also based on an application of Eqs. 7 and 8. Fitting X_2 to X_1 with Eq. 7 gives $\ln\gamma = 3.970$ and $k = 0.925$ ($r^2 = 0.94$, $n = 9$), and insertion of these values and the above basic estimates ($g_{41} = -0.19$, $g_{42} = 0.86$ and $\ln\alpha_4 = -0.19$) into Eq. 8 gives

$$g_{41} = 0.233\ln\alpha_4 - 0.144$$

$$g_{42} = -0.252\ln\alpha_4 + 0.813 \qquad (10)$$

$g_{42} = 1$ gives $g_{41} = -0.32$ and $\ln\alpha_4 = -0.74$, comparable to the values estimated from observed data.

Output from the model was again used to obtain a measure of the sensitivity of the estimates to observational error. 50 sets of 9 synthetic observations were generated from the output of the model and flux based estimation applied to each of these. The 50 sets of estimates for g_{42}, g_{41} and $\ln\alpha_4$ are shown in Fig. 3b along with the lines given by Eq. 10. Fig. 3b suggests the parameter ranges corresponding to $g_{42} = 1$ are $-1.3 < \ln\alpha_4 < -0.6$ and $-0.55 < g_{41} < -0.25$. Comparison with Section 5 again suggests replication has reduced the ranges.

7.2. Replication and allometry

Fig. 4 shows the time course of the relationship between X_1 and X_2 predicted by the model. This figure explicitly includes output at times 10, 14, 18, 24 and 50 years (shown as ●) and points resulting from an application of a random 10% relative error

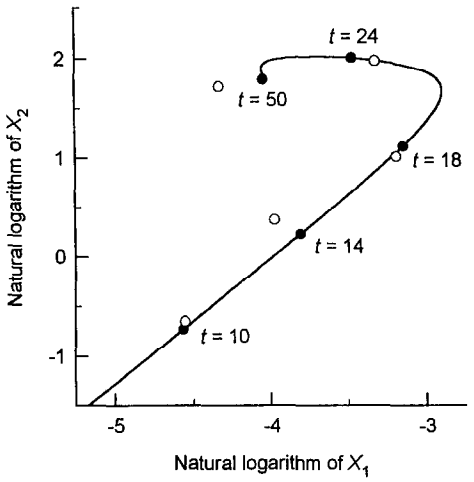

Fig. 4. Time course of the relationship between X_1 and X_2: (————), model output; (●), model output for ages 10, 14, 18, 24 and 50 years; (○), data generated by perturbing the points (●) to simulate observed data with 10% observational error.

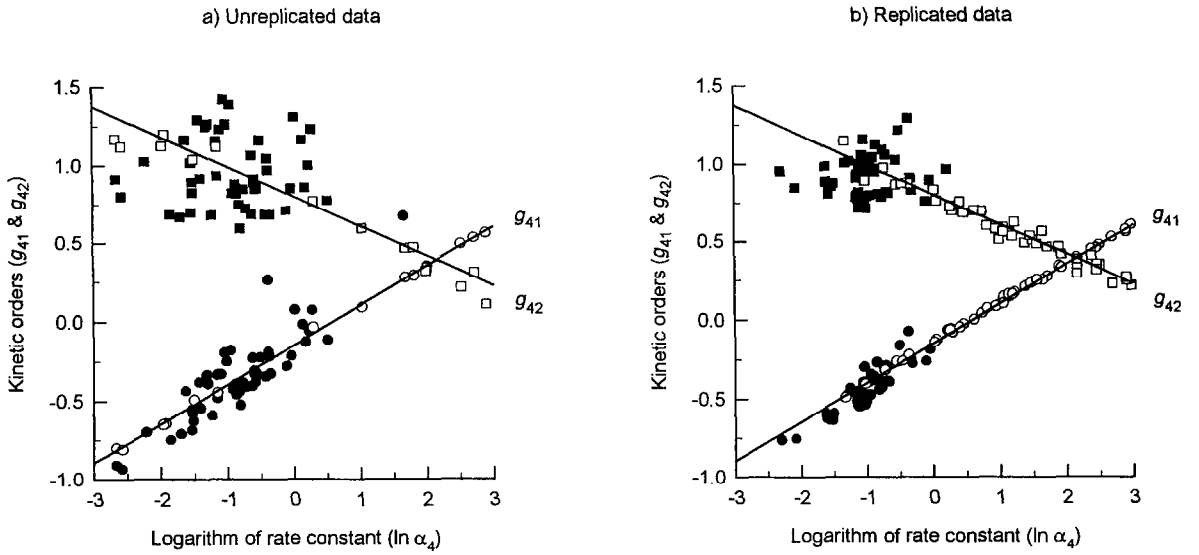

Fig. 5. (a) Flux-based estimates for g_{41}, g_{42} and ln α_4 from single replicates of three simulated observations obtained by applying 10% random relative error to model output: (○) and (□), basic estimates from simulated observations at ages 10, 14 and 18 years; (●) and (■), basic estimates from simulated observations at ages 18, 24 and 50; (————), linear relationships obtained by applying Eq. 8 to model output for ages 10, 14 and 18 years. (b) Same as in (a) but estimates are based on three replicates of three observations.

to these data (shown as ○). The latter are assumed to simulate observed data subject to 10% observational error. Both model output and data with simulated observational error are highly allometric for $t \leq 18$ years but not allometric for $t \geq 18$ years. It was found that only under very rare occasions does 10% random relative error induce any significant allometry between simulated observations at 18, 24 and 50 years, and when this is the case the resulting relationship is quite distinct from the true allometry characteristic of early years.

These data are used to further illustrate the effects of allometry and replication on flux-based estimates. Fig. 5a shows flux-based estimates of $\ln \alpha_4$, g_{41} and g_{42} obtained from a single replicate of simulated observations with 10% relative error based on model output for ages 10, 14 and 18 years (high allometry, open symbols) and ages 18, 24 and 50 (no allometry, solid symbols). Fig. 5b shows the same flux-based estimates but each is based on 3 replicates of simulated observed data with 10% observational error. In each case basic estimates were generated from 50 distinct simulated observations. Fig. 5 truncates data to the range $-3 \leq \ln \alpha_4 \leq 3$ and in the case of the high allometry data (Fig. 5a) many points fall outside this range. The lines are based on an application of Eqs. 7 and 8 to model output at ages 10, 14 and 18 years.

It is clear that the range of estimates which result from a given observational error is much higher when the observed data upon which these estimates are based are highly allometric than in the absence of allometry (cf. solid and open symbols in Fig. 5a). It is also clear that replication greatly reduces the range of the estimates (cf. Fig. 5a and b). Whereas the estimates are linearly related when the observed data are allometric, this need not be the case in the absence of allometry (e.g. g_{42}, solid symbols). However, the general trends of the estimates in the absence of allometry are again consistent with the pronounced linear relationships obtained in the presence of allometry.

8. Discussion

Parameter estimation in *S*-system models of biochemical, immunological or genetic control systems is usually based on data obtained when the system is in or close to steady state (Savageau, 1976; Voit et

al., 1990; and references therein). Recent work by Torsella and Razali (1991) and Voit and Sands (1995a,b) extends the domain of *S*-systems into ecology, where systems are rarely in steady state and estimation must be based on dynamic data. When developing our *S*-system model of forest growth (Voit and Sands, 1995b) we were handicapped by a paucity of data. We had only three sets of observations of biomass data, and these were inadequate to estimate parameters by fitting predicted biomasses to observed data. However, we did have observed values for all biomass influxes and effluxes corresponding to these biomass observations. This information was the minimum required to estimate parameters in biomass fluxes using the technique of flux-based estimation. Estimation from minimal data sets is important because it is not always possible or feasible to obtain additional data.

Flux-based estimation enables parameters in a flux to be estimated from observations of that flux for each of a number of sets of those variables which influence the flux. Because a flux is a power-law function of these variables, parameters in a flux can be estimated by multiple linear regression based on log-transforms of Eq. 2 (see Section 4.1). The independent variables are the logarithms of the variables in the flux, and the dependent variable is the logarithm of the flux. Ordinary least-squares linear regression assumes all independent variables are error free. This assumption is generally not satisfied in the context of Eq. 2 as F_i^+, F_i^- and the X_j are all observed data and hence estimates based on ordinary least-squares will be biased. Several authors (e.g. Madansky, 1959; Ketellapper, 1983) have devised techniques for estimation when there are errors in the variables, and which, under appropriate conditions, hold bias to a minimum. Similar problems have also been discussed in the context of structured equation modelling (Goldberger and Duncan, 1973; Hayduk, 1987). However, in a practical situation such as our forest growth data it is difficult to determine whether any of these methods can be legitimately used.

This paper ignores questions of bias and, instead, provides a practically feasible method for obtaining some estimates of reasonable quality. Our approach can be justified by arguing that in ecological contexts errors in observed fluxes are usually larger than the errors in observed variables so the assumptions for ordinary least-squares regression are approximately satisfied. Furthermore, if flux-based estimation is used to provide initial values for a conventional nonlinear least-squares parameter estimation technique, questions of bias in the initial estimates are irrelevant.

The use of flux-based parameter estimation breaks parameter estimation into manageable portions. If sufficient observed data are available to apply this method to each flux in the *S*-system, it is in principle possible to determine all parameters in the system. Caution is required as in this process each flux is considered in isolation whereas in the model the fluxes are linked through feedbacks inherent in the structure of the particular *S*-system. Thus, an optimum choice of parameters based on fluxes in isolation may be not be optimal for the *S*-system as a dynamic entity, and flux-based estimation should be complemented by an analysis of the dynamic behaviour of the model or by an application of a traditional parameter estimation technique. We examined the dynamic behaviour of our forest growth model with ESSYNS. A subjective assessment of quality of fit was used to choose final parameter values such that the dynamic behaviour of the system as a whole minimized differences between the predictions of the model and all observed states and fluxes.

Experience with the forest growth model showed that if the observed variables in a flux are allometrically related, flux-based estimation can lead to parameter estimates which are strongly correlated and very sensitive to observational error. Section 4.3 showed that these linear relationships can be determined from the allometric relationship between the variables using Eq. 8. In the case of the forest growth model the observations were from a period during which the state variables were strongly allometric; hence the high sensitivity and correlation of the estimates. The linear relationships between estimates obtained from Eq. 8 provide a powerful guide to the choice of biologically reasonable estimates. This is especially the case if a priori knowledge suggests a value for a single parameter, or bounds on its range.

Various features of flux-based estimation were illustrated using examples based on the *S*-system model of forest growth. *S*-system fluxes are power-

laws, see Eq. 5, and as the analyses of this paper focused on the estimation of the parameters in Eq. 5 the process of flux-based estimation and the conclusions drawn in this paper may also apply to the estimation of power-law relationships in general. However, in the context of *S*-systems the fluxes are embedded in a dynamic system. Successful estimation then requires that the observed dynamic behaviour of the system as a whole must be reproduced. This acts to reduce the ambiguity in acceptable estimates for parameter values that might otherwise be present if the flux were in fact a simple isolated power-law relationship.

Flux-based estimation can be applied to data from a number of distinct treatments observed at the same time (Section 5), or to data sets comprising time series of observations (Section 6), or to a combination of these. If there is the same number of sets of observations as parameters in the flux, the parameters are estimated by solving a set of linear equations. This was the case for our forest growth model. Although our initial estimates (Section 3.1) were biologically unrealistic and sensitive to small errors in the observations, we showed (Section 5) how useful parameter estimates could be obtained with even a minimum of observed data given some a priori knowledge of one of the parameters in the flux. We also illustrated the effects of replication on flux-based estimates (Section 7.1). Neither replication of a few observations, nor the availability of a larger number of observations, could guarantee straightforward estimation when the variables in the flux were allometrically related. However, the relationships obtained from Eq. 8 provide a powerful guide to the choice of biologically reasonable estimates.

Section 3 made reference to problems experienced during the application of conventional techniques for nonlinear estimation to *S*-systems, such as sensitivity of the estimation process to the initial parameter estimates, poor convergence, and convergence to a local minimum. These can be alleviated to some extent if good initial values for the parameters are available, and flux-based estimation can be a valuable tool for obtaining these.

Our experience with estimation in *S*-systems suggests that correlation between parameter estimates may be common. Problems encountered during parameter estimation may also arise because parameter values are correlated, or because of the particular manner in which the model is parameterized. These issues are considered by Ratkowsky (1983, 1990) in the context of common simple analytical models, and he shows that appropriate re-parameterization can lead to more robust estimation. However, his techniques cannot be applied in a routine manner to arbitrary nonlinear, process-based models. In such cases explicitly incorporating such correlations into the model structure may alleviate these difficulties.

We also showed that there may be considerable latitude in the choice of parameters: e.g. equally good fits may be obtained over a moderate range of kinetic orders by suitably choosing the rates. As a result we suggest that convergence problems could also be alleviated by assigning reasonable fixed values to key kinetic orders that are consistent with the observed or presumed dynamics of the system.

Convenient, robust parameter estimation of *S*-systems in an ecological context would be facilitated by the introduction into ESSYNS of a general nonlinear estimation technique that can place bounds on the parameter estimates, or by the introduction in modelling packages such as Scientist (MicroMath, 1995) or ModelMaker (SB Technology, 1995) of modules which explicitly implement *S*-systems and make use of the various numerical efficiencies which result from the canonical structure of *S*-systems. Then, when appropriate data is available, flux-based parameter estimation will provide a systematic process for obtaining the initial estimates required for nonlinear estimation.

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