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A Simultaneous Growth and Yield Model for Loblolly Pine

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Abstract. A model to project cubic-foot volume from initial age, site index, and basal area is developed for natural stands of loblolly pine. For a projection period of zero years, the model predicts current yield and is thus a simultaneous growth and yield model. Statistical procedures to obtain a single estimate of the parameters of this model, using data from remeasured plots, are considered under two assumptions about the error correlations of such data. Parameter estimates for each case are obtained and compared, using data from the Southeastern Forest Experiment Station's loblolly pine stand density study. Parameter estimates for a basal area projection model are also obtained. When taken together, these fitted models are a numerically consistent set of prediction equations for cubic-foot yield, cubic-foot volume projection, and basal area projection. ***Forest Sci. 18:76–86.***

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SOME nine years ago the junior author published a set of equations constituting an algebraically consistent model for basal area and cubic-foot volume development in even-aged loblolly pine stands (*Pinus taeda* L.) (Clutter 1963). This system was based on the consideration that a derivative-integral relationship must exist between the growth function and projection function for such quantities as stand volume and basal area. Several other recent mensurational systems have also been developed from this same premise (Buckman 1962, Curtis 1967, Moser and Hall 1969).

Two principal difficulties arise when actual data from permanent plots are used to estimate the parameters in various equations of such models:

1. The parameters in any one equation are not independent of those in other equations of the system. Unless this dependence is somehow recognized in the estimation process, the equations will not be numerically consistent when the parameter estimates are inserted in the model.
2. Successive measurements of such variables as stand volumes and basal areas on the same plot do not constitute statistically independent observations, and the estimation process should take this into account.

As a means of handling these difficulties, the present study develops a single linear model relating projected stand volume to initial stand age, projected age,

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site index, and initial basal area. If projected age is made equal to initial age, the model simplifies to a conventional yield equation. Procedures for estimating the parameters of the model system are developed and compared.

The Simultaneous Growth and Yield Model

The case we will consider involves two sets of measurements on each of n permanent plots. The models selected as a starting point are those for cubic-foot yield and projected basal area given by Clutter (1963). The yield model is a modification of an equation form suggested by Schumacher (1939) and applied to loblolly pine by MacKinney and Chaiken (1939).

The models for yield at both an initial and a subsequent measurement and for projected basal area can be written

$$E(y_1) = \beta_0 + \beta_1 S + \beta_2 A_1^{-1} + \beta_3 \ln B_1, \quad (1)$$

$$E(y_2) = \beta_0 + \beta_1 S + \beta_2 A_2^{-1} + \beta_3 \ln B_2, \text{ and} \quad (2)$$

$$E(\ln B_2) = (A_1/A_2) \ln B_1 + \alpha_1(1 - A_1/A_2) + \alpha_2(1 - A_1/A_2)S, \quad (3)$$

where

y_i = logarithm to the base e of inside-bark cubic-foot volume per acre of all pine stems including stump and top at the i^{th} measurement,
S = site index in feet,
A_i = stand age in years at the i^{th} measurement,
$\ln B_i$ = logarithm to the base e of basal area per acre in square feet at the i^{th} measurement, and
E indicates expected value.

Clutter (1963) listed the desirable features of this yield model as:

1. "The mathematical form of the variates implies relationships which agree with our biological concepts of even-aged stand development (Schumacher 1939).
2. "The use of $\ln V$ as the dependent variable rather than V will generally be more compatible with the statistical assumptions customarily made in regression analysis (linearity, normality, additivity, and homogeneity of variance).
3. "The use of $\ln V$ as the dependent variate is a convenient way to express mathematically the interaction of the independent variables in their effect on V. (For example, the change in expected volume occurring as a result of a change in site index from 60 to 70 would depend on the associated values of $\ln B$ and A^{-1}.)"

If we replace $\ln B_2$ in eq. (2) by the functional form of its expected value as represented by eq. (3), and simplify the resulting expression, we obtain

$$\begin{aligned} E(y_2) = \beta_0 + \beta_1 S + \beta_2 A_2^{-1} + \beta_3(A_1/A_2) \ln B_1 \\ + \beta_3 \alpha_1(1 - A_1/A_2) + \beta_3 \alpha_2(1 - A_1/A_2)S. \end{aligned} \quad (4)$$

If we let $\beta_4 = \beta_3 \alpha_1$ and $\beta_5 = \beta_3 \alpha_2$, (4) can be written as

$$\begin{aligned} E(y_2) = \beta_0 + \beta_1 S + \beta_2 A_2^{-1} + \beta_3(A_1/A_2) \ln B_1 \\ + \beta_4(1 - A_1/A_2) + \beta_5(1 - A_1/A_2)S. \end{aligned} \quad (5)$$

Eq. (5) is a model for projected cubic-foot volume as a function of site index, initial age, projection age, and initial basal area. When $A_2 = A_1$ (i.e. the projection period is zero years), eq. (5) reduces to the yield model (1), and is thus simultaneously a yield model for the observations at the first measurement and a projection or growth model for subsequent observations.

Given estimates for the parameters of (5), we may write prediction equations for cubic-foot yield as

$$\hat{V} = \exp [\hat{\beta}_0 + \hat{\beta}_1 S + \hat{\beta}_2 A^{-1} + \hat{\beta}_3 \ln B], \tag{6}$$

and for projected cubic-foot volume as

$$\hat{V}_2 = \exp [\hat{\beta}_0 + \hat{\beta}_1 S + \hat{\beta}_2 A_2^{-1} + \hat{\beta}_3 (A_1/A_2) \ln B_1 + \hat{\beta}_4 (1 - A_1/A_2) + \hat{\beta}_5 (1 - A_1/A_2) S]. \tag{7}$$

Recalling the definition of β_4 and β_5, we can obtain estimates of α_1 and α_2 as $\hat{\alpha}_1 = \hat{\beta}_4/\hat{\beta}_3$, and $\hat{\alpha}_2 = \hat{\beta}_5/\hat{\beta}_3$.

Projected basal area is then estimated by

$$\hat{B}_2 = \exp [(A_1/A_2) \ln B_1 + \hat{\alpha}_1 (1 - A_1/A_2) + \hat{\alpha}_2 (1 - A_1/A_2) S]. \tag{8}$$

While these are not the most efficient estimators of α_1 and α_2, this approach leads to a logically consistent system of prediction equations.

Two properties are desired in a set of prediction equations such as (6), (7), and (8). First, the projection equations should give predicted values which are invariant for different combinations of projection lengths. That is, if yield is projected from A_1 to A_2, and then from A_2 to A_3, the result should be identical with a projection from A_1 to A_3. To show that the basal area projection eq. (8) possesses this first property, let us consider a projection of B_3 from A_2, given by

$$\hat{B}_3 = \exp [(A_2/A_3) \ln B_2 + \hat{\alpha}_1 (1 - A_2/A_3) + \hat{\alpha}_2 (1 - A_2/A_3) S].$$

If a projection of B_2 from A_1 is substituted for B_2, we have

$$\hat{B}_3 = \exp \{[(A_1/A_2) \ln B_1 + \hat{\alpha}_1 (1 - A_1/A_2) + \hat{\alpha}_2 (1 - A_1/A_2) S] (A_2/A_3) + \hat{\alpha}_1 (1 - A_2/A_3) + \hat{\alpha}_2 (1 - A_2/A_3) S\}.$$

This will simplify to

$$\hat{B}_3 = \exp [(A_1/A_3) \ln B_1 + \hat{\alpha}_1 (1 - A_1/A_3) + \hat{\alpha}_2 (1 - A_1/A_3) S],$$

which would be obtained by projecting B_3 directly from A_1.

An equivalent invariance property for the cubic-foot volume projection equation is easily derived using an approach similar to that shown above.

The second desirable property is that using projected basal area in the cubic-foot yield equation should be equivalent to using initial basal area in the cubic-foot volume projection equation. This property follows from the derivation of the simultaneous growth and yield model.

The Problem of Correlated Errors

It is well known that data obtained by repeated measurements on an individual contain correlated errors. In practice, these correlations are frequently ignored and parameter estimates are obtained by the ordinary least-squares procedure. The resultant estimates of coefficients are unbiased, but have larger variances than would otherwise be the case. Also, the estimate of variance is biased downward, leading to an underestimate of the residual error.

Rao (1959) considered the estimation of parameters of a linear model from data containing correlated errors and derived estimators requiring an independent estimate of the covariance matrix. Elston and Grizzle (1962) compared Rao's procedure and the least-squares method. They suggested that least squares might, in some cases, provide adequate parameter estimates.

The specific problem of correlated errors in growth and yield data was considered by Swindel (1968). He dealt principally with situations where the pattern of correlations in the data is unknown, and emphasized hypothesis testing and construction of confidence intervals. Swindel suggested that least squares must be given due consideration for parameter estimation because of its simplicity. He noted that this technique is particularly attractive when the least-squares estimators are relatively efficient and parameter estimates rather than confidence intervals are desired.

Methods of Estimation

Case I: Independence. We desire a single estimate of the parameters of the simultaneous growth and yield model, using all observations from the first and second measurements. The simplest method of obtaining these estimates is to ignore any error correlations in the data and use ordinary least-squares procedures. However, this is strictly justified only when all $2n$ observations are mutually independent, which is usually not the case.

The models for the observations at the first and second measurement periods may be conveniently written in matrix notation as

$$E(\underline{y}_1) = X_1 \underline{\beta}, \text{ and} \tag{9}$$

$$E(\underline{y}_2) = X_2 \underline{\beta}, \tag{10}$$

where

- $\underline{y}_i$ is the $n \times 1$ vector of observations on the dependent variable at the i^{th} measurement,
- X_i is the $n \times 6$ $(n > 6)$ matrix of observations on the independent variables at the i^{th} measurement, and
- $\underline{\beta}$ is the 6×1 vector containing the parameters $[\beta_0, \beta_1, \ldots, \beta_5]$.

In (9) and (10) the j^{th} row of X_1 will be given by

$$[1 \quad S_j \quad A_{1j}^{-1} \quad \ln B_{1j} \quad 0 \quad 0 \quad],$$

and the j^{th} row of X_2 will be given by

$$[1 \quad S_j \quad A_{2j}^{-1} \quad (A_{1j}/A_{2j}) \ln B_{1j} \quad (1 - A_{1j}/A_{2j}) \quad (1 - A_{1j}/A_{2j}) S_j].$$

To simplify later derivations, we will assume that ordering of observations by plots is identical in (9) and (10).

If we now let

$$\underline{y}_{(2n\times 1)} = \begin{array}{c} (1) \\ \begin{bmatrix} \underline{y}_1 \\ \underline{y}_2 \end{bmatrix} \end{array} \begin{array}{c} \\ (n) \\ (n) \end{array}, \text{ and } X_{(2n\times 6)} = \begin{array}{c} (6) \\ \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} \end{array} \begin{array}{c} \\ (n) \\ (n) \end{array},$$

we can express (9) and (10) as a single linear model,

$$E(\underline{y}) = X\underline{\beta}. \tag{11}$$

If all the observations are mutually independent with constant variance, then

$$\text{var}\ (\underline{y}) = \sigma^2 I_{(2n \times 2n)}, \tag{12}$$

where I represents an identity matrix, and the appropriate estimator of $\underline{\beta}$ is the familiar least-squares estimator

$$\hat{\underline{\beta}} = (X'X)^{-1}X'\underline{y}. \tag{13}$$

That is, these are the usual estimates obtained by solving the normal equations in a multiple regression analysis.

Given the model (11) and assumption (12), the solution $\hat{\underline{\beta}}$ of (13) is the best (*i.e.*, minimum variance) linear unbiased estimator of $\underline{\beta}$. If, in addition to (11) and (12), we assume that errors about the model are normally distributed, these estimates are also maximum-likelihood estimates of $\underline{\beta}$. Even if (12) is false, the estimator given in (13) is still unbiased since

$$E(\hat{\underline{\beta}}) = E[(X'X)^{-1}X'\underline{y}] = (X'X)^{-1}X'E(\underline{y}) = (X'X)^{-1}X'X\underline{\beta} = \underline{\beta}.$$

Case II: Nonindependence. A more realistic assumption concerning var $(\underline{y})$ would be that the observations from the first measurement have a common variance, say σ_1^2; and that the observations from the second measurement also have a common variance, say σ_2^2. These are mild assumptions since the logarithmic transformation stabilizes the variance among observations within each measurement. We allow these two variances to differ because the model for the second measurement involves a projection of basal area from the first measurement. We further assume that the covariance between different plots is zero, which is guaranteed if the sample plots are randomly and independently selected. Finally, we assume that there is a common covariance between the first and second observations on the same plot.

With these assumptions we may write

$$\text{var}\ (\underline{y}) = \begin{array}{cc} (n) & (n) \\ \begin{bmatrix} \sigma_1^2 I & \sigma_{12} I \\ \sigma_{12} I & \sigma_2^2 I \end{bmatrix} & \begin{array}{c} (n) \\ (n) \end{array} \end{array} \tag{14}$$

where σ_{12} is the common covariance.

Noting that $\sigma_{12} = \rho\sigma_1\sigma_2$, where ρ is the correlation coefficient, we may write (14) as

$$\text{var}\ (\underline{y}) = \begin{bmatrix} \sigma_1^2 I & \rho\sigma_1\sigma_2 I \\ \rho\sigma_1\sigma_2 I & \sigma_2^2 I \end{bmatrix}. \tag{15}$$

Under these conditions, the estimator from (13) is neither the maximum-likelihood estimator of $\underline{\beta}$ nor a best linear unbiased estimator.

An estimator for the more general case, in which ρ is unknown and we allow two distinct variances for the respective measurements, can be derived by applying the principle of maximum likelihood. Assuming a bivariate normal distribution for a pair of observations from a plot, we may write

$$\begin{bmatrix} y_{1j} \\ y_{2j} \end{bmatrix} \sim N\left\{ \begin{bmatrix} x'_{1j}\underline{\beta} \\ x'_{2j}\underline{\beta} \end{bmatrix}, \quad \begin{bmatrix} \sigma_1^2 & \rho\sigma_1\sigma_2 \\ \rho\sigma_1\sigma_2 & \sigma_2^2 \end{bmatrix} \right\}$$

where

y_{ij} = the j^{th} element of $\underline{y}_i$, and
x'_{ij} = the j^{th} row of X_i.

The likelihood function for n pairs of such y's may be written as

$$L(\underline{\beta}, \sigma_1^2, \sigma_2^2, \rho; \underline{y}_1, \underline{y}_2, X_1, X_2) =$$
$$(2\pi)^{-n}[\sigma_1^2\sigma_2^2(1-\rho^2)]^{-n/2} \exp\left\{\frac{-1}{2(1-\rho^2)}\left[\frac{(\underline{y}_1 - X_1\underline{\beta})'(\underline{y}_1 - X_1\underline{\beta})}{\sigma_1^2} - \frac{2\rho(\underline{y}_1 - X_1\underline{\beta})'(\underline{y}_2 - X_2\underline{\beta})}{\sigma_1\sigma_2} + \frac{(\underline{y}_2 - X_2\underline{\beta})'(\underline{y}_2 - X_2\underline{\beta})}{\sigma_2^2}\right]\right\}.$$

If we expand the exponent and take the logarithm of L, we have

$$\ln L = -n \ln (2\pi) - n \ln \sigma_1 - n \ln \sigma_2 - n/2 \ln (1-\rho^2) - \frac{1}{2(1-\rho^2)}\left[\frac{\underline{y}'_1\underline{y}_1 - 2\underline{\beta}'X'_1\underline{y}_1 + \underline{\beta}'X'_1X_1\underline{\beta}}{\sigma_1^2} - \frac{2\rho(\underline{y}'_1\underline{y}_2 - \underline{\beta}'X'_1\underline{y}_2 - \underline{\beta}'X'_2\underline{y}_1 + \underline{\beta}'X'_1X_2\underline{\beta})}{\sigma_1\sigma_2} + \frac{\underline{y}'_2\underline{y}_2 - 2\underline{\beta}'X'_2\underline{y}_2 + \underline{\beta}'X'_2X_2\underline{\beta}}{\sigma_2^2}\right]. \tag{16}$$

By differentiating (16) with respect to the vector $\underline{\beta}$, and equating the result to the null vector $\underline{0}$, we can solve for $\hat{\underline{\beta}}$ as

$$\hat{\underline{\beta}} = [(1/\hat{\sigma}_1^2)X'_1X_1 + (1/\hat{\sigma}_2^2)X'_2X_2 - (\hat{\rho}/\hat{\sigma}_1\hat{\sigma}_2)(X'_1X_2 + X'_2X_1)]^{-1} [(1/\hat{\sigma}_1^2)X'_1\underline{y}_1 + (1/\hat{\sigma}_2^2)X'_2\underline{y}_2 - (\hat{\rho}/\hat{\sigma}_1\hat{\sigma}_2)(X'_1\underline{y}_2 + X'_2\underline{y}_1)]. \tag{17}$$

Similarly, we obtain

$$\hat{\sigma}_1^2 = (1/n)(\underline{y}'_1\underline{y}_1 - 2\hat{\underline{\beta}}'X'_1\underline{y}_1 + \hat{\underline{\beta}}'X'_1X_1\hat{\underline{\beta}}), \tag{18}$$

$$\hat{\sigma}_2^2 = (1/n)(\underline{y}'_2\underline{y}_2 - 2\hat{\underline{\beta}}'X'_2\underline{y}_2 + \hat{\underline{\beta}}'X'_2X_2\hat{\underline{\beta}}), \tag{19}$$

and

$$\hat{\rho} = \hat{\sigma}_{12}/(\hat{\sigma}_1\hat{\sigma}_2), \tag{20}$$

where

$$\hat{\sigma}_{12} = (1/n)(\underline{y}'_1\underline{y}_2 - \hat{\underline{\beta}}'X'_1\underline{y}_2 - \hat{\underline{\beta}}'X'_2\underline{y}_1 + \hat{\underline{\beta}}'X'_1X_2\hat{\underline{\beta}}). \tag{21}$$

To show that the estimator of $\underline{\beta}$ given by (17) is unbiased, let us consider σ_1^{2*}, σ_2^{2*}, and ρ^* to be any numerical guesses of $\hat{\sigma}_1^2$, $\hat{\sigma}_2^2$, and $\hat{\rho}$, respectively, where σ_1^{2*} and σ_2^{2*} are not equal to zero. Then

$$E(\underline{\beta}) = E\{[(1/\sigma_1^{2*})X'_1X_1 + (1/\sigma_2^{2*})X'_2X_2 - (\rho^*/\sigma_1^*\sigma_2^*)(X'_1X_2 + X'_2X_1)]^{-1} [(1/\sigma_1^{2*})X'_1\underline{y} + (1/\sigma_2^{2*})X'_2\underline{y}_2 - (\rho^*/\sigma_1^*\sigma_2^*)(X'_1\underline{y}_2 + X'_2\underline{y}_1)]\}$$
$$= [(1/\sigma_1^{2*})X'_1X_1 + (1/\sigma_2^{2*})X'_2X_2 - (\rho^*/\sigma_1^*\sigma_2^*)(X'_1X_2 + X'_2X_1)]^{-1} \{(1/\sigma_1^{2*})X'_1E(\underline{y}_1) + (1/\sigma_2^{2*})X'_2E(\underline{y}_2) - (\rho^*/\sigma_1^*\sigma_2^*)[X'_1E(\underline{y}_2) + X'_2E(\underline{y}_1)]\}$$

TABLE 1. Comparison of predicted cubic-foot yields for site index 90 using the different estimates of $\underline{\beta}$. Tabled values are the difference of predictions for Case I less predictions for Case II expressed in cubic feet.

Basal area	Age (years)						
	25	30	35	40	45	50	55
50	−17	−21	−24	−26	−28	−30	−32
70	− 7	−10	−12	−14	−16	−17	−18
90	6	5	4	3	3	2	1
110	22	23	25	24	24	26	25
130	41	44	47	49	50	52	53

$$= [(1/\sigma_1^{2*})X'_1X_1 + (1/\sigma_2^{2*})X'_2X_2 - (\rho^*/\sigma_1^*\sigma_2^*)(X'_1X_2 + X'_2X_1)]^{-1}$$
$$[(1/\sigma_1^{2*})X'_1X_1 + (1/\sigma_2^{2*})X'_2X_2 - (\rho^*/\sigma_1^*\sigma_2^*)(X'_1X_2 + X'_2X_1)]\underline{\beta}$$
$$= \underline{\beta}.$$

Since the estimator (17) is unbiased for any guesses of $\hat{\sigma}_1^2$, $\hat{\sigma}_2^2$ and $\hat{\rho}$, it is clearly unbiased if the maximum-likelihood estimates of these parameters are used.

An iterative procedure must be used to obtain numerical estimates. This is conveniently done by making initial guesses of $\hat{\sigma}_1^2$, $\hat{\sigma}_2^2$, and $\hat{\rho}$, and then solving for $\hat{\underline{\beta}}$ by eq. (17). (Setting $\hat{\sigma}_1^2$ equal to $\hat{\sigma}_2^2$ and $\hat{\rho}$ equal to zero implies the assumptions necessary for the least-squares estimators to be the maximum-likelihood estimators. If this is done, the result of the first calculation of $\hat{\underline{\beta}}$ will be identical to the least-squares estimate.) This estimate of $\hat{\underline{\beta}}$ is then used in eq. (18), (19), (20), and (21) to obtain estimates of σ_1^2, σ_2^2, and ρ. The latter estimates are then used to obtain a new estimate of $\underline{\beta}$, and this procedure is continued until the estimates of the elements of $\underline{\beta}$ no longer change (to the desired number of significant figures).

Application of Model and Estimation Techniques

Source of Data. The data used to illustrate application are those used by Clutter (1963). Wenger *et al.* (1958) and Nelson (1960) have also published analyses based on portions of these data.

These data are provided by 5- and 10-yr remeasurements of 102 thinned loblolly pine plots which are part of a large permanent plot study installed by the Southeastern Forest Experiment Station. The plots are located on the Hitchiti, Camp, Santee, and Westvaco Experimental Forests in Georgia, Virginia, and South Carolina. The plots, 0.25 acre in size and surrounded by 33-ft isolation strips, were selectively located to uniformly cover ranges of age, site, and density. At the time of the 5-yr remeasurements, the stand age varied from 21 to 69 yr; site index from 53 to 110; and basal area from 30 to 154 sq ft per acre.

Parameter Estimates. Using these data, the parameters of the simultaneous growth and yield model were estimated by the procedures appropriate to both variance models. The least-squares estimates for Case I give

$$\hat{y}_{LS} = 2.7600 + .014275S - 21.214A_2^{-1} + .98109(A_1/A_2)\ln B_1 + 3.4863(1 - A_1/A_2) + .024368(1 - A_1/A_2)S, \quad (22)$$

with $\hat{\sigma}_{LS}^2 = .0041782$.

The maximum-likelihood estimates for Case II give

$$\hat{y}_{ML} = 2.8837 + .014441S - 21.326A_2^{-1} + .95064(A_1/A_2)\ln B_1 + 3.2649(1 - A_1/A_2) + .025428(1 - A_1/A_2)S, \quad (23)$$

with $\hat{\sigma}^2_{1ML} = .0017866$, $\hat{\sigma}^2_{2ML} = .0065477$, and $\hat{\rho}_{ML} = .34811$.

The maximum-likelihood estimates were stable to five significant figures after six iterations, using initial guesses of $\hat{\sigma}_1^2 = \hat{\sigma}_2^2 = 1$, and $\hat{\rho} = 0$.

Yield equations using each set of estimates accounted for 99.2 percent of the variation around the mean cubic-foot volume on the 102 plots at the first measurement. Projection equations using the estimates from Case I and II, respectively, accounted for 98.0 and 97.9 percent of the variation at the second measurement. Table 1 compares predicted yields for site index 90, using the different estimates of β.

When confronted with estimation procedures as different in complexity as those considered here, it is natural to ask if the more involved procedure gives estimates which are really better in a practical sense, and if there are practical differences in the resulting equations. These are difficult questions, and more experience with similar models and other data will be required before final conclusions can be drawn. Obviously the assumptions of Case II are more reasonable and these estimates are theoretically superior. The differences, *in this case*, are probably not of practical significance for most applications. However, there is no guarantee that this same situation would occur with other bodies of data and, hence, use of the maximum-likelihood procedure given in Case II would always seem the prudent way to proceed.

The maximum-likelihood estimates for Case II will be used to illustrate features of the simultaneous growth and yield model, since the assumptions underlying their derivation are more tenable. First, we will obtain parameter estimates for basal area projection and compare the resulting equation with that obtained by Clutter (1963) using a least-squares fit of the same model. Recalling that β_4 was defined as $\beta_3\alpha_1$, and β_5 was defined as $\beta_3\alpha_2$, where α_1 and α_2 are the parameters of the basal area projection model, we can obtain estimates of α_1 and α_2 as

$$\hat{\alpha}_1 = \hat{\beta}_4/\hat{\beta}_3 = 3.4344, \qquad \text{and} \qquad \hat{\alpha}_2 = \hat{\beta}_5/\hat{\beta}_3 = .026748.$$

Projected basal area is then estimated by

$$\hat{B}_2 = \exp\left[(A_1/A_2)\ln B_1 + 3.4344(1 - A_1/A_2) + .026748(1 - A_1/A_2)S\right]. \quad (24)$$

To evaluate these estimates, predicted values from eq. (24) were compared with predicted values using the equation from Clutter's (1963) paper given by

$$\dot{B}_2 = \exp\left[(A_1/A_2)\ln B_1 + 4.6012(1 - A_1/A_2) + .013597(1 - A_1/A_2)S\right]. \quad (25)$$

Eq. (24) accounted for 96.6 percent of the variation about mean B_2 for the 102 loblolly plots, while eq. (25) accounted for 98.6 percent of the variation about mean B_2. It is apparent that eq. (24) is acceptable for basal area projection.

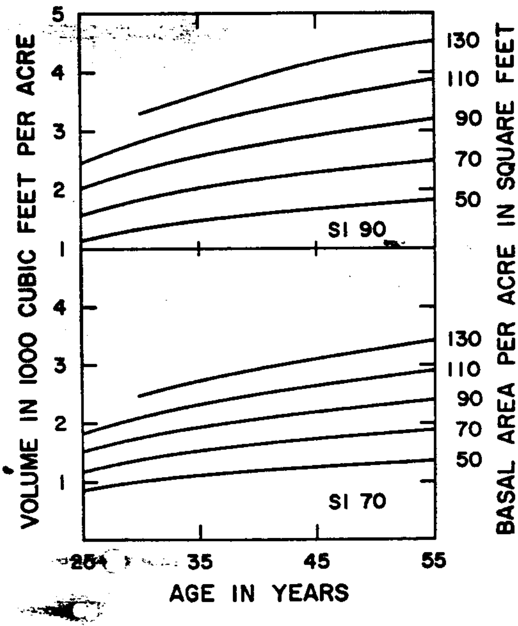

FIGURE 1. *Expected total cubic-foot yields at various ages and densities for site indices 70 and 90.*

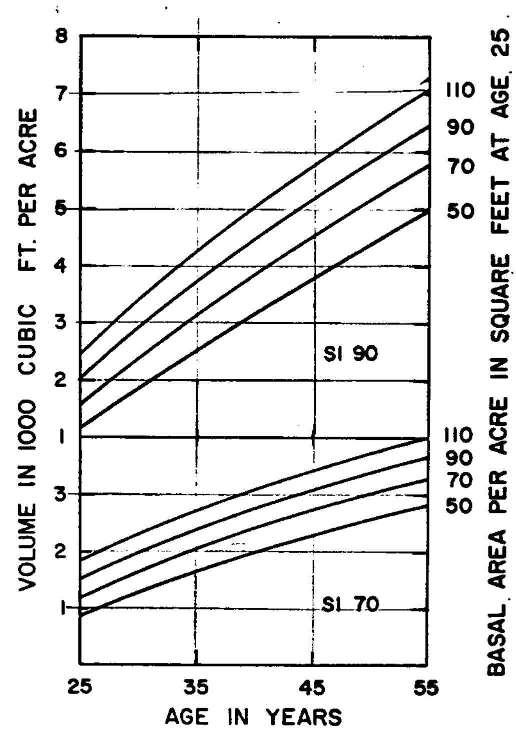

FIGURE 2. *Projected per acre cubic-foot volume at various ages and initial densities for site indices 70 and 90.*

The final equations may be presented as follows:

1. Cubic-foot yield is given by

ln V = 2.5576 + .0151085 S − 21.863 A⁻¹ + .95931 (ln B)

$$\hat{V} = \exp\,[2.8837 + .014441S - 21.326A^{-1} + .95064 \ln B]. \quad (26)$$

2. Projected cubic-foot volume is given by

$$\hat{V}_2 = \exp\,[2.8837 + .014441S - 21.326A_2^{-1} + .95064(A_1/A_2)\ln B_1 + 3.2649(1 - A_1/A_2) + .025428(1 - A_1/A_2)S]. \quad (27)$$

3. Projected basal area is given by

$$\hat{B}_2 = \exp\,[(A_1/A_2)\ln B_1 + 3.4344(1 - A_1/A_2) + .026748(1 - A_1/A_2)S]. \quad (28)$$

To illustrate the use of these equations, assume that information for a particular stand indicates that it is on site index 70 land with 100 sq ft of basal area per acre and a current age of 25 yr.

The predicted basal area at age 35 may be calculated from (28) as

$$\hat{B}_{35} = \exp\,[(25/35)\ln(100) + 3.4344(1 - 25/35) + .026748(1 - 25/35)70]$$
$$= 122.2 \text{ sq ft}$$

This projected basal area may be used in eq. (26) to predict the cubic-foot volume at age 35 as

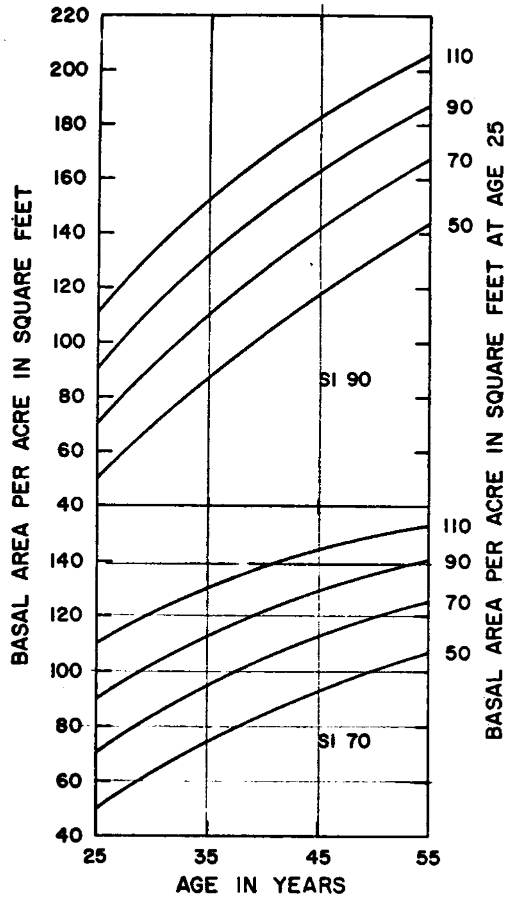

FIGURE 3. *Projected per acre basal areas at various ages and initial densities for site indices 70 and 90.*

$$\hat{V}_{35} = \exp\,[2.8837 + .014441(70) - 21.326/35 + .95064 \ln (122.2)]$$
$$= 2575 \text{ cu ft}$$

Alternatively, we may calculate cubic-foot volume at age 35 directly from the projection eq. (27) to obtain

$$\hat{V}_{35} = \exp\,[2.8837 + .014441(70) - 21.326/35 + .95064(25/35) \ln (100) + 3.2649(1-25/35) + .025428(1-25/35)70]$$
$$= 2575 \text{ cu ft}$$

Figure 1 shows solutions of eq. (26) for various ages and basal areas, and site indices of 70 and 90. Figure 2 gives solutions of eq. (27) showing the expected development of cubic-foot volume in stands of various initial basal areas for site indices 70 and 90. Figure 3 presents solutions of eq. (28) depicting expected development of basal area in stands of various initial basal areas for site indices 70 and 90.

Extension to More Than Two Data Sets

The estimation techniques developed here can be readily extended to cases in which data from more than two measurements are available. If, for example, we had data from three measurements we might use those from the first period as yield information, and those from the second and third periods as projections from the first period.

The derivation of the maximum-likelihood estimates for Case II would be more complicated with three data sets than with two. In addition to dealing with a three-variable distribution, a reasonable assumption about the variances of these observations would involve at least two covariances and three variances. For example, if the three sets of measurements were taken at equal intervals, we might assume a constant covariance between the first and second and second and third observations on the same plot; but we would assume another distinct covariance between the first and third observations on the same plot. As the number of data sets grow larger, the derivation of the maximum-likelihood estimates will become increasingly difficult.

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