

LATERAL STABILITY OF BEAMS WITH INITIAL IMPERFECTIONS

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ABSTRACT: The design of large flat-roof systems necessitates a means of analyzing the required strength of bracing members. If the braced beam is initially perfectly straight, it will remain straight until it buckles; the prebuckling stresses in the braces are theoretically zero. If the braced beam has initial imperfections, they will grow under load and so will the stresses in the braces. Thus, there is need to analyze the buckling of beams with initial imperfections. This was done in the context of linear small-deflection theory; the results are simple and suitable for use in design where loads are well below buckling and initial deformations are small. This paper presents a variational derivation of a set of linear differential equations and boundary conditions and applies them to the calculation of forces in midspan diagonal bracing members on a simply supported beam under constant bending. A sample calculation for a typical wood roof application shows the forces in the bracing to be quite small. Only rectangular beams are treated in detail, but the extension to singly symmetric I-beams is sketched.

INTRODUCTION

The widespread use of large flat-roof systems raises questions about the need for lateral bracing and the required strength of bracing members when needed. It is easier to answer questions about the required stiffness and spacing of such restraints and several classical eigenvalue analyses have been performed (3). These analyses contain stiffnesses as coefficients and predict the buckling load as a function of stiffness and geometry. The required strength of bracing members is another matter. Linear small-deflection theory is silent about postbuckling forces, and the prebuckling forces in elastic restraints are theoretically zero since the member is assumed to be initially straight. Thus it becomes apparent that there is a need for a treatment of imperfections.

In the presence of initial imperfections, deflections and stresses grow continuously with load and equilibrium is always stable. The phenomenon of "buckling" appears as the indefinite growth of deflections and stresses as the load approaches its critical value. If there are bracing members present, they will exert an elastic restraint on the beam and the restraint deflections and forces will grow continuously with load and

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approach infinity at buckling. It thus becomes possible to solve for non-zero prebuckling forces in the braces.

While a complete treatment of the problem would employ nonlinear large-deflection theory, it was decided here to pursue a course similar to the small-deflection analysis of columns under eccentric load. Thus the present results resemble the so-called "secant" formula for columns in that prebuckling stresses and deflections approach infinity as the load asymptotically approaches its classical eigenvalue. While this may be physically unrealistic it can nevertheless provide suitable design information at loads well below buckling if the initial deformations are small. Several investigators have presented solutions to one or two cases of simply supported beams with initial imperfections, namely constant bending (2,6,7,8) and concentrated center load (2,8,10). They all presented their differential equations without derivation or attribution other than to Timoshenko's book (9) which does not consider initial imperfections. Their differential equations are herein rigorously derived and generalized.

The method employed is variational. It is similar to that of Bleich (1) for perfect beams except that the treatment of work done by initial stresses is new. Part I shows the derivation for rectangular beams with initial imperfections and sketches the extension to singly symmetric I-beams. Part II discusses the application to required strength of bracing.

PART I. THEORY

Figure 1 shows the member, coordinate axes x (longitudinal), y (vertical), z (lateral), the corresponding displacements of the centroidal axis $v(x)$ (vertical), $w(x)$ (lateral), and angle of twist $\beta(x)$. The initial imperfections are a lateral deformation (bow) and twist:

$$w_i(x)$$

$$\beta_i(x) \dots \dots \dots (1)$$

The total deformations under load are the sum of the initial deformations and the deflections due to load

$$w_t(x) = w_i(x) + w(x) \dots \dots \dots (2)$$

$$\beta_t(x) = \beta_i(x) + \beta(x) \dots \dots \dots (3)$$

The vertical deflections due to load are assumed to be much less than the lateral deflections w and $c\beta$, and will be ignored. This assumption is conservative, since their effect is known to be stabilizing.

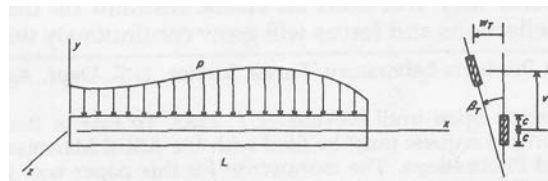


FIG. 1.—Sketch of Beam Showing Axes and Displacement Sign Convention. Initial Imperfections Not Shown

The well-known treatise by Bleich (1) discusses the lateral stability of initially perfect beams and employs a key simplifying assumption: That the vertical displacement of the centroidal axis of the beam remains constant during movement from the prebuckled but critically loaded state to the buckled state. This reasonable-looking assumption eliminates the vertical displacement from his formulation and produces a pair of differential equations involving only the lateral displacement of the centroidal axis and twist. Since Bleich imposes equilibrium by minimizing the total potential energy, this assumption can be viewed in terms of function space, i.e., the search for a minimum is constrained to a subspace in which the vertical displacement is held constant. The resulting minimum is not exact but only the best approximation within the subspace. The magnitude of the resulting error is known to be small, especially if the beam is given an initial camber, as is common in timber construction.

Since the search was confined to a subspace, it was convenient to choose a datum within the subspace: The prebuckled loaded state. As that was not a stress-free datum, it was necessary to include in the total energy the work done by the datum-state stresses in going from datum to buckled equilibrium. Bleich does not treat this work as part of the strain energy; instead he considers it to be part of the potential energy of the load. In this paper it will be considered as strain energy and will be evaluated in a straightforward manner by integrating the product of datum stress and incremental strain (11).

In the presence of initial imperfections there is no neutral equilibrium corresponding to the critical load; instead, the lateral deflection increases continuously as vertical load is applied and becomes indeterminate at the critical load. We are forced to analyze the stable equilibrium state at subcritical load. Nevertheless, Bleich's key assumption can be adapted to this case:

1. Assume that the vertical displacement is the same as would have occurred were the beam initially perfect.
2. Constrain the minimization process to a subspace in which vertical displacement and associated stresses are constant and equal to the value assumed in 1.
3. Choose a datum in this subspace. The most convenient datum is one in which the lateral bending moment and twisting moment are zero, i.e. in which the lateral displacement and twist have their initial values. Note this is not an equilibrium state, but that is all right since the choice of datum for potential energy is entirely arbitrary.

Strain Energy.—The strain energy associated with lateral bending and twist is

$$U_1 = \frac{1}{2} \int_0^L EI_y (w'')^2 dx + \frac{1}{2} \int_0^L GK (\beta')^2 dx \dots\dots\dots (4)$$

in which EI_y = lateral flexural rigidity, GK = torsional rigidity, w = lateral deflection of centroidal axis, β = angle of twist, x = axial coordinate, and primes denote differentiation with respect to x . This is the usual elementary form, due to the choice of datum in which lateral bending

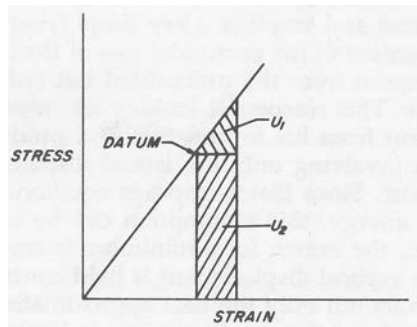


FIG. 2.—Diagrammatic Sketch of Strain Energy when Datum State is Not Stress Free

and twist are initially stress free. The strain energy of vertical bending is constant in this subspace and can be ignored. The work done by the datum stresses in going from datum to equilibrium must be accounted for. Call it U_2 . Then

$$\text{total strain energy} = U_1 + U_2 \dots\dots\dots (5)$$

This is sketched diagrammatically in Fig. 2. The next few sections are aimed at obtaining an expression for U_2 .

Euler-Bernoulli Assumption. For the purpose of calculating U_2 the warping of the cross section due to shear and torsion can be ignored. Then plane transverse cross sections remain plane and the displacements $\bar{u}$, $\bar{v}$, $\bar{w}$ of a material point at x, y, z can be related to the deformations as follows (subscript t is total, D is datum):

$$\bar{u}_t(x, y, z) = -yv'_t - zw'_t \approx -zw'_t \dots\dots\dots (6)$$

$$\bar{v}_t(x, y, z) = v_t - z\beta_t \approx -z\beta_t \dots\dots\dots (7)$$

$$\bar{w}_t(x, y, z) = w_t + y\beta_t \dots\dots\dots (8)$$

$$\bar{u}_D(x, y, z) = -yv'_D - zw'_D \approx -zw'_D \dots\dots\dots (9)$$

$$\bar{v}_D(x, y, z) = v_D - z\beta_D \approx -z\beta_D \dots\dots\dots (10)$$

$$\bar{w}_D(x, y, z) = w_D + y\beta_D \dots\dots\dots (11)$$

Datum Stresses.—These are due solely to vertical deflections. They are

$$\sigma_x^0 = -\frac{M_z y}{I_z} \dots\dots\dots (12)$$

$$\tau_{xy}^0 = -\frac{3}{2A} M'_z \left(1 - \left(\frac{y}{c} \right)^2 \right) \dots\dots\dots (13)$$

in which M_z is the bending moment about the z axis due to vertical loading, I_z = principal moment of inertia (about the z axis), A = cross sectional area, and c = half depth.

Incremental Strain.—Since U_1 is a second order quantity (stress and strain are both small) U_2 must also be accurate to second order. But U_2

is the product of datum stress and incremental strain, and the datum stresses are finite. Thus the incremental strain must be accurate to the second order. Therefore we use the Lagrangian strain-displacement relations:

$$\epsilon_x = \frac{\partial \bar{u}}{\partial x} + \frac{1}{2} \left(\frac{\partial \bar{u}}{\partial x} \right)^2 + \frac{1}{2} \left(\frac{\partial \bar{v}}{\partial x} \right)^2 + \frac{1}{2} \left(\frac{\partial \bar{w}}{\partial x} \right)^2 \dots \dots \dots (14)$$

$$\gamma_{xy} = \frac{\partial \bar{v}}{\partial x} + \frac{\partial \bar{u}}{\partial y} + \frac{\partial \bar{u}}{\partial x} \frac{\partial \bar{u}}{\partial y} + \frac{\partial \bar{v}}{\partial x} \frac{\partial \bar{v}}{\partial y} + \frac{\partial \bar{w}}{\partial x} \frac{\partial \bar{w}}{\partial y} \dots \dots \dots (15)$$

Since $\bar{u}$ is in the direction of the long axis of the member, $\bar{u}$ is very much less than $\bar{v}$ and $\bar{w}$. Thus products of derivatives of $\bar{u}$ will be neglected in Eqs. 14 and 15 for strain. This follows Von Karman's treatment of the large deflection of plates (4).

Using Eqs. 6–11 in Eqs. 14 and 15, and taking the difference between total strain and datum strain yields the following incremental strains:

$$\epsilon_x = -zw'' + \frac{1}{2} [z^2(\beta_i')^2 - z^2(\beta_i)^2 + (w_i' + y\beta_i')^2 - (w_i' + y\beta_i)^2] \dots \dots \dots (16)$$

$$\gamma_{xy} = -z\beta' + (w_i' + y\beta_i')\beta_i - (w_i' + y\beta_i)\beta_i \dots \dots \dots (17)$$

Calculation of U_2

$$U_2 = \int_0^L \int_A (\sigma_x^o \epsilon_x + \tau_{xy}^o \gamma_{xy}) dA \, dx \dots \dots \dots (18)$$

$$= - \int_0^L [w_i'(M_z \beta_i)' - w_i'(M_z \beta_i)'] dx \dots \dots \dots (19)$$

after substituting Eqs. 12, 13, 16, and 17 into Eq. 18 and integrating over the cross sectional area. This agrees with Bleich if w_i and β_i are set to zero and the remaining term is integrated by parts

$$- \int_0^L w_i'(M_z \beta_i)' dx = \int_0^L w_i'' M_z \beta_i dx - [w_i' M_z \beta_i]_0^L \dots \dots \dots (20)$$

Bleich does not get this integrated term in Eq. 20 because he derived his result with the assumption of simple support at each end. Note that Eq. 19 is correct for any boundary condition however.

Potential Energy of Load. Let p be a downward distributed load on the top of the beam. The change in potential energy of the load in going from datum to equilibrium is due only to the change in twist

$$U_L = - \int_0^L pc(\cos \beta_i - \cos \beta_i) dx \approx - \int_0^L \frac{1}{2} pc(\beta_i^2 - \beta_i^2) dx \dots \dots \dots (21)$$

$$U = U_1 + U_2 + U_L \dots \dots \dots (22)$$

$$= \frac{1}{2} \int_0^L [EI_y(w'')^2 + GK(\beta')^2 - 2w_i'(M_z \beta_i)' + 2w_i'(M_z \beta_i)' - pc(\beta_i^2 - \beta_i^2)] dx \dots \dots \dots (23)$$

This result can be made to agree with earlier investigators (2,6,7,10) after some integration by parts and assumption of appropriate boundary conditions (except for some terms whose first variation is zero). Those results were stated without derivation and evidently obtained by adding appropriate terms to Bleich's result without noting the restriction on boundary conditions.

Differential Equations and Boundary Conditions. Equilibrium can now be imposed by setting the first variation of U to zero

$$\delta U = 0 \dots\dots\dots (24)$$

This yields, after two integrations by parts, the following Euler-Lagrange differential equations:

$$EI_y w'' + M_z(\beta + \beta_i) = C_1 x + C_2 \dots\dots\dots (25)$$

$$GK\beta'' - M_z(w'' + w_i'') + pc(\beta + \beta_i) = 0 \dots\dots\dots (26)$$

and boundary conditions

$$\{EI_y w''' + [M_z(\beta + \beta_i)]'\} \delta w|_0^L = C_1 \delta w|_0^L = 0 \dots\dots\dots (27)$$

$$w'' \delta w'|_0^L = 0 \dots\dots\dots (28)$$

$$[GK\beta' - M_z(w' + w_i')] \delta \beta|_0^L = 0 \dots\dots\dots (29)$$

i.e., six conditions for C_1 , C_2 , and four other integration constants. In Eqs. 27 to 29 the symbol $|_0^L$ indicates that everything preceding that symbol is to be evaluated at $x = L$ in the first of each pair of equations and at $x = 0$ in the second.

Extension to I-Beams. With a few modifications the preceding result can be extended to I-beams. Suppose the cross section is a singly symmetric I shape as shown in Fig. 3. The origin is at the centroid, and dimensions a_1 , a_2 , b_1 , b_2 , c_1 , c_2 , d , t_1 , t_2 , and t_3 are defined on the figure.

The term U_1 , which represents the strain energy of lateral bending

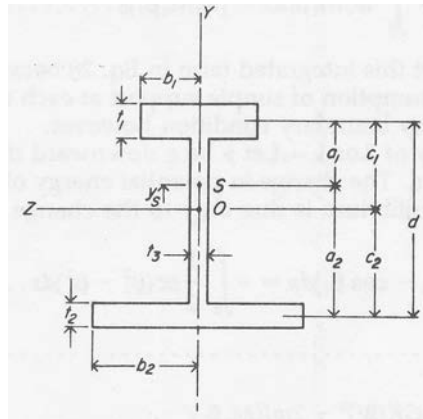


FIG. 3. — Cross Section of Singly Symmetric I-Beam. S is the Shear Center (Center of Rotation) and C is the Centroid

and twist, must here be augmented to account for an additional torsion term first discussed by Timoshenko (8) and due to restraints against warping of the cross section. The complete expression for U_1 becomes

$$U_1 = \frac{1}{2} \int_0^L [EI_y (w'')^2 + GK(\beta')^2 + E\Gamma(\beta'')^2] dx \dots\dots\dots (30)$$

in which, for the section shown in Fig. 3, Γ is

$$\Gamma = \frac{d^2 t_2 b_2^3 t_1 b_1^3}{t_2 b_2^3 + t_1 b_1^3} \dots\dots\dots (31)$$

Equation 18 for U_2 must be augmented by a term for the horizontal shear in the flanges

$$U_2 = \int_0^L \int_A (\sigma_x^o \epsilon_x + \tau_{xy}^o \gamma_{xy} + \tau_{xz}^o \gamma_{xz}) dA dx \dots\dots\dots (32)$$

The horizontal shear strain in the flanges is derived by using the Lagrangian strain-displacement relation

$$\gamma_{xz} = \frac{\partial \bar{w}}{\partial x} + \frac{\partial \bar{u}}{\partial z} + \frac{\partial \bar{u}}{\partial x} \frac{\partial \bar{u}}{\partial z} + \frac{\partial \bar{v}}{\partial x} \frac{\partial \bar{v}}{\partial z} + \frac{\partial \bar{w}}{\partial x} \frac{\partial \bar{w}}{\partial z} \dots\dots\dots (33)$$

where the product of derivatives of $\bar{u}$ will be neglected as before. Using Eqs. 6-11 in Eq. 33 and taking the difference between total strain and datum strain, the incremental shear strain becomes

$$\gamma_{xz} = y\beta'_i + z\beta'_i - y\beta'_i - z\beta'_i \dots\dots\dots (34)$$

The datum stresses are

$$\sigma_x^o = -\frac{M_z y}{I_z} \dots\dots\dots (35)$$

$$\tau_{xy}^o = \left\{ \begin{array}{l} -\frac{M'_z}{I_z} \left[2b_1 c_1 \frac{t_1}{t_3} + \frac{1}{2} (c_1^2 - y^2) \right], \quad 0 \leq y \leq c_1 \\ -\frac{M'_z}{I_z} \left[2b_2 c_2 \frac{t_2}{t_3} + \frac{1}{2} (c_2^2 - y^2) \right], \quad -c_2 \leq y \leq 0 \\ 0, \quad \text{in flanges} \end{array} \right\} \dots\dots\dots (36)$$

$$\tau_{xz}^o = \left\{ \begin{array}{l} -\frac{M'_z}{I_z} c_1 (b_1 - z), \quad 0 \leq z \leq b, \quad y = c_1 \\ \frac{M'_z}{I_z} c_1 (b_1 + z), \quad -b_1 \leq z \leq 0, \quad y = c_1 \\ \frac{M'_z}{I_z} c_2 (b_2 - z), \quad 0 \leq z \leq b_2, \quad y = -c_2 \\ -\frac{M'_z}{I_z} c_2 (b_2 + z), \quad -b_2 \leq z \leq 0, \quad y = -c_2 \\ 0, \quad \text{in web} \end{array} \right\} \dots\dots\dots (37)$$

Putting these datum stresses and the incremental strains, Eqs. 16, 17, and 34 into Eq. 32 for U_2 and integrating over the cross section area yields

$$U_2 = - \int_0^L \{ (w'_i + \kappa \beta'_i)(M_z \beta_i)' - (w'_i + \kappa \beta'_i)(M_z \beta_i) \} dx \dots\dots\dots (38)$$

$$\text{in which } \kappa \equiv \frac{1}{2I_z} \int_A y(y^2 + z^2) dA$$

$$= \frac{\left\{ b_1 c_1^3 t_1 - b_2 c_2^3 t_2 + \frac{t_3}{8} (c_1^4 - c_2^4) + \frac{1}{3} (b_1^3 c_1 t_1 - b_2^3 c_2 t_2) \right\}}{I_z} \dots\dots\dots (39)$$

Finally, the expression for the potential energy of the load U_L must be modified since the center of rotation in this case is the shear center rather than the centroid. The distance to the shear center is given by

$$y_s = \frac{t_1 b_1^3 c_1 - t_2 b_2^3 c_2}{t_1 b_1^3 + t_2 b_2^3} \dots\dots\dots (40)$$

and since a_1 denotes the distance from the shear center to the top flange, U_L becomes

$$U_L = - \frac{1}{2} \int_0^L p a_1 (\beta_i^2 - \beta_i'^2) dx \dots\dots\dots (41)$$

The total potential energy is

$$U = U_1 + U_2 + U_L = \frac{1}{2} \int_0^L [EI_y (w'')^2 + GK (\beta')^2 + E\Gamma (\beta'')^2 - 2(w'_i + \kappa \beta'_i)(M_z \beta_i)' + 2(w'_i + \kappa \beta'_i)(M_z \beta_i) - p a_1 (\beta_i^2 - \beta_i'^2)] dx \dots\dots\dots (42)$$

and the Euler-Lagrange differential equations and boundary conditions are

$$EI_y w'' + M_z \beta_i = C_1 x + C_2 \dots\dots\dots (43)$$

$$E\Gamma \beta^{IV} - GK \beta'' + M_z w'' + \kappa [M_z \beta_i'' + (M_z \beta_i)'] - p a_1 \beta_i = 0 \dots\dots\dots (44)$$

$$[EI_y w''' + (M_z \beta_i)'] \delta w|_0^L \equiv C_1 \delta w|_0^L = 0 \dots\dots\dots (45)$$

$$w'' \delta w'|_0^L = 0 \dots\dots\dots (46)$$

$$[E\Gamma \beta''' - GK \beta' + M_z w' + \kappa M_z' \beta_i] \delta \beta|_0^L = 0 \dots\dots\dots (47)$$

$$\beta'' \delta \beta'|_0^L = 0 \dots\dots\dots (48)$$

i.e., eight conditions for C_1 , C_2 , and six other integration constants.

This result can be compared with that of Gjelsvik (5) who considered an unsymmetric I-beam without initial imperfections. The agreement is close but not exact. Instead of the term

$$\kappa [M_z \beta'' + (M_z \beta)'] \dots\dots\dots (49)$$

Gjelsvik gets (in my notation)

$$2\kappa(M_z\beta')' \dots\dots\dots (50)$$

$$\text{Since } M_z\beta'' + (M_z\beta)' = M_z''\beta + 2(M_z\beta')' \dots\dots\dots (51)$$

it can be seen that the net difference is a term

$$\kappa M_z\beta'' \dots\dots\dots (52)$$

present here but not in Gjelsvik's result. This difference is due to the different simplifying assumptions used in the two derivations.

PART II. APPLICATION

The result derived above is here applied to the calculation of the forces induced in diagonal bracing between the bottom edge of a beam and purlins (see Fig. 4). For simplicity a rectangular beam will be considered. Suppose the bracing is at midspan and place the origin there. Its effect is to exert an elastic restraint against rotation

$$T = -R\beta_0 \dots\dots\dots (53)$$

in which T = restraint torque, and R = restraint stiffness. Assume the loading is symmetric about midspan, and let L be the half length as shown in Fig. 5. Then it is only necessary to consider one-half of the total system, namely from $x = 0$ to $x = L$.

Adding one-half the potential energy of the restraint to U , namely $1/4 R\beta_0^2$, and setting the first variation to zero yields the same differential equations and boundary conditions given in Eqs. 25–32 except that Eq. 31 is replaced by

$$\left[\frac{1}{2} R\beta - GK\beta' + M_z(w' + w_i) \right] \delta\beta = 0 \text{ at } x = 0 \dots\dots\dots (54)$$

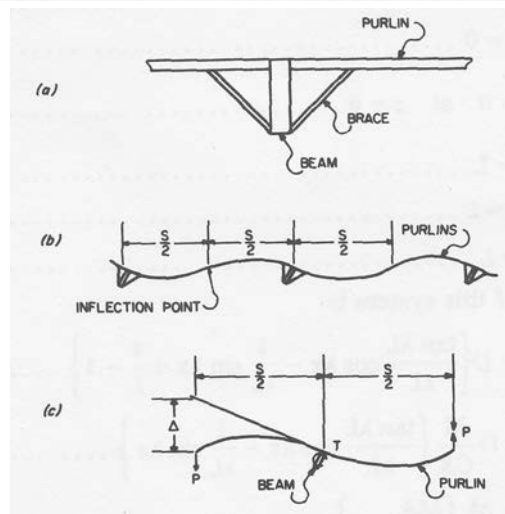


FIG. 4.—(a) Typical Elastic Restraint against Tipping. The Bottom Edge of the Beam is Braced against Transverse Purlins by 45° Braces. (b) Roof Beams Buckling Congruently. (c) Analysis of Torque Exerted by Restraint

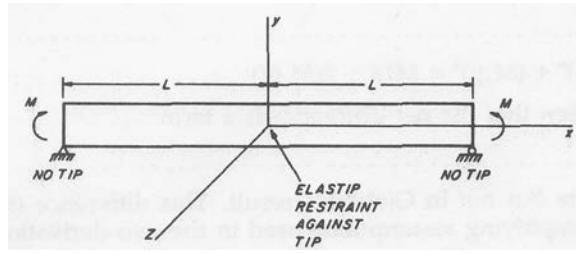


FIG. 5.—Simply Supported Rectangular Beam under Constant Bending Moment. Ends are Rigidly Restrained against Tipping. An Elastic Restraint against Rotation Acts at the Center

For this example, take the loading to be a constant bending moment M and use conditions of simple support at $x = \pm L$. Assume that the initial deformations are in the form of the eigenfunctions for this case

$$w_i = A \cos \mu x \dots\dots\dots (55)$$

$$\beta_i = B \cos \mu x \dots\dots\dots (56)$$

$$\text{in which } \mu = \frac{\pi}{2L} \dots\dots\dots (57)$$

Then the differential equations and boundary conditions become

$$EI_y w'' + M\beta = -MB \cos \mu x + C_1 x + C_2 \dots\dots\dots (58)$$

$$GK\beta'' - Mw'' = -MA\mu^2 \cos \mu x \dots\dots\dots (59)$$

$$C_1 = 0 \dots\dots\dots (60)$$

$$w' = 0 \text{ at } x = 0 \dots\dots\dots (61)$$

$$\frac{1}{2}R\beta - GK\beta' = 0 \text{ at } x = 0 \dots\dots\dots (62)$$

$$w = 0 \text{ at } x = L \dots\dots\dots (63)$$

$$w'' = 0 \text{ at } x = L \dots\dots\dots (64)$$

$$\beta = 0 \text{ at } x = L \dots\dots\dots (65)$$

The solution of this system is

$$w = \hat{A} \cos \mu x + D \left\{ \frac{\tan \lambda L}{\lambda L} \cos \lambda x - \frac{1}{\lambda L} \sin \lambda x + \frac{x}{L} - 1 \right\} \dots\dots\dots (66)$$

$$\beta = \hat{B} \cos \mu x + D \frac{M}{GK} \left\{ \frac{\tan \lambda L}{\lambda L} \cos \lambda x - \frac{1}{\lambda L} \sin \lambda x \right\} \dots\dots\dots (67)$$

$$\text{in which } \hat{A} = \frac{\frac{M}{EI_y} \left\{ \frac{MA}{GK} + B \right\}}{\mu^2 - \lambda^2} \dots\dots\dots (68)$$

$$\hat{B} = \frac{\frac{M}{GK} \left\{ \mu^2 A + \frac{MB}{EI_y} \right\}}{\mu^2 - \lambda^2} \dots\dots\dots (69)$$

$$\lambda^2 = \frac{M^2}{EI_y GK} \dots\dots\dots (70)$$

and D is an integration constant to be determined from Eq. 62. This will now be determined for the following three cases.

Case 1: No Elastic Restraint. Setting $R = 0$ and applying Eq. 62 yields $D = 0$ and

$$w = \hat{A} \cos \mu x \dots\dots\dots (71)$$

$$\beta = \hat{B} \cos \mu x \dots\dots\dots (72)$$

Note that as $\lambda \rightarrow \mu$, both w and β approach infinity with $\hat{A}$ and $\hat{B}$. This agrees with the critical buckling load of a perfect beam. The result given by Eqs. 68–72 agrees with Refs. 2, 6, 7, and 10.

Case 2: No Initial Imperfections. When both A and B are zero, both $\hat{A}$ and $\hat{B}$ are zero also. Thus D must be nonzero at buckling. Equation 62 produces the eigenvalue equation

$$\tan \lambda L = -\frac{2GK\lambda}{R} \dots\dots\dots (73)$$

Since the elastic restraint stiffens the systems, the buckling load is higher here than in case 1. Thus λ exceeds μ . The critical value of λL from Eq. 73 lies between $\pi/2$ and π . As $R \rightarrow 0$, $\lambda L \rightarrow \pi/2$ which agrees with Bleich (1). As $R \rightarrow \infty$, $\lambda L \rightarrow \pi$ which is like having another simple support at the center or like cutting the length in half.

Case 3: General Case.—In the general case, prebuckling deformations approach infinity as the load approaches the buckling load of case 2. Equation 62 yields

$$D = -\frac{\frac{1}{2}R\hat{B}}{\frac{RM \tan \lambda L}{2GK\lambda L} + \frac{M}{L}} \dots\dots\dots (74)$$

from which, using Eq. 67, the restraint torque can be found to be

$$R\beta_o = \frac{R\mu^2 MA + R\lambda^2 GKB}{(\mu^2 - \lambda^2) \left[GK + \frac{RL \tan \lambda L}{2\lambda L} \right]} \dots\dots\dots (75)$$

Numerical Example. Consider a wood beam with a 30-ft (approx 9 m) span. Then half length $L = 180$ in. (approx 4.5 m) and suppose further that width $b = 5.125$ in. (130 mm), depth $h = 18$ in. (457 mm), beam spacing $S = 96$ in. (2.44 m), and modulus of elasticity $E = 1.8 \times 10^6$ pounds per square inch (lb/in.²) (6.9 GPa). For many wood species G is nearly $E/16$ and for narrow rectangular cross sections K is approximately $4 I_y$. Thus for this wood beam assume

$$GK \approx \frac{1}{4} EI_y \dots\dots\dots (76)$$

Assume that the center restraint consists of 45° braces from the bottom of the beam to nominal 2- × 6-in. wood purlins (Fig. 4(a)). Figure 4(b) shows purlin bending when all the main beams in the roof system buckle congruently, as they must if they buckle at all. Figure 4(c) extracts one period from this periodic system. On this free body diagram the restraint torque is

$$T = PS \dots\dots\dots (77)$$

and the angle of rotation is

$$\beta = \frac{2\Delta}{S} \dots\dots\dots (78)$$

$$\text{in which } \Delta = \frac{p \left(\frac{S}{2}\right)^3}{3(EI)_{\text{purlin}}} \dots\dots\dots (79)$$

In this example, the flexural rigidity of a wood 2 × 6 purlin is

$$(EI)_{\text{purlin}} = (1.8 \times 10^6)(20.8) = 37.4 \times 10^6 \text{ lb-in.}^2 \text{ (107 N-m}^2\text{)} \dots\dots\dots (80)$$

and the restraint stiffness is, from Eqs. 77–80,

$$R \equiv \frac{T}{\beta} = 4.68 \times 10^6 \text{ in.-lb/radian (13.4 KN-m/rad)} \dots\dots\dots (81)$$

Assume that the beam has a rated bending strength of $f = 2,100 \text{ lb/in.}^2$ (14.5 MPa). Then the bending moment at design load is

$$M = \frac{1}{6} bh^2 f = 581,000 \text{ in.-lb (1.67 KN-m)} \dots\dots\dots (82)$$

Substituting these numeric values into Eq. 75 the torque exerted by the constraint becomes

$$R\beta_o = 5,590A + 117,000B \dots\dots\dots (83)$$

If we take $A = 6 \text{ in. (152 mm)}$ and $B = 15^\circ$, which is about three times larger than what could reasonably be expected, then

$$R\beta_o = 64,300 \text{ in.-lb (185 N-m)} \dots\dots\dots (84)$$

From the free body diagram in Fig. 6, this torque corresponds to a force F in the bracing of

$$F = \frac{R\beta_o}{\sqrt{2} h} = 2,520 \text{ lb (11.2 KN)} \dots\dots\dots (85)$$

Thus a nominal 2 × 4 would be sufficient. The stress in a 2 × 4 brace would be only

$$f = \frac{2,520}{(1.5)(3.5)} = 481 \text{ lb/in.}^2 \text{ (3.3 MPa)} \dots\dots\dots (86)$$

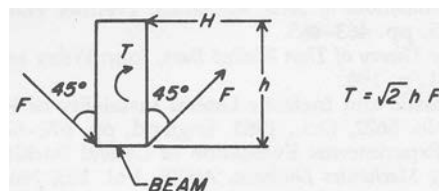


FIG. 6.—Free-Body Diagram of Beam Section at Restraint Showing how Restraint Torque Relates to Forces in 45° Braces. Force H is Exerted by Purlin against Top Edge of Beam

If only the compression brace is effective, this stress should be doubled. This example strongly suggests that bracing is valuable primarily for the stiffening which it provides and need not be very strong; a result that seems intuitively correct.

SUMMARY AND REMARKS

Equations 25–29 present the differential equations and boundary conditions for the lateral buckling of a rectangular beam with initial imperfections. The corresponding result for singly symmetric I-beams is presented in Eqs. 43–48. An example application to a rectangular beam braced at midspan by an elastic tipping restraint showed that the restraint torque is dependent upon the applied bending moment M , the restraint stiffness R , and the magnitude of the initial bow and twist of the beam. A numerical calculation for an 18 in. (457 mm) deep timber beam diagonally braced into 2 × 6 purlins showed the required bracing strength to be very small. Nominal 2 × 4 members were more than sufficient.

This theory is based on the assumption that total lateral displacements w_i and $c\beta_i$ are small. Yet the results approach infinity as the load approaches the Euler critical value. Therefore, any large results for w or β should not be taken literally except to indicate approaching buckling. An exact elastic buckling analysis requires the use of nonlinear large-deflection theory. This theory is a compromise in the same spirit as the “secant” formula for eccentrically loaded columns and suffers exactly the same inconsistency: “small” deflections that approach infinity. Its virtues are: (1) It qualitatively describes the onset of buckling in the presence of initial deformations; (2) it accurately estimates prebuckling stresses due to w_i and β_i when those deflections are small; and (3) it permits one to calculate prebuckling stresses in ancillary parts of the system such as elastic restraints.

APPENDIX I.—REFERENCES

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APPENDIX II.- NOTATION

The following symbols are used in this paper:

$A, \hat{A}$	=	amplitude of w_i , w , in. (m);
a_1, a_2	=	locate shear center of I-beam. See Fig. 3, in. (m);
$B, \hat{B}$	=	amplitude of β_i , β , radian;
C_1, C_2	=	integration constants;
c	=	half depth of rectangular beam, in. (m);
D	=	integration constant;
d	=	depth of I-beam, in. (m);
E	=	modulus of elasticity, lb/in. ² (N/m ²);
f	=	stress, lb/in. ² (N/m ²);
F	=	force in brace, lb (N);
G	=	shear modulus, lb/in. ² (N/m ²);
h	=	depth of rectangular beam, in. (m);
I, I_y, I_z	=	moment of inertia, in. ⁴ (m ⁴);
K	=	cross section shape factor for Saint Venant torsion, in. ⁴ (m ⁴);
L	=	length of beam, in. (m);
M_z	=	moment about z axis, in.-lb (N-m);
p	=	distributed load, lb/in. (N/m);
R	=	restraint stiffness, in.-lb/radian (N-m/rad);
S	=	beam spacing, in. (m);
T	=	torque, in.-lb (N-m);
t_1, t_2, t_3	=	flange and web thicknesses, see Fig. 3, in. (m);
U, U_1, U_2, U_L	=	energy, in.-lb (N-m);
$\bar{u}, \bar{v}, \bar{w}$	=	displacements of a point at (x, y, z) , in. (m);
v	=	vertical deflection of longitudinal axis (line joining shear centers), in. (m);
w, w_i, w_t	=	displacement of shear center in z -direction, in. (m);
x, y, z	=	coordinate axes, see Fig. 1;
β, β_i, β_t	=	angle of twist, radian;
γ_{xy}, γ_{xz}	=	shear strain;

Γ	=	cross section shape factor for Timoshenko torsion, in. ⁵ (m ⁵);
δ	=	variation operator or purlin deflection, in. (m);
ϵ_x	=	strain;
κ	=	cross section property defined by Eq. 42, in. (m);
λ	=	load parameter $M/\sqrt{EI_y GK}$, in. ⁻¹ (m ⁻¹);
μ	=	eigenvalue of A, in. ⁻¹ (m ⁻¹); $\pi/2L$; and
$\sigma_x^\circ, \tau_{xy}^\circ, \tau_{xz}^\circ$	=	datum stresses, lb/in. ² (N/m ²).