

COMBINED-LOAD STABILITY CRITERION FOR WOOD BEAM-COLUMNS

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ABSTRACT: An analysis of elastic buckling in beam-columns was undertaken to develop a general interaction equation for use in timber design. The derivation of a general buckling criterion and the determination of moment modification factors following the method of Southwell (1922) are presented. Biaxial bending, axial compression, and water ponding on flat roofs are all considered. Simple support is assumed, but results may be applicable to other support cases. The results are of interest to code developers and to designers of space frames, towers, and trusses, where biaxial bending and axial compression are combined, or whenever axial compression and water ponding may be combined.

INTRODUCTION

The present design criterion for wood beam-columns, as contained in the *National Design Specification* section 3.10 (1986), is inaccurate and incomplete. The specification is inaccurate because it does not properly account for the interaction between bending moment and axial compression. Southwell (1922) showed that the bending moment is modified by the presence of an axial compressive load, P . The moment modification factor is $1/(1 - P/P_1)$ where P_1 is the elastic buckling value of P . This term has been used in the *Manual of Steel Construction* (1986) for decades, but it is still not used for wood. The specification is incomplete because it does not answer (1) How does ponding collapse of flat roof beams interact with end compression? (2) How do biaxial bending moments interact? Concerning question 2, Galambos (1968) showed that, under constant bending moments M_x and M_y about the major and minor axes, respectively, the minor-axis moment is modified by the presence of the major-axis moment. The minor moment modification factor is $1/(1 - M_x^2/M_1^2)$ where M_1 is the elastic lateral-torsional buckling value of M_x . The design of electric transmission towers and top chords of trusses with lateral loads, two situations where biaxial bending commonly occurs, show a clear need for a more general biaxial design criterion.

In this paper an elastic buckling criterion is derived for the case of simply supported members under any combination of strong-axis bending loads, weak-axis bending loads, axial compression, equal end couples, and water ponding. In addition, the growth of prebuckling deformations is approximately represented by the method Southwell (1922) applied to initially crooked columns. These deformations are used to solve for the secondary moments caused by axial load, ponding, and other-axis bending if there is bending about both axes. Simple moment modification factors are derived for use in design. The enable the designer to model buckling as a material rupture caused by the

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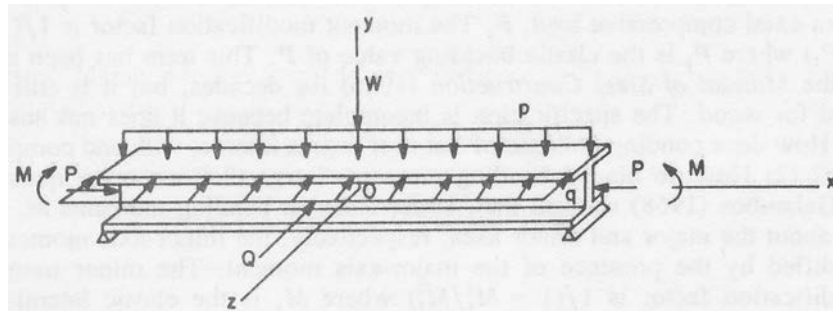
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large deformations at prebuckling loads. The application of these ideas to the design of wood beam-columns is outlined.

There are very few combined-mode buckling criteria in the literature. The combination of end couples and axial compression has received the most attention (Chen and Atsuta 1977; Culver 1966a, 1966b; Home 1954; Salvadori 1956; Syal and Sharma 1970; Timoshenko 1936), but almost no other cases have been investigated. Prescott (1920) considered the combination of concentrated load and uniformly distributed load. Most modern work has focused on plastic buckling or the interaction of local and global buckling (Fukumoto and Galambos 1966; Lay or Galambos 1965; Syal and Sharma 1971; Vinnakota and Aystö 1974).

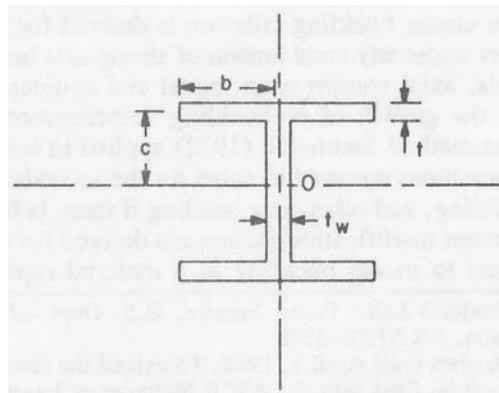
DIFFERENTIAL EQUATIONS AND BOUNDARY CONDITIONS

The beam-column is assumed to be a doubly symmetric I-beam. Fig. 1 shows the loading, and Fig. 2 shows the cross-sectional shape. To exploit symmetry, the origin O is placed at the center of the beam. The loading consists of an end load P , an end couple M , distributed loads p and q , and concentrated loads W and Q at midspan. Fig. 3 shows the sign convention for internal moments, shear, torque, and angle of twist.



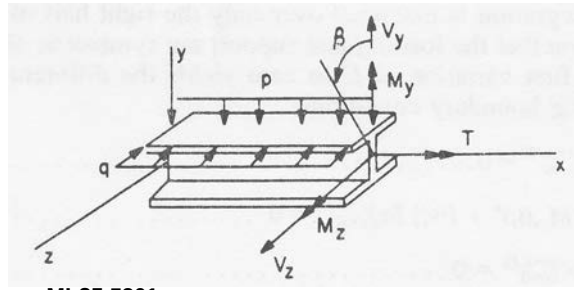
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FIG. 1. Doubly Symmetric I-Beam under Biaxial Lateral Loads and End Load



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FIG. 2. Beam Cross Section. O is Centroid and Shear Center



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FIG. 3. Sign Convention for Internal Shears, Moments, and Angle of Twist

The basic differential equations are from Gjelsvik (1981) with the term $\rho S v_t$ added in cases where this is a load caused by ponded water:

$$EI_z(v_t'''' - v_e''''') + (M_{ye}\beta_t)'' + P v_t'' - \rho S v_t = 0 \dots\dots\dots (1)$$

$$EI_y(w_t'''' - w_e''''') + (M_{ze}\beta_t)'' + P w_t'' = 0 \dots\dots\dots (2)$$

$$EI_w(\beta_t'''' - \beta_e''''') - GK(\beta_t'' - \beta_e'') + M_{ye}v_t'' + M_{ze}w_t'' + P r_0^2 \beta_t'' - p c \beta_t = 0 \dots\dots\dots (3)$$

in which EI_z and EI_y are flexural rigidities; EI_w = torsional warping rigidity; GK = Saint Venant torsional rigidity; v and w are displacements in the y and z directions; β is the angle of twist; M_y and M_z are internal bending moments as in Fig. 3; P is compressive axial force; p is the weight density of water; S is the roof beam spacing in flat roofs; r_0 is the polar radius of gyration about the shear center; p is distributed load; c is the distance from the shear center to the point of application of the distributed loads p and q ; primes denote differentiation with respect to x ; the subscript t denotes actual (total); and the subscript e denotes that part of the total which can be obtained from elementary linear theory. It should be emphasized that the actual displacements are assumed to be "small" in the derivation of these equations, although nonlinear terms are included in the equilibrium analysis. The resulting buckling analysis resembles the "secant formula" for eccentrically loaded columns in that "small" deformations are later seen to approach infinity as the critical loading condition is approached. Such theories can only describe the onset of buckling and cannot be taken literally at large deflections. For a material such as wood, however, in which brittle failure usually occurs before the deflections become excessively large, such a theory can form a useful basis for ultimate strength design, as will be shown in this paper.

It is not difficult to show that Gjelsvik's equations can be obtained by variational minimization of the following expression for total potential energy, U :

$$U = \frac{1}{2} \int_0^{L/2} \{EI_z(v_t'' - v_e'')^2 + EI_y(w_t'' - w_e'')^2 + EI_w(\beta_t'' - \beta_e'')^2 + GK(\beta_t' - \beta_e')^2 - 2v_t'(M_{ye}\beta_t)' - 2w_t'(M_{ze}\beta_t)' - P[(v_t')^2 + (w_t')^2 + (r_0\beta_t')^2] - p c \beta_t^2 - \rho S v_t^2\} dx - \frac{1}{4} W c \beta_t^2|_{x=0} \dots\dots\dots (4)$$

in which the integration is extended over only the right half of the member on the assumption that the loading and support are symmetric about the center. Setting the first variation of U to zero yields the differential Eqs. 1–3 and the following boundary conditions:

$$\{EI_z(v''_t - v''_e)\}\delta v'_t|_{x=0}^{x=L/2} = 0 \dots\dots\dots (5)$$

$$\{EI_z(v'''_t - v'''_e) + (M_{ye}\beta_t)' + Pv'_t\}\delta v_t|_{x=0}^{x=L/2} = 0 \dots\dots\dots (6)$$

$$\{EI_y(w''_t - w''_e)\}\delta w'_t|_{x=0}^{x=L/2} = 0 \dots\dots\dots (7)$$

$$\{EI_y(w'''_t - w'''_e) + (M_{ze}\beta_t)' + Pw'_t\}\delta w_t|_{x=0}^{x=L/2} = 0 \dots\dots\dots (8)$$

$$EI_\omega(\beta''_t - \beta''_e)\delta\beta'_t|_{x=0}^{x=L/2} = 0 \dots\dots\dots (9)$$

$$\begin{aligned} & -\frac{1}{2} W_c\beta_t\delta\beta_t|_{x=0} + \{EI_\omega(\beta'''_t - \beta'''_e) - GK(\beta'_t - \beta'_e) + M_{ye}v'_t + M_{ze}w'_t \\ & + Pr_0^2\beta'_t\}\delta\beta_t|_{x=0}^{x=L/2} = 0 \dots\dots\dots (10) \end{aligned}$$

SECONDARY MOMENTS

We shall present solutions of Eqs. 1–3 by the Ritz method, representing all deflections as simple cosine functions. Our interest centers in a design equation for beam-columns under biaxial bending. In particular, we wish to find simple expressions for the “secondary moments” by which we mean those terms which must be added to the elementary moments to give the actual moments:

$$M_{zt} = M_{ze} + \text{“secondary } M_z\text{”} \dots\dots\dots (11)$$

$$M_{yt} = M_{ye} + \text{“secondary } M_y\text{”} \dots\dots\dots (12)$$

Actual moments could be obtained from the approximate Ritz solution by taking second derivatives, but the accuracy of an approximate solution worsens with each successive differentiation. Consider instead the differential Eqs. 1 and 2. Integrating $\int_0^{L/2} \int_0^x \dots dx dx$ and applying the boundary conditions 5–8, Eqs. 1 and 2 become

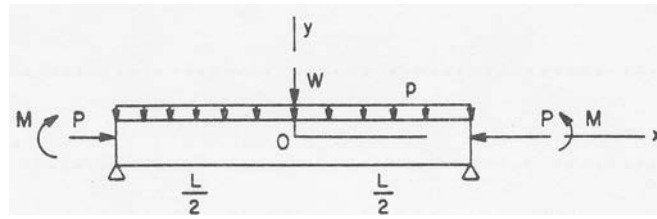
$$M_{zt0} - M_{ze0} + M_{ye0}\beta_{r0} + Pv_{r0} + \rho S \int_0^{L/2} v_t(x) \left[\frac{L}{2} - x \right] dx = 0 \dots\dots\dots (13)$$

$$M_{yt0} - M_{ye0} - M_{ze0}\beta_{r0} - Pw_{r0} = 0 \dots\dots\dots (14)$$

where the subscript zero denotes evaluation at $x = 0$. Eqs. 13 and 14 give the secondary moments directly without any need to differentiate. To obtain the total center deflections we next apply the Ritz method.

Approximate Solution by Ritz Method

In this analysis, the loading in the z direction shall be represented only by the moment $M_y(x)$ and displacement $w(x)$ that it produces. Otherwise, the loads are as shown in Fig. 4.



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FIG. 4. Simply Supported Beam under Axial Load, End Couples, Water Ponding, and Biaxial Bending. Water Pond and Loading in z Direction Exists But Is Not Shown

Assume that

$$v(x) = \delta_y \cos \frac{\pi x}{L} \dots \dots \dots (15)$$

$$w(x) = \delta_z \cos \frac{\pi x}{L} \dots \dots \dots (16)$$

$$\beta(x) = \phi \cos \frac{\pi x}{L} \dots \dots \dots (17)$$

where δ_y , δ_z , and ϕ are center deflections. The subscripts t and e shall be appended to these scalars as needed and shall have their usual meaning.

To obtain an upper bound to the buckling criterion, we substitute Eqs. 15–17 into Eq. 4 for U and minimize U directly. This yields three equations:

$$\frac{\partial U}{\partial \delta_{yt}} = 0 \dots \dots \dots (18)$$

$$\frac{\partial U}{\partial \delta_{zt}} = 0 \dots \dots \dots (19)$$

$$\frac{\partial U}{\partial \phi_t} = 0 \dots \dots \dots (20)$$

After some simplification, Eqs. 18–20 can be written

$$\left(1 - \frac{P}{P_1} - \frac{S}{S_1}\right) \delta_{yt} - \frac{m_2}{P_1} \phi_t = \delta_{ye} \dots \dots \dots (21)$$

$$\left(1 - \frac{P}{P_2}\right) \delta_{zt} - \frac{m_1}{P_2} \phi_t = \delta_{ze} \dots \dots \dots (22)$$

$$-\frac{m_2}{r_0^2 P_t} \delta_{yt} - \frac{m_1}{r_0^2 P_T} \delta_{zt} + \left(1 - \frac{P}{P_T} - 2.93v \frac{p}{\hat{p}_0} - 3.62v \frac{W}{\hat{W}_0}\right) \phi_t = \phi_e \dots \dots \dots (23)$$

in which

$$P_1 \equiv \frac{\pi^2 E I_z}{L^2} \dots \dots \dots (24)$$

$$P_2 \equiv \frac{\pi^2 EI_y}{L^2} \dots\dots\dots (25)$$

$$S_1 \equiv \frac{\pi^4 EI_z}{\rho L^4} \dots\dots\dots (26)$$

$$P_T \equiv \frac{1}{r_0^2} \left[GK + \frac{\pi^2 EI_\omega}{L^2} \right] \dots\dots\dots (27)$$

$$m_1 \equiv 0.869 \frac{pL^2}{8} + 0.703 \frac{WL^2}{4} + M \dots\dots\dots (28)$$

$$m_2 \equiv \frac{4}{\pi} \int_0^{L/2} \sin \frac{\pi x}{L} \left[M_{ye} \cos \frac{\pi x}{L} \right]' dx \dots\dots\dots (29)$$

$$v \equiv \frac{c}{L} \sqrt{\frac{EI_y}{GK + \frac{\pi^2 EI_\omega}{L^2}}} \dots\dots\dots (30)$$

$$\hat{p}_0 \equiv \frac{28.9 \sqrt{\left[GK + \frac{\pi^2 EI_\omega}{L^2} \right] EI_y}}{L^3} \dots\dots\dots (31)$$

$$\hat{W}_0 \equiv \frac{17.9 \sqrt{\left[GK + \frac{\pi^2 EI_\omega}{L^2} \right] EI_y}}{L^2} \dots\dots\dots (32)$$

We now recognize that the quantities defined in Eqs. 24–27, 31, and 32 are single-load critical values. P_1 and P_2 are associated with column buckling about the strong and weak axes, respectively, and P_T is associated with torsional column buckling. S_1 is the smallest beam spacing at which ponding collapse can occur. These values are exact because the cosine happens to be the exact eigenfunction for end load and ponding. The values of $\hat{p}_0$ and $\hat{W}_0$ are upper bound approximations for the critical values of these loads when acting alone at the shear center. The exact values are known to be

$$p_0 \equiv \frac{28.30 \sqrt{\left[GK + \frac{\pi^2 EI_\omega}{L^2} \right] EI_y}}{L^3} \dots\dots\dots (33)$$

$$W_0 \equiv \frac{16.93 \sqrt{\left[GK + \frac{\pi^2 EI_\omega}{L^2} \right] EI_y}}{L^2} \dots\dots\dots (34)$$

so that $\hat{p}_0$ and $\hat{W}_0$ are within a few percent of being exact. The terms containing v are an approximate account of the effect of raising p and W above the shear center. Again, comparison with exact solutions (21) shows them to be within $\pm 1\%$ of exact.

Deflections

The solution of Eqs. 21-23 is

$$\delta_{y_i} = \frac{1}{\theta} \left\{ (\zeta_2 \zeta_T - \mu_1^2) \delta_{ye} + \mu_1 \mu_2 \sqrt{\frac{P_2}{P_1}} \delta_{ze} + \zeta_2 \mu_2 \sqrt{\frac{P_T}{P_1}} r_0 \phi_e \right\} \dots \dots \dots (35)$$

$$\delta_{z_i} = \frac{1}{\theta} \left\{ (\zeta_1 \zeta_T - \mu_2^2) \delta_{ze} + \mu_1 \mu_2 \sqrt{\frac{P_1}{P_2}} \delta_{ye} + \zeta_1 \mu_1 \sqrt{\frac{P_T}{P_2}} r_0 \phi_e \right\} \dots \dots \dots (36)$$

$$\phi_i = \frac{1}{\theta} \left\{ (\zeta_1 \zeta_2 \phi_e + \zeta_2 \mu_2) \sqrt{\frac{P_1}{P_T}} \frac{\delta_{ye}}{r_0} + \zeta_1 \mu_1 \sqrt{\frac{P_2}{P_T}} \frac{\delta_{ze}}{r_0} \right\} \dots \dots \dots (37)$$

in which

$$\theta \equiv \zeta_1 \zeta_2 \zeta_T - \zeta_1 \mu_1^2 - \zeta_2 \mu_2^2 \dots \dots \dots (38)$$

$$\zeta_1 \equiv 1 - \frac{P}{P_1} - \frac{S}{S_1} \dots \dots \dots (39)$$

$$\zeta_2 \equiv 1 - \frac{P}{P_2} \dots \dots \dots (40)$$

$$\zeta_T \equiv \lambda - \frac{P}{P_T} \dots \dots \dots (41)$$

$$\lambda \equiv 1 - 2.93\nu \frac{p}{p_0} - 3.62\nu \frac{W}{W_0} \dots \dots \dots (42)$$

$$\mu_1 \equiv \frac{m_1}{M_1} \dots \dots \dots (43)$$

$$\mu_2 \equiv \frac{m_2}{M_2} \dots \dots \dots (44)$$

$$M_1 \equiv P_2 r_0^2 P_T \equiv \frac{\pi \sqrt{\left[GK + \frac{\pi^2 EI_\omega}{L^2} \right]} EI_y}{L} \dots \dots \dots (45)$$

$$M_2 \equiv P_1 r_0^2 P_T \equiv \frac{\pi \sqrt{\left[GK + \frac{\pi^2 EI_\omega}{L^2} \right]} EI_z}{L} \dots \dots \dots (46)$$

The buckling criterion is

$$\theta = 0 \dots \dots \dots (47)$$

Uncoupled Approximation

These equations can be simplified by noting that

$$\mu_2 \ll \mu_1 \dots \dots \dots (48)$$

which follows from the fact that the bending moment about the weak axis is less than that about the strong axis,

$$m_2 < m_1 \dots\dots\dots (49)$$

and that

$$M_2 >> M_1 \dots\dots\dots (50)$$

If terms containing μ_2 are neglected, Eqs. 35–38 uncouple into one equation that describes in-plane buckling and a pair of equations that describe lateral-torsional buckling:

$$\delta_{yt} = \frac{\delta_{ye}}{\theta_1} \dots\dots\dots (51)$$

$$\delta_{zt} = \frac{1}{\theta_2} \left\{ \zeta_T \delta_{ze} + \mu_1 \sqrt{\frac{P_T}{P_2}} r_0 \phi_e \right\} \dots\dots\dots (52)$$

$$\phi_t = \frac{1}{\theta_2} \left\{ \zeta_2 \phi_e + \mu_1 \sqrt{\frac{P_2}{P_T}} \frac{\delta_{ze}}{r_0} \right\} \dots\dots\dots (53)$$

in which

$$\theta_1 \equiv \zeta_1 \equiv 1 - \frac{P}{P_1} - \frac{S}{S_1} \dots\dots\dots (54)$$

$$\theta_2 \equiv \zeta_2 \zeta_T - \mu_1^2 \equiv \left(1 - \frac{P}{P_2} \right) \left(\lambda - \frac{P}{P_T} \right) - \left(\frac{m_1}{M_1} \right)^2 \dots\dots\dots (55)$$

Now the buckling criterion is

$$\theta_1 \theta_2 = 0 \dots\dots\dots (56)$$

Waving obtained the total deflections, we return to Eqs. 13 and 14. Putting the assumed cosine displacements 15–17 into Eqs. 13 and 14, the secondary moments become

$$M_{zt0} - M_{ze0} = -M_{ye0} \phi_t - \left[P + \rho S \frac{L^2}{\pi^2} \right] \delta_{yt} \dots\dots\dots (57)$$

$$M_{yt0} - M_{ye0} = M_{ze0} \phi_t + P \delta_{zt} \dots\dots\dots (58)$$

Following the method of Southwell (1922), we shall ultimately write the actual moments in terms of the elementary moments times a modification factor. Eqs. 51–53 give the total deflections in terms of the elementary deflections. Therefore, we must express the elementary deflections in terms of the elementary moments. This is done in the next section.

Elementary Deflections

The vertical deflection is proportional to M_{ze0} and can be written in the form

$$\delta_{ye} = -\frac{M_{ze0}(1 + \alpha_1)}{P_1} \dots\dots\dots (59)$$

in which P_1 has been used to represent $\pi^2 EI_z / L^2$ and α_1 is a dimensionless

Load (1)	α_1 Or α_2 (2)
Uniform	0.028
Concentrated, center span	- 0.178
End couples	0.234

constant that depends on the type of vertical loading. Table 1 shows the value of α_1 for a few cases.

In similar fashion the horizontal deflection is written

$$\delta_{ze} = -\frac{M_{ye0}(1 + \alpha_2)}{P_2} \dots\dots\dots (60)$$

with α_2 dependent on the type of lateral loading.

The elementary twist is often small enough to be ignored. However, there is an elementary differential equation for the torsion of thin-walled beams:

$$-EI_{\omega}(\beta_e'''' - \beta_i''') + GK(\beta_e'' - \beta_i'') = qc \dots\dots\dots (61)$$

in which the subscript i denotes initial value.

By this theory the center deflection is

$$\phi_e = \frac{qL^2}{8} \cdot \frac{c}{GK} \left[1 + \frac{2 - 2 \cosh \mu}{\mu^2 \cosh \mu} \right] + \frac{QL}{4} \cdot \frac{c}{GK} \left[1 - \frac{\tanh \mu}{\mu} \right] \dots\dots\dots (62)$$

in which

$$\mu \equiv \frac{L}{2} \sqrt{\frac{GK}{EI_{\omega}}} \dots\dots\dots (63)$$

This result can be approximated within $\pm 11\%$ by

$$\phi_e \approx \frac{qL^2}{8} \frac{c}{GK} \left[\frac{1}{1 + \frac{2.4}{\mu^2}} \right] + \frac{QL}{4} \frac{c}{GK} \left[\frac{1}{1 + \frac{3}{\mu^2}} \right] \dots\dots\dots (64)$$

Just as P_1 was used in Eq. 59 to represent the flexural rigidity, so too P_T shall be used here to approximately represent the combinations of GK and μ . We have

$$r_0^2 P_T \equiv GK + \frac{\pi^2 EI_{\omega}}{L^2} \equiv GK \left[1 + \frac{2.467}{\mu^2} \right] \dots\dots\dots (65)$$

so that Eq. 64 can be written

$$\phi_e \approx \left(\frac{qL^2}{8} + \frac{QL}{4} \right) \frac{c}{r_0^2 P_T} = \frac{M_{ye0} c}{r_0^2 P_T} \dots\dots\dots (66)$$

Although Eq. 66 was derived for the particular loading q and Q , we shall assume that it holds generally.

Actual Moments M_x and M_y

Substituting Eqs. 59, 60, and 66 for elementary deflections into Eqs. 51–53 for actual deflections, and then substituting the results into Eqs. 57 and 58 for the actual moments, we get

$$M_{x0} = M_{xe0} - \frac{M_{ye0}^2}{\theta_2} \left[\frac{c}{r_0^2 P_T} \left(1 - \frac{P}{P_2} \right) + \frac{(1 + \alpha_1) m_1}{P_2 r_0^2 P_T} \right] + \frac{M_{ze0}}{\theta_1} \left(\frac{P}{P_1} + \frac{S}{S_1} \right) (1 + \alpha_1) \dots \dots \dots (67)$$

$$M_{y0} = M_{ye0} + \frac{M_{ye0}}{\theta_2} \left\{ \frac{M_{ze0}}{M_1} \left[\left(1 + \frac{P}{P_2} \right) \pi v + \frac{m_1}{M_1} \right] + \frac{P}{P_2} \left[1 - 2.93 v \frac{p}{\hat{p}_0} - 3.62 v \frac{W}{\hat{W}_0} - \frac{P}{P_T} + \pi v \frac{m_1}{M_1} \right] \right\} \dots \dots \dots (68)$$

The term containing c in Eq. 67 can conservatively be ignored if c is positive (lateral loads applied above the shear center as shown in Fig. 1), because then it is always opposite in sign to M_{ze} and m_1 . Even when c is negative, the net contribution of this term is always very small. Therefore it shall be neglected. Then Eq. 67 becomes

$$M_{x0} = \frac{M_{ze0}}{\theta_1} \left[1 + \alpha_1 \left(\frac{P}{P_1} + \frac{S}{S_1} \right) \right] \dots \dots \dots (69)$$

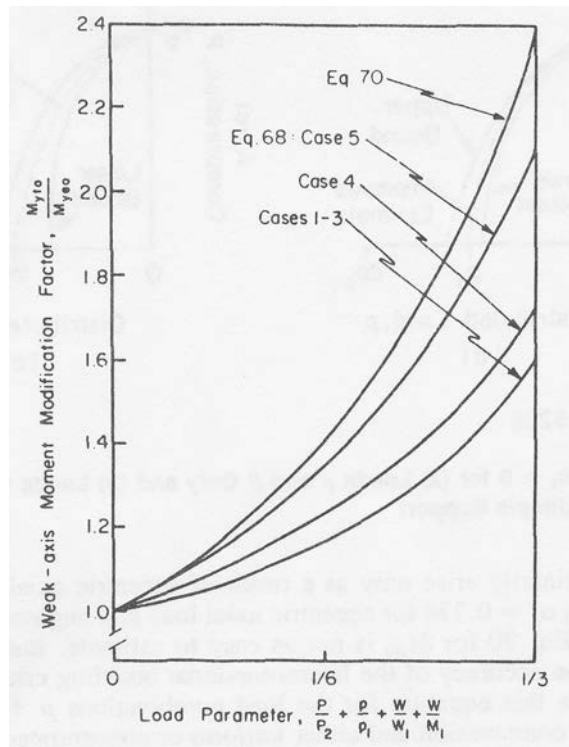
Eq. 68 can be bounded from above by the simple expression

$$M_{y0} < \frac{M_{ye0}}{1 - \frac{P}{P_2} - \frac{p}{p_1} - \frac{W}{W_1} - \frac{M}{M_1}} \dots \dots \dots (70)$$

in which p_1 and W_1 are critical values of p and W acting above the shear center. Their values are available in the literature (Zahn 1973). To investigate the accuracy of Eq. 70, Eqs. 68 and 70 are compared in Fig. 5 for the five cases listed in Table 2. In each case the load ratios were adjusted to maximize the difference between Eq. 70 and Eq. 68. These resulting load ratios are also included in Table 2. It is seen from Fig. 5 that Eq. 10 has an error that is always conservative and no greater than about 50% for load combinations that are less than one-third of the way to buckling. Since M_y is not the primary moment, the crude estimate of Eq. 70 is sufficient for design calculations.

The important results to note here are the right-hand sides of Eqs. 57 and 58. These show how biaxial bending moments interact, each contributing a secondary component to the other's value. The effect of M_{ze} on M_{yt} is large and additive and was retained. The effect of M_{ye} on M_{xt} , however, is small and ordinarily subtractive and was ignored. End compression affects both bending moments in the way one would expect, and ponding affects only the strong-axis bending moment M_x . The actual moments can be estimated by Eqs. 69 and 70.

The quantity α_1 in Eq. 69 can be neglected when it is small or negative. According to Table 1, this means that only the case of end couples (eccentric



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FIG. 5. Comparison of Eqs. 68 and 70 for Five Cases Listed in Table 2

end load) necessitates the retention of $\alpha_1 = 0.234$. The results in Eqs. 69 and 70 will be used later to obtain an interaction equation for use in design.

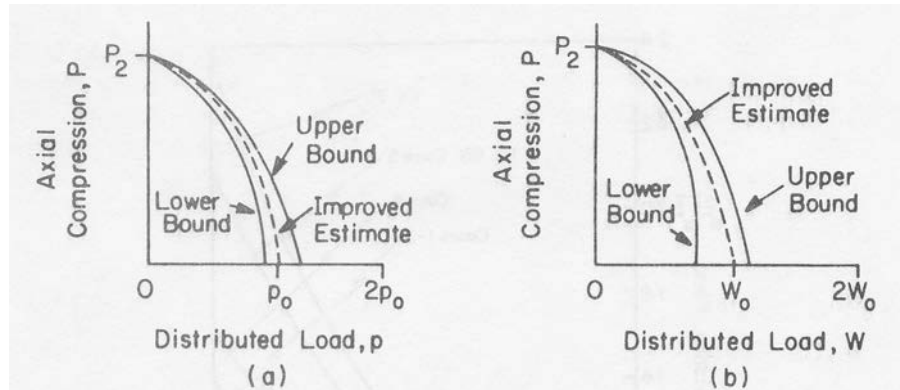
Accuracy

The combination of uniaxial bending with ponding and axial compression can be treated by an exact solution of Eq. 1 with $M_{ye} = 0$. Comparison of Eq. 69 with the exact solution shows an error of less than 1% when $\theta_1 > 0.5$. Thus, the approximation will carry one-half way to buckling with an error less than 1%.

If α_1 is neglected for every case other than end couples, the error is conservative and equal to about 10%. If α_1 were neglected for end couples, the error for that case would be unconservative and equal to about 12%. Since

TABLE 2. Cases Shown in Fig. 5

Case (1)	Loading (2)	Cross section (3)	r_o/c (4)	p/P (5)	W/P (6)	M/P (7)
1	P and p	Rectangle	0.577	4.30	—	—
2	P and W	Rectangle	0.577	—	3.14	—
3	P and M	Rectangle	0.577	—	—	4.30
4	P and W	I-beam	1.0	—	3.85	—
5	P and M	I-beam	1.0	—	—	1.83



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FIG. 6. Plot of $\theta_2 = 0$ for (a) Loads p and P Only and (b) Loads W and P Only, Both Assuming Simple Support

end couples ordinarily arise only as a result of eccentric axial load, it may be wise to retain $\alpha_1 = 0.234$ for eccentric axial load and neglect it otherwise.

The error in Eq. 70 for M_{y0} is not as easy to estimate. Instead we shall only examine the accuracy of the lateral-torsional buckling criterion $\theta_2 = 0$. Fig. 6 illustrates this equation for the load combinations $p + P$ and $W + P$, that is, axial compression and either uniform or concentrated lateral load. Three parabolas are shown in each case: (1) An upper bound (Ritz solution with $\hat{p}_0$ or W_0); (2) an improved estimate (Ritz solution with p_0 or W_0); and (3) a lower bound (exact solution with M_{xe} replaced by its maximum value as end couples). The "improved estimate" is quite accurate for uniform load and still acceptable for concentrated load.

Another test of θ_2 is to determine how accurately it models the effect of raising p or W above the shear center. Comparing this with data from Zahn (1973), in which $P = 0$, shows an error of less than 1% in this case.

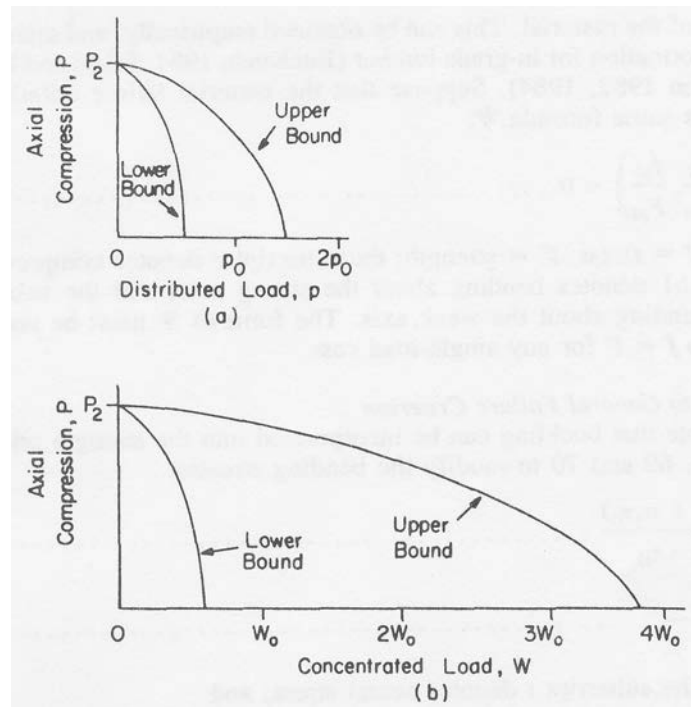
As for Eq. 70, it is, of course, an upper bound and quite conservative. However, design must be very conservative to cover the possibility of initial imperfections (bow and twist) in the member.

CANTILEVERS

The result of applying to cantilevers the same method used previously for simply supported beams is a buckling criterion of the same general form. This suggests that the same design criterion may be applicable to both cantilevers and simple beams. However, the accuracy is an open question because the simple bounds shown in Fig. 6 turn out to be very wide for cantilevers, as is shown in Fig. 7, where the assumed displacements had the form

$$v(x) = \delta_y \left[\cos \frac{\pi x}{2L} - 1 \right] \dots \dots \dots (71)$$

$$w(x) = \delta_z \left[\cos \frac{\pi x}{2L} - 1 \right] \dots \dots \dots (72)$$



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FIG. 7. Plot of $\theta_2 = 0$ for Cantilever Beam with (a) Loads p and P Only and (b) Loads W and P Only

$$\beta(x) = \theta \left[\cos \frac{\pi x}{2L} - 1 \right] \dots\dots\dots (73)$$

As can be seen, these bounds are too wide to be of any practical value. This attempt failed because the exact buckled shape is no longer closely approximated by a single trigonometric term. The accuracy of Eqs. 69 and 70 for cantilevers requires further investigation.

WOOD ENGINEERING DESIGN

The results in Eqs. 69 and 70 have important implications in the design of beam-columns. Galamos (1968) discussed the application to the design of steel members. Eqs. 69 and 70 are a generalization of his result to include ponding and biaxial bending. In this paper the application will be restricted to wood.

The quadratic form of θ_2 (Eq. 55) is well supported by data for clear wood. Newlin and Trayer's (1941) data for clear Sitka spruce beam-columns confirm the result with an error of less than 2%.

Material Failure Criterion

To design beam-columns of any length, including members too short to buckle, one must possess a criterion for the intersection of bending and crushing

strengths of the material. This can be obtained empirically, and there is much recent information for in-grade lumber (Buchanan 1984; Johns and Buchanan 1982; Zahn 1982, 1984). Suppose that the material failure criterion is expressed as some formula Ψ :

$$\Psi \left(\frac{f_c}{F_c}, \frac{f_{b1}}{F_{b1}}, \frac{f_{b2}}{F_{b2}} \right) = 0 \dots\dots\dots (74)$$

in which f = stress; F = strength; the subscript c denotes compression; the subscript $b1$ denotes bending about the strong axis; and the subscript $b2$ denotes bending about the weak axis. The form of Ψ must be such that it reduces to $f = F$ for any single-load case.

Extension to General Failure Criterion

Now note that buckling can be incorporated into the strength criterion by using Eqs. 69 and 70 to modify the bending stresses:

$$f_{bt} \equiv \frac{f_{b1}(1 + \alpha_1 \eta_1)}{1 - \eta_1} \dots\dots\dots (75)$$

$$f_{bt} \equiv \frac{f_{b2}}{1 - \eta_2} \dots\dots\dots (76)$$

in which the subscript t denotes actual stress, and

$$\eta_1 \equiv \frac{P}{P_1} + \frac{S}{S_1} \dots\dots\dots (77)$$

$$\eta_2 \equiv \frac{P}{P_2} + \frac{p}{p_1} + \frac{W}{W_1} + \frac{Pe}{M_1} \dots\dots\dots (78)$$

$$\alpha_1 = \begin{cases} 0.234 & \text{if } f_{b1} \text{ is caused by eccentricity} \\ 0 & \text{otherwise} \end{cases} \dots\dots\dots (79)$$

If the modified bending stresses are used in Ψ , instead of the nominal bending stresses obtained by elementary theory, $\Psi = 0$ will model both buckling and strength interactions simultaneously. Eqs. 75 and 76 accomplish this by modeling "buckling failure" as a prebuckling strength failure.

This scheme fails to model elastic buckling if there are no bending stresses, however. Therefore, the crushing strength F_c should be replaced by the column strength F'_c obtained from the semiempirical column design formula. This describes plastic buckling of intermediate length columns, as well as elastic buckling of long columns. It is appropriate to replace F_{b1} by F'_{b1} also, so as to describe the lateral-torsional buckling of intermediate length beams. Then the general interaction equation takes the form

$$\Psi \left(\frac{f_c}{F'_c}, \frac{f_{b1}(1 + \alpha_1 \eta_1)}{F'_{b1}(1 - \eta_1)}, \frac{f_{b2}}{F_{b2}(1 - \eta_2)} \right) = 0 \dots\dots\dots (80)$$

in which the single prime denotes "reduced for slenderness."

Generalization to Several Lateral Loads

The only loads treated here were uniform load and concentrated load at midspan. However, since any lateral load in the y direction would cause a

buckled shape closely resembling the cosine, Eq. 78 may safely be generalized to any sum of lateral loads:

$$\eta_2 \equiv \frac{P}{P_2} + \sum_i \frac{L_i}{L_{1i}} \dots \dots \dots (81)$$

in which L_i = any lateral load in y direction (including end couples about the z axis); and L_{1i} = its critical value.

SUMMARY

The interaction of elastic buckling modes of simply supported beam-columns is described in Eqs. 35-47 and with very small error by the uncoupled Eqs. 51-56.

The actual moments, including secondary moments induced by axial compression, water ponding, and other-axis bending moment, are described in Eqs. 69 and 70. It is important to note that the secondary effects of axial compression and water ponding should be combined as shown here, and not checked separately as in current practice.

Biaxial bending rarely occurs in practice; but when it does, the secondary effects can be important. Even without any axial compression or ponding, the weak-axis moment is increased by a secondary contribution from the strong-axis moment, as shown in Eq. 68. A simple conservative estimate of this effect is given in Eq. 70.

A general interaction design equation can be derived simply from a given strength interaction criterion by using the moment modification factors presented in this paper. The resulting design criterion has the form of Eq. 80 and models both strength and buckling interactions simultaneously.

These results have been derived for simply supported beam-columns, but the general design equation may provide a useful design expedient for other support conditions as well. The accuracy of the equation in such cases remains to be investigated but may turn out to be quite good.

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APPENDIX II NOTATION

The following symbols are used in this paper:

A	=	cross-sectional area;
b	=	half-width of flange, Fig. 2;
c	=	half-depth of beam, Fig. 2;
e	=	elementary theory (subscript);
EI_y, EI_z	=	flexural rigidities;
EI_w	=	torsional warping rigidity;
F	=	design strength;
f	=	design stress;
GK	=	St. Venant torsional rigidity;
I_z, I_y	=	moments of inertia;
L	=	length;
M	=	end couple;
M_1, M_2	=	critical values equal end couples about z and y axes, Eqs. 45 and 46;

M_y, M_z	=	bending moments;
m_1, m_2	=	bending moments, Eqs. 28 and 29;
P	=	axial compression;
P_1, P_2, P_T	=	critical values of P , Eqs. 24, 25, and 27;
p	=	distributed load in negative y direction;
p_1	=	critical value of p acting above shear center;
$\hat{p}_0, p_0$	=	critical values of p acting at shear center, Eqs. 31 and 33;
q	=	distributed load in negative z direction;
Q	=	concentrated load in negative z direction;
r_0	=	polar radius of gyration of cross section about shear center;
S	=	beam spacing in flat roof system;
S_1	=	critical value of S for ponding collapse, Eq. 26;
T	=	torque;
T	=	torsion (subscript);
t	=	total, i.e., actual, value (subscript);
U	=	potential energy;
u, v, w	=	displacements in x , y , and z directions;
V_y, V_z	=	internal shear force;
W	=	concentrated force in negative z direction;
W_1	=	critical value of W acting above shear center;
$\hat{W}_0, W_0$	=	critical value of W acting at shear center, Eqs. 32 and 34;
x, y, z	=	coordinate axes, Fig. 1;
α_1, α_2	=	see Eqs. 59 and 60;
β	=	rotation displacement about x axis;
δ_y, δ_z	=	center deflections;
η_1, η_2	=	see Eqs. 77, 78, and 81;
$\zeta_1, \zeta_2, \zeta_T$	=	see Eqs. 39–41;
θ	=	buckling criterion, Eq. 38;
θ_1, θ_2	=	see Eqs. 54 and 55;
λ	=	see Eq. 42;
μ	=	see Eq. 63;
μ_1, μ_2	=	see Eqs. 43 and 44;
ν	=	see Eq. 30;
ρ	=	weight density of water;
σ	=	normal stress;
τ	=	shear stress;
ϕ	=	center axial rotation; and
Ψ	=	strength criterion, see Eq. 74.