EMPIRICAL FAILURE CRITERIA WITH CORRELATED RESISTANCE VARIABLES

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ABSTRACT: When a member is subjected to combined loading, such as combined bending and compression, the failure criterion contains more than one resistance variable. This paper addresses the practical effects of correlation among those resistance variables, both during the experimental inference of the failure criterion itself and during subsequent reliability analyses that employ it. There are two main findings. The first, of interest to experimental researchers. is that the mean failing load of a population is always less than the load that would be obtained from the exact (but as yet unknown) failure criterion applied to an individual member, provided that the failure criterion is concave downward, as all realistic engineering failure criteria are likely to be. The second finding, of interest to reliability analysts, is that the reliability of a member under combined loading is at a minimum when the resistance variables are maximally correlated. It is therefore conservative to assume perfect correlation. In that case, the problem is effectively univariate, that is, the effects of less-than-perfect correlation may be safely and conveniently ignored.

INTRODUCTION

The wood industry and the American Society of Civil Engineers' Committee on Wood are currently engaged in efforts to produce reliability-based design codes for engineered wood construction. This paper discusses the effect, or lack thereof, of correlations among the resistance variables when the design equation contains more than one resistance variable.

Generally speaking, the form of the design equation follows that of the failure criterion upon which it is based. Usually this is just a simple linear equation with only one resistance variable, e.g.

$$R - S = 0 \quad (1)$$

where R = resistance; and S = load (or sum of loads). Occasionally, however, the failure criterion expresses the interaction between two or more resistance variables. A typical interaction equation might have the form

$$\left(\frac{P}{R_1}\right)^a + \left(\frac{M}{R_2}\right)^b = 1 \quad (2)$$

where P = axial compressive load effect caused by the loads; M = bending moment load effect caused by the loads; R_1 = axial force resistance capacity; R_2 = moment resistance capacity; and a and b = empirically fitted parameters. In this case, it is likely that the two resistance variables are correlated, and the question arises as to how this affects the reliability of the member.

The main finding of the study is that such correlations can safely be ignored. Since the degree of correlation affects the means of the data from

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which the failure criterion was inferred, one might argue that the effect of the correlation was already modeled at that time and need not be accounted for again during reliability analysis. However, that argument is not valid, because reliability depends solely on the low tail of the joint distribution and mean value depends on the entire distribution. Generally speaking, correlation fattens both tails of the distribution. While this will always depress the reliability, its effect on the mean is unpredictable until the exact form of the failure criterion is taken into account.

Therefore, this study will examine the effects of correlation during two distinct operations: (1) The fitting of an empirical failure criterion to the means of populations tested at various load combinations; and (2) the reliability analysis of an individual member under combined load.

FITTING FAILURE CRITERION TO DATA

Testing the same member to destruction under more than one combined loading is impossible. It is therefore impossible to determine the combined-load failure criterion of a single member. Instead, we determine the average failure criterion of a population of members. At each combined load, a large sample of members is tested to failure and the results are averaged. We might assume that when such means are plotted, they would agree with the failure criterion of an individual member, at least in the limit as sample size approached infinity. It will be shown, however, that this is not necessarily the case.

To be specific, let us consider the case of combined bending and compression and assume that the individual failure criterion is known a priori to be the familiar linear interaction equation

$$\frac{M}{M_b} + \frac{P}{P_c} = 1 \quad (3)$$

in which M_b = bending moment resistance capacity; and P_c = compression resistance capacity. This equation has the form of Eq. 2 with $a = 1$ and $b = 1$. Other examples are considered in this paper, and some speculative general conclusions are drawn.

In this example, M_b and P_c are the resistance variables. The "linear" interaction equation is nonlinear in these variables because they are in the denominators. We shall assume this failure criterion is exact and applies to every member of the population. Any variation in the failure criterion from one individual member to another is completely accounted for by the variability in these two properties, i.e., by their variances and their correlation. To keep the notation compact, let us write $y \equiv M_b$ and $z \equiv P_c$. We call the machine load x. At a particular combination of load, let us say that the test machine load is $x = P$ and the bending moment is $y \equiv Pe$, in which e = the eccentricity of the axial compression. Then Eq. 3 becomes

$$x = g(y, z) \quad (4)$$

in which g = the failure criterion defined as

$$g(y, z) \equiv \frac{1}{\left(\dfrac{e}{y} + \dfrac{1}{z}\right)} \quad (5)$$

To find an approximation to the mean of x, we first expand $g(y, z)$ in a Taylor series about the mean of (y, z), i.e., about the point $y = \bar{y}, z = \bar{z}$:

$$x = g(\bar{y},\bar{z}) + g_y(\bar{y},\bar{z}) \cdot (y - \bar{y}) + g_z(\bar{y},\bar{z}) \cdot (z - \bar{z}) + \frac{1}{2} g_{yy}(\bar{y},\bar{z}) \cdot (y - \bar{y})^2$$

$$+ g_{yz}(\bar{y},\bar{z}) \cdot (y - \bar{y})(z - \bar{z}) + \frac{1}{2} g_{zz}(\bar{y},\bar{z}) \cdot (z - \bar{z})^2 + \ldots \quad (6)$$

in which a subscript denotes partial differentiation with respect to that variable; and an overbar denotes the mean value.

Taking the expected value in Eq. 6 yields

$$\bar{x} = g(\bar{y},\bar{z}) + \frac{1}{2} g_{\bar{y}\bar{y}}\sigma_y^2 + g_{\bar{y}\bar{z}}\rho\sigma_y\sigma_z + \frac{1}{2} g_{\bar{z}\bar{z}}\sigma_z^2 + \ldots \quad (7)$$

in which σ_y and σ_z = standard deviations; and ρ = the correlation coefficient. Call the second-order terms ϵ

$$\epsilon \equiv \frac{1}{2} g_{\bar{y}\bar{y}}\sigma_y^2 + g_{\bar{y}\bar{z}}\rho\sigma_y\sigma_z + \frac{1}{2} g_{\bar{z}\bar{z}}\sigma_z^2 \quad (8)$$

and neglect the higher order terms. Then, for the specific example in Eqs. 3 and 5, Eq. 7 becomes

$$\bar{x} = \frac{1}{\left(\frac{e}{\bar{y}} + \frac{1}{\bar{z}}\right)} + \epsilon \quad (9)$$

$$\epsilon \equiv \frac{-Q\overline{\alpha z}}{(1 + \bar{\alpha})^3} \quad (10)$$

$$Q \equiv V_y^2 - 2\rho V_y V_z + V_z^2 \quad (11)$$

$$\bar{\alpha} \equiv \frac{e\bar{z}}{\bar{y}} \quad (12)$$

in which V = the coefficient of variation (standard deviation over mean).

We see that $\bar{x}$ (the failure criterion that one would infer from the means of the tested samples) is not the same as $g(\bar{y},\bar{z})$ (the individual failure criterion evaluated at the means of the controlling strength properties). The error that would be made by inferring g from $\bar{x}$ is approximately given by ϵ. Note that ϵ depends on two things: (1) The covariance matrix of the controlling strength properties y and z; and (2) the second derivatives of the failure surface $g(y, z)$. All these quantities can be estimated fairly accurately, with the exception of ρ. Nevertheless, the form of Eq. 11 shows that, because strength properties tend to be equally variable and because their correlation is likely to be quite high $(\rho \approx 1)$, $Q \approx 0$, i.e., E is small. Fig. 1 shows a plot of Eq. 9 with $V_v = V_z = 0.3$ and ρ = 1, 0, or -1. The curve with ρ = 1 is also the exact failure criterion in this case. The other two curves are the failure criteria we would mistakenly infer from the data if ρ = 0 or -1. Our customary assumption that $g = \bar{x}$ is fairly accurate, with the error on the conservative side. (We assume that the error due to neglect

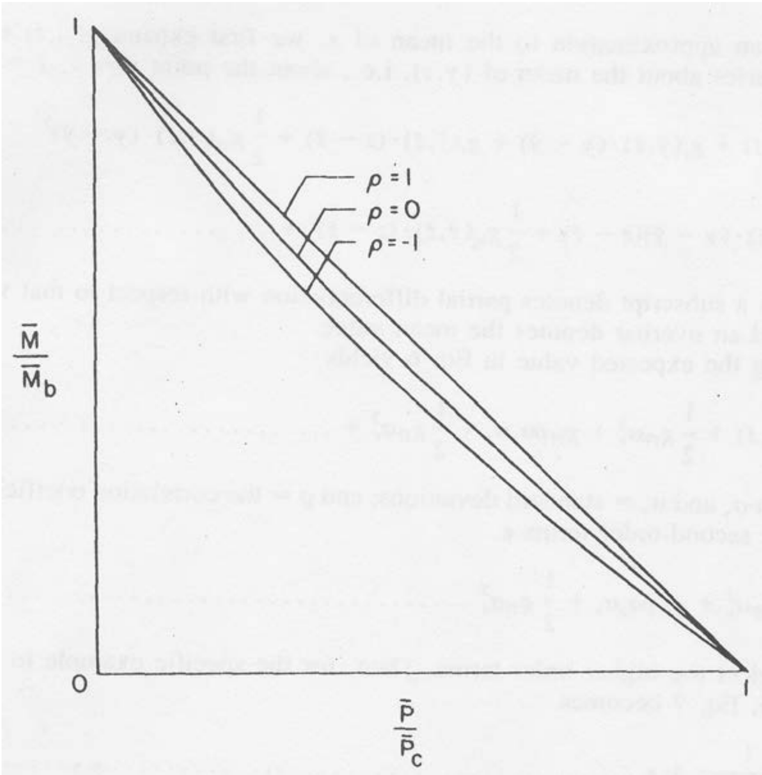

FIG. 1. Plot of Eq. 9 Showing Mean Aggrate Behavior of Members of which Individual Failure Criterion is Given by Eq. 3; Here $y = M_b$, $z = P_c$, $V_y = V_z =$ **0.3**

of the higher order terms in Eq. 7 is even smaller.)

We have verified this conclusion for two other cases: Eq. 2 with $a = 1$ and $b = 2$, and Eq. 2 with $a = 2$ and $b = 1$. In both cases, the result has the form

$$\bar{x} = g(\bar{y},\bar{z}) - f(\bar{\alpha})Q \quad (13)$$

in which $f(\bar{\alpha})$ = probability density function of $\bar{\alpha}$ and Q is given by Eq. 11. If $a = 2$ and $b = 1$, Eq. 2 is an exact failure criterion for mixed-mode buckling of a long slender beam-column, provided that $y \equiv$ critical buckling moment and $z \equiv$ critical axial compressive force (Zahn 1986). In that case, y is a multiple of $\sqrt{EG}$ and z is a multiple of E, where E = the modulus of elasticity; and G = the shear modulus.

Note that it would be possible to actually measure y and z nondestructively and obtain the covariance matrix. This case is also otic in which the exact failure criterion is the same for every member of the population and is known in advance of any testing. In other cases, we cannot know the failure criterion in advance and must assume the population is homogeneous. i.e., that it has only one failure criterion that applies to all members.

Finally, we have also examined the column failure criterion of Ylinen (1956):

$$P = \frac{P_c + P_E}{2c} - \sqrt{\left(\frac{P_c + P_E}{2c}\right)^2 - \frac{P_c P_E}{c}} \quad (14)$$

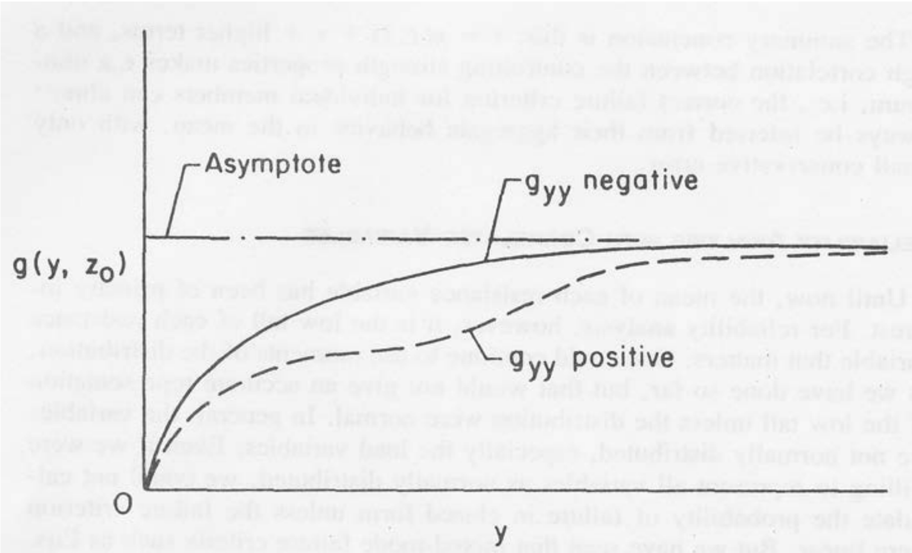

FIG. 2. Typical Failure Criterion $g(y,z)$ Plotted as Function of y for Fixed z; Solid Line is Realistic; Dashed Line Shows Exotic and Unlikely Behavior

in which P_c = the crushing load capacity of a short member; P_E = load value obtained by applying Euler's buckling formula; and c = an adjustable constant. Here we found that

$$\bar{x} = g(\bar{y},\bar{z}) + \epsilon \qquad (15)$$

$$g(y,z) \equiv \frac{y+z}{2c} - r \qquad (16)$$

$$r \equiv \sqrt{\left(\frac{y+z}{2c}\right)^2 - \frac{yz}{c}} \qquad (17)$$

$$\epsilon \equiv -\frac{y^2z^2(1-c)}{8c^3r^3}\,Q \qquad (18)$$

in which $y \equiv P_c$; and $z \equiv P_E$. Once again the statistical measures of y and z appear as the factor Q (Eq. 11).

In general, g_{yy} and g_{zz} should be negative for a realistic empirical failure criterion. To understand why, consider Fig. 2, which shows g as a function of y for some fixed z. The function must contain the origin because a small strength property y implies a small failing load x. Likewise, the function must approach some asymptotic value dependent only on z as y becomes large, because if one strength property y becomes very large, the failing load x is entirely determined by the value of the other strength property z. Thus, the shape must be like the solid line in Fig. 2, a line for which g_{yy} is negative. Of course, g_{yy} could still be positive in some local interval, as shown in the dashed line. However, such a line would probably never be fitted to data, because engineering failure criteria are the basis of design and must of necessity be kept its simple as possible. This result suggests that even if the mixed partial g_{yz} is positive, the net correction E is likely to be negative, as in the preceding examples.

The summary conclusion is this: $\bar{x} = g(\bar{y}, \bar{z}) + \epsilon +$ higher terms, and a high correlation between the controlling strength properties makes ϵ a minimum, i.e., the correct failure criterion for individual members can almost always be inferred from their aggregate behavior in the mean, with only small conservative error.

Reliability Analysis with Correlated Variables

Until now, the mean of each resistance variable has been of primary interest. For reliability analysis, however, it is the low tail of each resistance variable that matters. One could continue to use moments of the distribution, as we have done so far, but that would not give an accurate representation of the low tail unless the distribution were normal. In general, the variables are not normally distributed, especially the load variables. Even if we were willing to represent all variables as normally distributed, we could not calculate the probability of failure in closed form unless the failure criterion were linear. But we have seen that mixed-mode failure criteria such as Eqs. 3 and 14 are nonlinear. Therefore, we must resort to an approximate method of analysis, such as the "extended level two" method (Thoft-Christensen and Baker 1982). In that method, two very accurate approximations are introduced: (1) Local linearization of the failure criterion (Hasofer-Lind reliability index); and (2) normalization of nonnormal distributions (Rackwitz-Fiessler algorithm).

Hasofer-Lind Reliability Index

The Hasofer-Lind reliability index ß (Hasofer and Lind 1974; Thoft-Christensen and Baker 1982) is defined as the minimum distance from the origin to the failure surface in a space of reduced variables. The point on the failure surface is called the "design point." (See also Appendix I, where a search algorithm for the calculation of ß is presented.) For our purposes, it is sufficient to note here that if and only if the failure criterion is linear and all variables are normal, ß reduces to the well-known second-moment reliability index. In that case, ß bears a simple relationship to the probability of failure P_f:

$$P_f = \Phi(-\beta) \qquad (19)$$

in which Φ = the standard normal cumulative distribution function (CDF). If the failure surface, $g = 0$, is nonlinear and all variables are normal, Eq. 19 is approximately true in the sense that the failure surface has been replaced by a local linearization obtained from a Taylor series at the design point. The error is equal to the integral of the multivariate normal CDF of the resistance variables times the probability density function (PDF) of the load over the region between $g = 0$ and $g' = 0$, where g' = the linear Taylor series approximation. If one were intent upon calculating the exact probability of failure, it would be advisable to start with Eq. 19 and integrate numerically only for the error correction. This would be far more accurate than direct numerical evaluation of the multiple infinite integral for the probability of failure. In summary, the Hasofer-Lind reliability index is more than merely convenient. In most applications it is highly accurate as well.

Rackwitz-Fiessler Algorithm

The Rackwitz-Fiessler algorithm (Rackwitz and Fiessler 1977) is a straightforward approximation of the marginal PDF of a non-normal variable with a normal PDF that has the same ordinate and slope at the design point. In this way, the approximating normal PDF is given a low tail that is nearly the same as that of the actual PDF. The normal variable so obtained can then be reduced and employed in the Hasofer-Lind definition of the reliability index. Again, the approximation error is quite small, and it is doubtful whether a more accurate result could be obtained by numerical integration. Unfortunately, this does not mean that Eq. 19 is always fairly accurate. However, most of the error comes not from the method of analysis but from the crudeness of the load and resistance distributions that must be employed. Owing to the lack of sufficient data, the upper tails of the nominal load distributions should not be regarded as literally true, yet they exert a powerful influence on the calculated reliability index, especially when combined with resistance distributions that have fat lower tails. For this reason, the reliability index is not a true measure of reliability, especially for comparing materials with widely dissimilar distributions. It is, however, a useful index for comparing materials with similar distributions. For example, it can be used to calibrate a code for a given material or to homogenize a code. However, comparisons across materials with widely dissimilar distributions are probably not meaningful and should not be relied upon.

Effect of Correlation

We are interested in examining the effect of correlation upon reliability in cases where the variables are nonnormal. It is perhaps worth examining the meaning of correlated nonnormal variables. The statistical literature contains very few multivariate distributions of which the marginals are different nonnormal types, and none of these employ the usual definition of correlation. Of course, one could always define correlation as the covariance divided by the product of the standard deviations, but for nonnormal variates, this quantity may possess a theoretical maximum value of less than one. Bear in mind, we do not actually know the covariance matrix; we are interested only in studying its effect upon reliability. If we cannot predict its maximum value, that may be difficult to do. But since we are going to use the Rackwitz-Fiessler algorithm, it is sufficient to impose a correlation matrix upon the approximating normal distributions as if they were the marginals of a multivariate normal. The relation of this correlation matrix to the covariance matrix of the original variables is, of course, unknown, but one can hope that as the normal correlation ranges from -1 to $+1$, the qualitative effect upon reliability will be the same as if the covariance of the original variables ranged from its minimum to its maximum value. (We shall see that this hope is only roughly realized.)

While all the necessary theory for nonlinear failure surfaces and for correlated variables is contained in Thoft-Christensen and Baker (1982), the combined case of both nonlinear failure surface and correlated variables is not treated there. For the reader's convenience, Appendix I includes a general search algorithm. Here we show the results of applying that algorithm to the following two cases:

1. Case One: All variables normal. In this case, the correlation matrix can be interpreted literally.

2. Case Two: Nonnormal variables. In this case, the meaning of the correlation matrix is not as clear, but the effect upon reliability will bc seen to be qualitatively similar to case one, within the limits of accuracy imposed by correlating the Rackwitz-Fiessler normals rather than the original variables.

For both cases, the specific example to be considered will be a nominal wood 2 x 6 under eccentric axial load. We suppose that the failure criterion has the form of Eq. 3. We imagine we are constructing a design code in which the nominal resistance values are the fifth percentiles of the distributions. Thus, the member properties shall be: area A = 8.25 sq in. (0.00532 m^2); section modulus Z = 7.5625 cu in. (0.0001239 m^3); nominal compressive strength F_{cn} = 1,900 lb/sq in. (13.1 MPa); and nominal bending strength F_{bn} = 2,280 lb/sq in. (15.7 MPa).

The nominal load shall be taken to be the mean load. This is the usual way of treating live loads. The new design code formula is

$$\frac{P_n}{\phi_c P_{cn}} + \frac{M_n}{\phi_b M_{bn}} \leq 1 \qquad (20)$$

in which $P_n \equiv$ nominal axial force caused by the load, $M_n \equiv$ nominal bending moment caused by the load; ϕ_c and $\phi_b \equiv$ resistance factors; $P_{cn} \equiv AF_{cn}$; and $M_{bn} \equiv ZF_{bn}$.

We shall suppose that the target reliability index of the new code is ß = 2.5 and that the coefficients of variation of the basic variables are: for compression, V_c = 0.25; for bending, V_b = 0.28; and for load, V_L = 0.25.

Under eccentric axial load

$$M_n = P_n e \qquad (21)$$

We shall examine several values of e and plot ß as a function of e to see the effect of different load combinations upon reliability.

Reduction to Single-Load Cases

If e = 0, then Eq. 20 reduces to

$$P_n - \phi_c P_{cn} \leq 0 \qquad (22)$$

which determines ϕ_c. Likewise, under pure bending, Eq. 20 reduces to

$$M_n - \phi_b M_{bn} \leq 0 \qquad (23)$$

which determines ϕ_b. With these values of the resistance factors we shall analyze Eq. 20 for the reliability index ß at various load combinations and degrees of correlation.

Case One: Normal Variables. For normal distributions, we have nominal = mean (1 - 1.645V). Thus we find that: mean compressive strength $\bar{F}_c$ = 3,227 lb/sq in. (22.3 MPa); and mean bending strength $\bar{F}_b$ = 4,227 lb/sq in. (29.2 MPa).

Consider first the single-load case of compression alone, and employ the search algorithm from Appendix I. Applying the algorithm in design mode with ß = 2.5 and failure surface given by

$$P - P_c = 0 \qquad (24)$$

we find that the mean load is $\bar{P}$ = 9,049 lb (40.2 kN). Therefore ϕ_c = 0.577 from Eq. 22. Likewise, for bending alone, with failure surface given by

$$M - M_b = 0 \qquad (25)$$

we find that the mean load is $\bar{M}$ = 8,908 in.-lb (1,006 Nm). Therefore ϕ_b = 0.517 from Eq. 23. Here we have been specifying the member size and solving for the maximum allowable load. In the normal design use of the equations, the designer would specify the load and solve for member size. In the next step, we shall specify both the load and member size and analyze the reliability by employing the search algorithm in the analysis mode. Solving Eq. 20 for the required area yields

$$A = \frac{P_n}{\phi_c F_{cn}} + \frac{P_n e A}{\phi_b F_{bn} Z} \qquad (26)$$

Without loss of generality, we may specify the nominal load to be unity, which is equivalent to an arbitrary change of units. Then the input means and standard deviations become the following: for load, $\bar{P} = 1$, $\sigma_P = 0.25$; required area $A = 0.0009116(1 + 1.016e)$; for compressive strength, $\bar{P}_c \equiv \bar{F}_c A = 2.94(1 + 1.016e)$, $\sigma_c = 0.25\bar{P}_c$; and for bending strength, $\bar{M}_b \equiv F_b Z = 3.64(1 + 1.016e)$, $\sigma_b = 0.28\bar{M}_b$.

The correlation matrix is

$$\mathbf{R} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & \rho \\ 0 & \rho & 1 \end{bmatrix} \qquad (27)$$

in which the order of the variables is load. compression, and bending. Employing the search algorithm in the analysis mode, with eccentricity in the range of $0 \leq e \leq 1$ and various values for the correlation p, yields the results shown in the left half of Fig. 3. The right half of Fig. 3 was obtained by solving Eq. 20 for the required section modulus and specifying the nominal moment load to be unity. This yields

$$Z = \frac{1}{\phi_b F_{bn}} + \frac{Z}{e\phi_c F_{cn} A} \qquad (28)$$

and input statistics as follows: for load, $M = 1$, $\sigma_M = 0.25$; required section modulus $Z = 0.001613(1 + 0.984/e)$; for compressive strength, $\bar{P}_c \equiv \bar{F}_c A = 2.90(1 + 0.984/e)$, $\sigma_c = 0.25\bar{P}_c$; and for bending strength, $\bar{M}_b \equiv \bar{F}_b Z = 3.59(1 + 0.984/e)$, $\sigma_b = 0.28\bar{M}_b$.

Using these values and taking the eccentricity in the range $1 \geq 1/e \geq 0$ yielded the results shown in the right half of Fig. 3. Note that perfcct correlation produces a lower bound ß equal to the value prescribed for single-load cases. Any less correlation can only increase the reliability. This seems reasonable if one considers that perfect correlation effectively reduces the number of independent resistance variables to one and, therefore, produces the same reliability as that specified for a single load. Less correlation means a degree of independence between the two resistance variables. This makes it less likely that both of them will be small when they are selected at random, i.e., that the member will fail. This powerful influence of correlation stands in sharp contrast to the minor influence found earlier. It is because

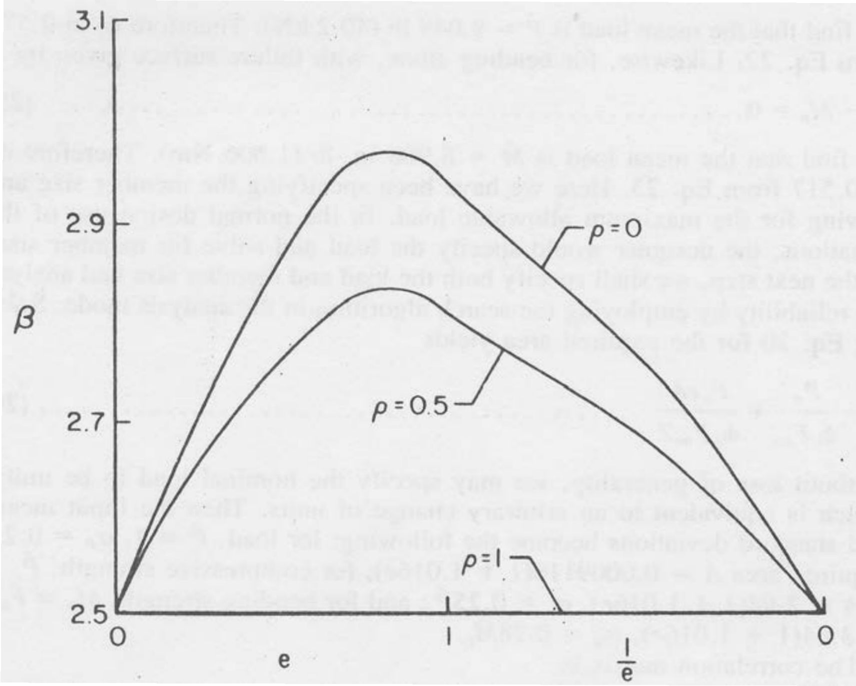

FIG. 3. Reliability index as Function of Load Combination Using Eq. 3 as individual Failure Criterion; All Variables Have Normal Distributions

the low tails of the resistance distributions are controlling here, whereas the means were the controlling values in the earlier case.

Case Two: Nonnormal Variables. For this case, we shall choose a variety of distributions. Let the load be a Gumbel distribution, the compressive strength be a Weibull distribution, and the bending strength be a lognormal distribution. We shall use the same coefficients of variation as in case one. Then, following all the same steps as in case one, we find that: $\bar{F}_c$ = 1,756 lb/sq in. (12.1 MPa) for a Weibull distribution with V_c = 0.25; and $\bar{F}_b$ = 1,958 lb/sq in. (13.5 MPa) for a lognormal distribution with V_b = 0.28.

Using a target ß = 2.5, the single-load cases yield ϕ_c = 0.580 and ϕ_b = 0.637. Analyzing Eq. 20, with eccentricity in the range $0 \leq e \leq 1$, input statistics $\bar{P} = 1$, $\sigma_P = 0.25$; $\bar{F}_c = 3.03(1 + 0.828e)$, $\sigma_c = 0.25\bar{F}_c$; $\bar{F}_b$ = 2.12(1 + 0.828e), $\sigma_b = 0.28\bar{F}_b$; and various values for p yields the left half of Fig. 4. Analyzing Eq. 20 further, with eccentricity in the range $1 \geq 1/e \geq 0$, input statistics $\bar{M}$ = 1, σ_M = 0.25; $\bar{F}_c$ = 3.66(1 + 1.208/e), $\sigma_c = 0.25\bar{F}_c$; $\bar{F}_b$ = 2.56(1 + 1.208/e), $\sigma_b = 0.28F_b$; and various values for p yields the right half of Fig. 4. The behavior is qualitatively similar to case one in that a degree of independence between the resistance variables strongly increases the reliability. One cannot say that "perfect" correlation produces a uniform reliability, as it did in case one, but then perfect correlation is difficult to define when the marginal distributions are of different types and different variances. Additionally, we have not correlated the original variables, only their normal approximations obtained from the Rackwitz-Fiessler algorithm. Despite these considerations, Fig. 4 shows variations in ß of less

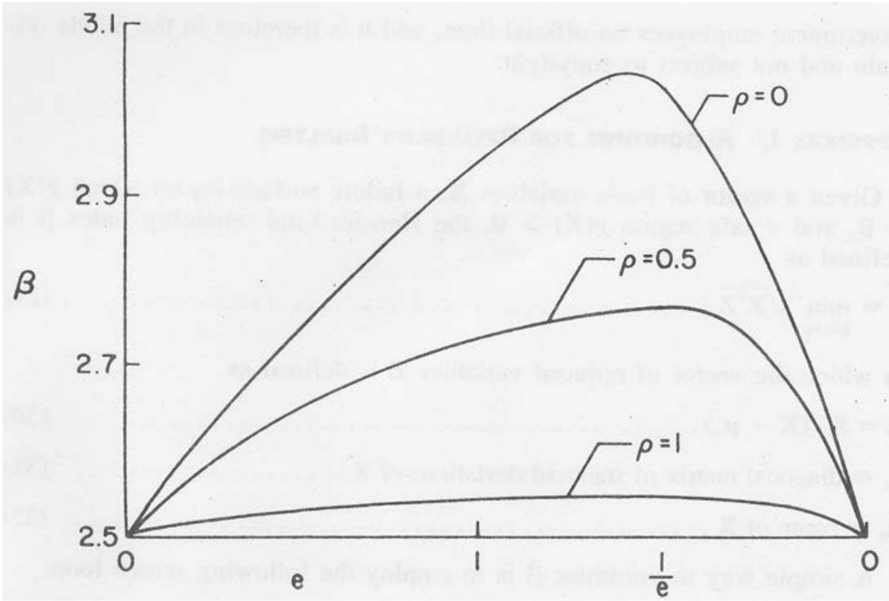

FIG. 4. Reliability Index as Function of Load Combination Using Eq. 3 as individual Failure Criterion; Each Variable Has Different Nonnormal Distribution

than 0.5% when p = 1. It seems likely that the conclusions from case one are robust, i.e., true even when the variables are not normally distributed.

SUMMARY AND CONCLUSIONS

In this paper we examined the question of fitting an empirical failure criterion to the means of test data obtained at various load combinations and found it gives a close approximation to the failure criterion that controls the behavior of individual members. The error is always small, always conservative, and smallest whenever the strength-controlling properties are most strongly correlated.

Then we examined the effect of correlation on reliability under combined load and found the reliability is increased by any degree of independence (lack of perfect correlation) that may exist among the strength-controlling properties. There is, therefore, no need to consider the effects of correlated resistance variables when constructing a new design code using reliability-based methods. Because the target reliability index employed in obtaining the resistance factors for single-load cases is always a lower bound on the reliability under combined load, the effects of partial correlation among interacting resistance properties may safely be ignored. This is true for all types of resistance-variable interaction. The worst case that could occur is that the interacting variables would be perfectly correlated, and in that case, the reliability index is uniform throughout the entire space of their interaction.

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APPENDIX I. ALGORITHMS FOR RELIABILITY ANALYSIS

Given a vector of basic variables **X,** a failure surface $\partial\omega$ on which g(X) = 0, and a safe region g(X) > 0, the Hasofer-Lind reliability index ß is defined as

$$\beta \equiv \min_{\mathbf{Z}\in\partial\omega} \sqrt{\mathbf{Z}^T\mathbf{Z}} \quad (29)$$

in which the vector of reduced variables Z is defined as

$$\mathbf{Z} \equiv \mathbf{S}_x^{-1}(\mathbf{X} - \boldsymbol{\mu}_x) \quad (30)$$

$$\mathbf{S}_x \equiv \text{diagonal matrix of standard deviations of } \mathbf{X} \quad (31)$$

$$\boldsymbol{\mu}_x \equiv \text{mean of } \mathbf{X} \quad (31)$$

A simple way to minimize ß is to employ the following search loop:

Initialize: $c_i := \pm 1; \quad \beta := \beta_0; \quad \mathbf{c} := \mathbf{S}_x\mathbf{c}$

Step one: $|\mathbf{c}| := \sqrt{\mathbf{c}^T\mathbf{c}}$

$$\boldsymbol{\alpha} := -\frac{c}{|\mathbf{c}|}$$

$$\mathbf{X} := \boldsymbol{\mu}_x + \mathbf{S}_x\boldsymbol{\alpha}\beta$$

Evaluate: $g(\mathbf{X})$ and $\mathbf{b} \equiv \dfrac{\partial g}{\partial \mathbf{X}}$ (subroutine)

Solve: $\dot{g}(\mathbf{X}) = 0$ for β

$$\mathbf{c} := \mathbf{S}_x\mathbf{b}$$

Iterate from step one until convergence is obtained.

The := sign in this loop is a "replacement" operator like the = sign in Fortran and some other computer languages. The initial values of c_i should be +1 for loads and -1 for resistances. The multiplication by S_x during initialization gives a slightly better first guess for the design point. This was done because the search loop may not converge if g is highly nonlinear and the first guess is not sufficiently close to the correct answer.

In this search loop

$$\mathbf{Z} = \boldsymbol{\alpha}\beta \quad (33)$$

in which $\boldsymbol{\alpha}$ = a unit vector in the direction of Z; and ß = its magnitude. The converged value of ß is the Hasofer-Lind reliability index.

Instead of solving a possibly nonlinear equation for ß each time through the loop, it is sufficient to employ the replacement statement

$$\beta := \beta + \frac{g}{|\mathbf{c}|} \quad (34)$$

This simple result is easily derived by replacing g by its two-term Taylor series expansion about the design point, i.e., the point at which ß is a minimum.

Solving for ß is the analysis problem. The corresponding design problem is one in which ß is specified and one of the means of the random variables is not. Instead of solving $g = 0$ for ß, one solves it for μ_j, the unknown mean. Again, replacing g with a linear Taylor series approximation results in a simpler replacement statement

$$\mu_j := \mu_j - \frac{g}{b_j} \qquad (35)$$

Nonnormal Variables

If any X_j is nonnormal, the Rackwitz-Fiessler algorithm may be inserted in the loop just after updating ß or μ_j

$$z = \Phi^{-1}[F(X_j)] \qquad (36)$$

$$S_{xjj} = \frac{\phi(z)}{f(X_j)} \qquad (37)$$

$$\mu_{xj} = X_j - S_{xjj}z \qquad (38)$$

in which F and f = the CDF and PDF of X_j; and Φ and ϕ = the standard normal CDF and PDF. Here Eq. 36 equates the ordinates and Eq. 37 equates the slopes of F and its locally approximating normal. Eq. 38 merely defines the dummy variable z. The resulting values of S_{xjj} and μ_{xj} belong to the approximating normal and should be used in place of the actual values of S_{xjj} and μ_{xj} throughout the rest of the loop.

Correlated Basic Variables

If the basic variables are correlated, Thoft-Christensen and Baker (1982) recommend transforming the problem to a space of new variables that are uncorrelated and thereafter minimizing ß in that space. This requires one to solve an eigenvector-eigenvalue problem associated with the given covariance matrix. Herein we derive an algorithm that merely appeals to the existence of such a space. without requiring the user to leave the original space of correlated variables. When using this algorithm, no eigenvalue analysis need ever be performed, thereby enabling a more compact coding and a greatly reduced computing time.

Given a covariance matrix C_x and its associated correlation matrix R

$$\mathbf{C}_x \equiv \mathbf{S}_x\mathbf{R}\mathbf{S}_x \qquad (39)$$

there exists a “similarity transformation” matrix A such that

$$\mathbf{C}_x\mathbf{A} = \mathbf{A}\mathbf{\Lambda} \qquad (40)$$

in which the columns of A are the orthonormal eigenvectors of the matrix C_x, and the diagonal elements of diagonal matrix Λ are its eigenvalues. For such a matrix A, the transpose is also the inverse

$$\mathbf{A}^T = \mathbf{A}^{-1} \qquad (41)$$

If a transformed vector of basic variables Y is defined such that

$$\mathbf{X} = \mathbf{A}\mathbf{Y} \quad (42)$$

it follows that $\mathbf{\Lambda}$ is the covariance matrix of Y. Define the Hasoler-Lind reliability index ß in the Y space. Then the reduced variables are

$$\mathbf{Z} = \mathbf{\Lambda}^{-1/2}(\mathbf{Y} - \mathbf{A}^T\boldsymbol{\mu}_x) \quad (43)$$

and the direction of Z is once again given by

$$\boldsymbol{\alpha} = -\frac{\mathbf{c}}{|\mathbf{c}|} \quad (44)$$

in which

$$\mathbf{c}^T \equiv \left(\frac{\partial g}{\partial \mathbf{Z}}\right)^T \quad (45a)$$

$$\mathbf{c}^T = \left(\frac{\partial g}{\partial \mathbf{X}}\right)^T \frac{\partial \mathbf{X}}{\partial \mathbf{Y}} \frac{\partial \mathbf{Y}}{\partial \mathbf{Z}} \quad (45b)$$

$$\mathbf{c}^T = \mathbf{b}^T\mathbf{A}\mathbf{\Lambda}^{1/2} \quad (45)$$

Therefore, the square of the magnitude of c reduces to

$$\mathbf{c}^T\mathbf{c} = \mathbf{b}^T\mathbf{A}\mathbf{\Lambda}^{1/2}\mathbf{\Lambda}^{1/2}\mathbf{A}^T\mathbf{b} \text{ (by Eq. 45)} \quad (46a)$$

$$\mathbf{c}^T\mathbf{c} = \mathbf{b}^T\mathbf{C}_x\mathbf{b} \text{ (by Eqs. 40 and 41)} \quad (46b)$$

Let d be defined as

$$\mathbf{d} \equiv \mathbf{S}_x\mathbf{b} \quad (47)$$

and use this and Eq. 39 to write the square of the magnitude of c as

$$|\mathbf{c}|^2 = \mathbf{d}^T\mathbf{R}\mathbf{d} \quad (48)$$

We wish to remain in the X space. Therefore we say

$$\mathbf{X} = \mathbf{A}\mathbf{Y} \quad (49a)$$

$$\mathbf{X} = \mathbf{A}\boldsymbol{\mu}_y + \mathbf{A}\mathbf{\Lambda}^{1/2}\boldsymbol{\alpha}\beta \text{ (by Eqs. 43 and 33)} \quad (49b)$$

$$\mathbf{X} = \boldsymbol{\mu}_x + \mathbf{A}\mathbf{\Lambda}^{1/2}\frac{-\mathbf{\Lambda}^{1/2}\mathbf{A}^T\mathbf{b}}{|\mathbf{c}|} \cdot \beta \text{ (by Eqs. 44 and 45)} \quad (49c)$$

Examine the coefficient of ß/|c|. It reduces to

$$-\mathbf{C}_x\mathbf{b} \quad (50)$$

by definition of $\mathbf{A}$ and $\mathbf{\Lambda}$ and further reduces to

$$-\mathbf{S}_x\mathbf{R}\mathbf{d} \quad (51)$$

by definition of R and d (Eqs. 39 and 47). Now define

$$\boldsymbol{\gamma} \equiv -\frac{\mathbf{R}\mathbf{d}}{|\mathbf{c}|} \quad (52)$$

Then we can write X as

$$\mathbf{X} = \boldsymbol{\mu}_x + \mathbf{S}_x\boldsymbol{\gamma}\beta \quad (53)$$

in which $\boldsymbol{\gamma}$ is determined from d (Eq. 52). and |c| is determined from Eq. 48. Note that $\boldsymbol{\gamma}$ in X space is analogous to $\boldsymbol{\alpha}$ in Y space, but is not a unit vector. Likewise, d in X space is analogous to c in Y space, but whereas c is normal to the failure surface, d is not. One might think of R as the metric tensor of the X space. If X is uncorrelated, then R is the unit matrix and the X space has a Euclidean metric. If X is correlated, then R is not a unit matrix and the X space has a distorted metric. However, we can still minimize ß in that space, provided we employ the proper metric R when measuring the distance to the failure surface.

The search loop can now be written as follows:

Initialize: $d_i := \pm 1;\ \beta := \beta_0;\ \mathbf{d} := \mathbf{S}_x\mathbf{d}$

Step one: $|\mathbf{c}| := \sqrt{\mathbf{d}^T\mathbf{R}\mathbf{d}}$

$$\boldsymbol{\gamma} := -\frac{\mathbf{R}\mathbf{d}}{|\mathbf{c}|}$$

$$\mathbf{X} := \boldsymbol{\mu}_x + \mathbf{S}_x\boldsymbol{\gamma}\beta$$

Evaluate: $g(\mathbf{X})$ and $\mathbf{b} \equiv \dfrac{\partial g}{\partial \mathbf{X}}$ (subroutine)

Either $\beta := \beta + \dfrac{g}{|\mathbf{c}|}$ (analysis)

or $\mu_j := \mu_j - \dfrac{g}{b_j}$ (design)

$$\mathbf{d} := \mathbf{S}_x\mathbf{b}$$

Iterate from step one until convergence is obtained.

Finally, if the Rackwitz-Fiessler algorithm is inserted just after updating ß or μ_j, this search loop can be used with nonnormal variables. However, the correlation matrix R must be understood as defining the correlation in an X space of normal variables, some of which are now local approximations obtained from the Rackwitz-Fiessler algorithm. As such, they have fictitious means and variances and may be expected to have a substantially different covariance matrix than that possessed by the original variables. In other words, the R matrix is fictitious and not easily related to the covariance matrix of the original variables X.

APPENDIX II. REFERENCES

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Appendix III. Notation

The following symbols are used in this paper.

A = cross-sectional area;
$\mathbf{A}$ = similarity transformation matrix in transformation algorithm;
a, b = interaction equation coefficients;
$\mathbf{b}$ = vector of partial derivatives of failure function g;
$\mathbf{C}_x$ = covariance matrix of basic variables;
c = fitted parameter of Ylinen's column formula;
$\mathbf{c}$ = trial vector in search algorithm;
$\mathbf{d}$ = trial vector in search algorithm for correlated variables;
E = modulus of elasticity;
e = eccentricity of axial compression;
F = strength property, if subscripted, or cumulative distribution function of basic variable;
f = probability density function of basic variable;
G = shear modulus;
g = failure criterion, function of basic variables;
M = bending moment load effect, property if subscripted;
P = axial compressive load effect, property if subscripted;
P_f = probability of failure;
R = resistance;
$\mathbf{R}$ = correlation matrix;
S = load;
$\mathbf{S}_x$ = diagonal matrix of standard deviations of X;
V = coefficient of variation;
$\mathbf{X}$ = vector of basic variables;
x = test machine load;
$\mathbf{Y}$ = transformed vector of basic variables (uncorrelated):
y, z = strength properties;
Z = section modulus of bending member;
$\mathbf{Z}$ = vector of reduced basic variables;
$\boldsymbol{\alpha}$ = unit vector in direction of Z in search algorithm;
β = Hasofer-Lind reliability index;
$\boldsymbol{\gamma}$ = analog of $\boldsymbol{\alpha}$;
ϵ = error of using aggregate behavior as approximation for individual behavior;
Λ = diagonal matrix of eigenvalues;
$\mu, \boldsymbol{\mu}$ = mean values;
ρ = correlation coefficient;
σ = standard deviation;
Φ = cumulative distribution function of standard normal distribution; and
ϕ = probability density function of standard normal distribution.

Subscripts

b = bending;
c = compression; and
n = nominal.