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On Asymmetric Discrete Laplace Distributions and Processes

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Abstract

After reviewing two asymmetric discrete Laplace models, some asymmetric discrete Laplace processes are introduced and investigated. The problem of developing bivariate discrete Laplace processes is also addressed.

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1 Introduction

Laplace distributions and their Asymmetric relatives have been found to be useful for describing a wide variety of data configurations. The flexibility inherent in the asymmetric models makes them particularly attractive models since the more classical symmetric models are included as special cases which will be useful in relatively rare cases where symmetry assumptions are scientifically justified. However, these are continuous models, In practice, most data is collected in a discretized form, due to rounding or because of available measurement limitations. This observation justifies investigation of discrete analogs of the symmetric and asymmetric Laplace models, which is the focus of the present paper. Just as in the continuous case, the symmetry exhibited by the discrete Laplace distribution will be a handicap when dealing with many real world data situations in which asymmetry is frequently encountered. Some simple variations of the discrete Laplace model will allow it to adapt to asymmetric data configurations. We begin by discussing the basic definitions and attributes of asymmetric discrete Laplace distributions in Section 2. In Section 3, we review results on geometric compounding of discrete asymmetric Laplace models, which are used in later sections in the development of discrete symmetric and asymmetric Laplace

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processes. In Sections 4 and 5, two distinct but related asymmetric discrete Laplace process models are exhibited. Section 6 briefly discusses generalized adaptations of the two types of processes. Lastly, we visit strategies for constructing bivariate versions of these processes in Section 7.

2 Asymmetric Discrete Laplace Distributions

For the basic asymmetric discrete model (introduced by Kozubowski and Inusah, 2006), we begin with two independent geometric variables V_1 and V_2 with $V_i \sim geo(p_i)$, $i = 1, 2$. We then define

$$X = V_1 - V_2. \tag{1}$$

If X has a representation of this form, we write $X \sim ADL(p_1, p_2)$. Note that in this paper, unless otherwise stated, geometric random variables are defined to have support $\{0, 1, 2, ...\}$. Kozubowski and Inusah (2006) first defined the univariate asymmetric discrete Laplace density as a discretized version of the asymmetric Laplace density. They then identified its characteristic function in the following form.

$$\phi_X(t) = p_1 p_2 (1 - (1 - p_1)e^{it})^{-1}(1 - (1 - p_2)e^{-it})^{-1}, \tag{2}$$

where $p_1, p_2 \in (0, 1)$. However, the equivalent definition, $X = V_1 - V_2$, is often, though not always, more convenient.

The possible values of X are $\{..., -3.-2.-1, 0, 1, 2, 3, ...\}$, and its discrete density can be derived as follows (using the notation $q_i = 1 - p_i, \quad i = 1, 2$ here and subsequently to simplify expressions).

For $x \geq 0$,

$$\begin{aligned} P(X = x) &= \sum_{v_2=0}^{\infty} P(V_1 = x + v_2, V_2 = v_2) \\ &= \sum_{v_2=0}^{\infty} p_1 q_1^{x+v_2} p_2 q_2^{v_2} \\ &= p_1 p_2 q_1^{x} (1 - q_1 q_2)^{-1} \end{aligned}$$

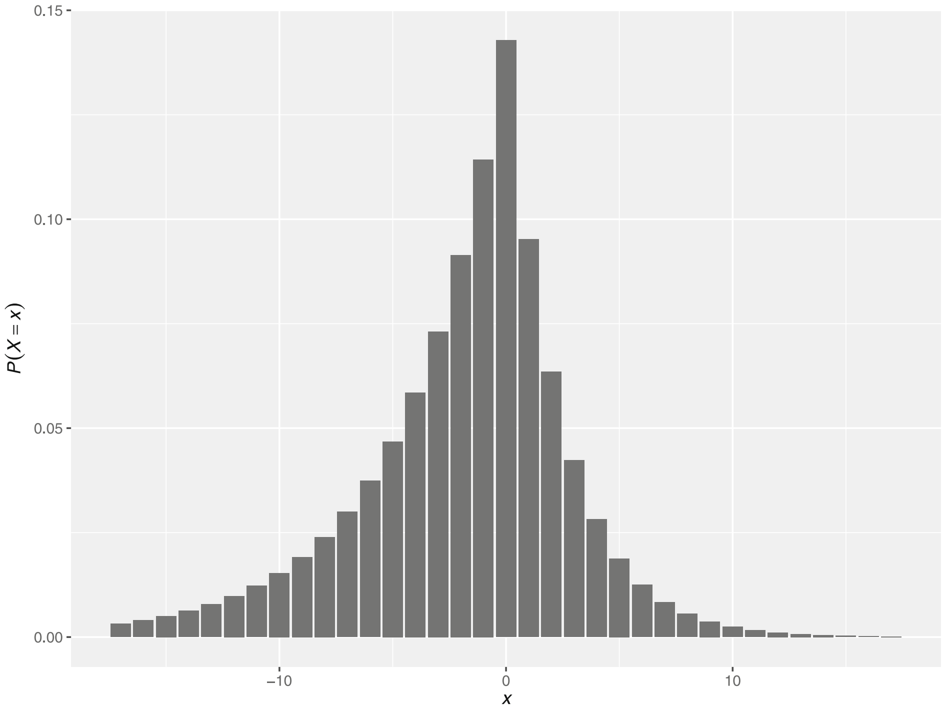

Figure 1: Example of ADL density with parameters $p_1 = \frac{1}{3}$ and $p_2 = \frac{1}{5}$

Analogously, for $x < 0$,

$$\begin{aligned} P(X = x) &= \sum_{v_1=0}^{\infty} P(V_1 = v_1, V_2 = v_1 - x) \\ &= \sum_{v_2=0}^{\infty} p_1 q_1^{v_1} p_2 q_2^{v_1 - x} \\ &= p_1 p_2 q_2^{-x} (1 - q_1 q_2)^{-1}. \end{aligned}$$

This distribution, called the asymmetric discrete Laplace distribution (ADL), is a discrete parallel to the asymmetric Laplace model $Y = U_1 - U_2$ where the U_i's are independent with $U_i \sim exp(\lambda_i), \quad i = 1, 2$. A typical ADL density is displayed in Fig. 1.

Remark 1 The fact that asymmetric discrete Laplace variables are infinitely divisible follows from the infinite divisibility of geometric variables.

We define a mixture alternative parallel to the generalized asymmetric Laplace (GAL) model as follows

$$Y = IV_1 + (1-I)(-V_2), \tag{3}$$

where $V_i \sim geo(p_i)$, $i = 1,2$, are independent and I is an independent Bernoulli random variable with $P(I=1) = \pi$.

In this case Y is said to have a generalized asymmetric discrete Laplace (GADL) distribution, and has a discrete density of the following form:

$$\begin{aligned} P(Y=y) &= \pi p_1 q_1^y, \quad y = 1,2,..., \\ &= \pi p_1 + (1-\pi)p_2, \quad y = 0, \\ &= (1-\pi)p_2 q_2^{-y}, \quad = -1,-2,-3,..., \end{aligned} \tag{4}$$

where p_1, p_2 and π are parameters ranging over the interval $(0,1)$. A typical GADL density is displayed in Fig. 2.

The corresponding characteristic function is of the form;

$$\phi_Y(t) = \pi p_1(1-(1-p_1)e^{it})^{-1} + (1-\pi)p_2(1-(1-p_2)e^{-it})^{-1}. \tag{5}$$

Using the characteristic function, we find, for the GADL distribution,

$$E(Y) = \frac{\pi(1-p_1)}{p_1} - \frac{(1-\pi)(1-p_2)}{p_2}$$

and

$$var(Y) = \left\{\frac{\pi(1-p_1)(2-p_1)}{p_1^2} + \frac{(1-\pi)(1-p_2)(2-p_2)}{p_2^2}\right\} - \left\{\frac{\pi(1-p_1)}{p_1} - \frac{(1-\pi)(1-p_2)}{p_2}\right\}^2.$$

In the case in which $X \sim ADL(p_1, p_2)$, the moments simplify to become

$$E(X) = (1-p_1)/p_1 - (1-p_2)/p_2$$

and

$$var(X) = (1-p_1)/p_1^2 + (1-p_2)/p_2^2.$$

Parametric inference for samples from the ADL distribution are investigated in Kozubowski and Inusah (2006) while inference for GADL samples is discussed in Arnold and Arvanitis (2022).

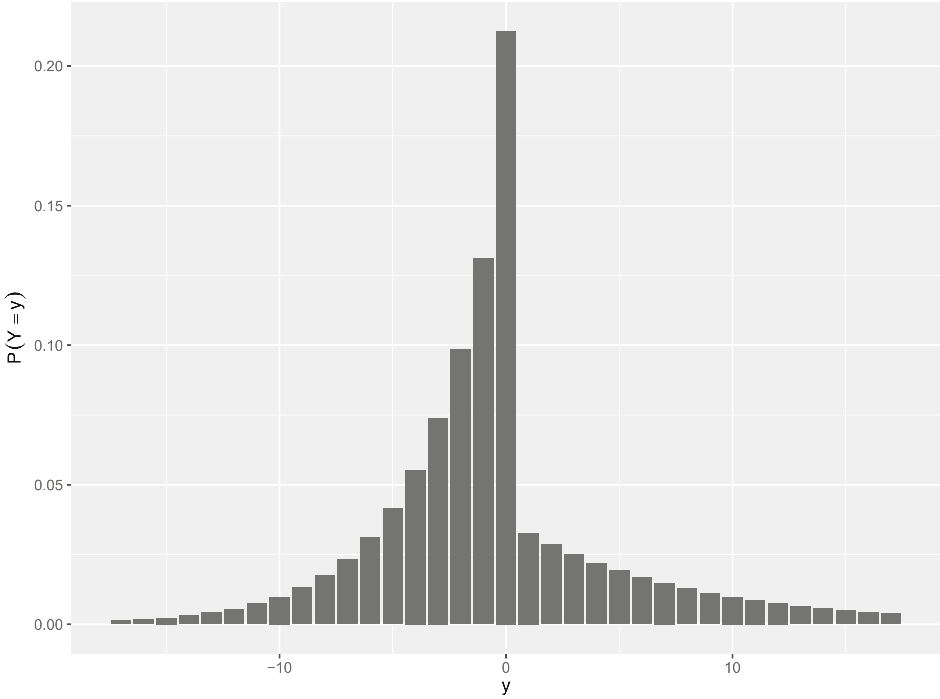

Figure 2: Example of a GADL density with parameters $p_1 = \frac{1}{8}$ and $p_2 = \frac{1}{4}$, and $\pi = \frac{3}{10}$

3 Regarding Geometric Sums of Discrete Asymmetric Laplace Variables

It is well known that geometric sums of exponential variables are again exponentially distributed. In parallel fashion, geometric sums of geometric variables are again geometrically distributed. Less obviously, but also true, is the fact that geometric sums of asymmetric (or symmetric) Laplace variables have again asymmetric (or symmetric) Laplace distributions (Kozubowski and Podgórski 2000).

These observations naturally inspire us to investigate whether a parallel situation exists involving discrete asymmetric Laplace variables and generalized discrete asymmetric Laplace variables. First, consider the ADL case (Fig. 1). Suppose that If $X_i,\ \ i = 1, 2, ...$ are i.i.d. $ADL(p_1, p_2)$ variables and N independent of the X_i's has a geometric distribution with support $\{1, 2, ...\}$ and success probability π, i.e., $P(N = j) = \pi(1-\pi)^{j-1}$, and define

$$Z = \sum_{i=1}^{N} X_i. \tag{6}$$

It is not at all obvious that Z has a characteristic function analogous to that of the ADL distribution, as in Eq. 2. Specifically we have

$$\begin{aligned}\phi_Z(t) &= \frac{\pi\phi_X(t)}{1-(1-\pi)\phi_X(t)} \\ &= \frac{\pi p_1 p_2 (1-(1-p_1)e^{it})^{-1}(1-(1-p_2)e^{-it})^{-1}}{1-(1-\pi)p_1 p_2(1-(1-p_1)e^{it})^{-1}(1-(1-p_2)e^{-it})^{-1}},\end{aligned} \tag{7}$$

However, Kozubowski and Inusah (2006), in their Proposition 4.2, were able to verify that $\phi_Z(t)$ is of the factorizable form of Eq. 2, with p_1 and p_2 replaced by new values p_1^* and p_2^*, which are complicated functions of p_1, p_2 and π, as displayed in their equation (31). Thus they have confirmed that geometric sums of ADL variables are again of the ADL form.

Turning to consider geometric sums of GDAL variables, we may argue as follows.

Suppose that $X_i,\ i = 1, 2, ...$ are i.i.d. $GADL(\pi, p_1, p_2)$ variables and that N independent of the X_i's has a geometric distribution with support $\{1, 2, ...\}$ and success probability δ, i.e., $P(N = j) = \delta(1-\delta)^{j-1}$, and define

$$Z = \sum_{i=1}^{N} X_i. \tag{8}$$

It follows that the characteristic function of Z is of the form:

$$\begin{aligned}\phi_Z(t) &= \frac{\delta\phi_X(t)}{1-(1-\delta)\phi_X(t)} \\ &= \frac{\delta\left[\frac{\pi p_1}{1-(1-p_1)e^{it}} + \frac{(1-\pi)p_2}{1-(1-p_2)e^{-it}}\right]}{\left\{1-(1-\delta)\left[\frac{\pi p_1}{1-(1-p_1)e^{it}} + \frac{(1-\pi)p_2}{1-(1-p_2)e^{-it}}\right]\right\}}\end{aligned} \tag{9}$$

It can be verified that this characteristic function is not of the form Eq. 5 and thus Z does not have a GADL distribution.

This rules out the possibility of defining discrete asymmetric Laplace processes analogous to the several available asymmetric Laplace processes that use, in their construction, the fact that geometric sums of asymmetric Laplace variables are again of the asymmetric Laplace form.

Although we can construct discrete asymmetric Laplace processes by directly using a geometric sum approach (as was done when constructing asymmetric Laplace processes in Arnold and Arvanitis, 2025), the complicated relationship between the basic parameters and the parameters in the summed version (as in equation (31) in Kozubowski and Inusah, 2006) may make interpretation of model parameters less convenient. Instead, we propose to construct ADL and GADL processes as differences (and weighted differences) of independent geometric processes. Two examples are provided in the following Sections.

4 A Type I Asymmetric Discrete Laplace Process

If $X_i,\ i = 1, 2, ...,$ are i.i.d. $Geometric(\pi)$ variables with support $\{1, 2, ...\}$ and N independent of the X_i's has a geometric distribution with support $\{1, 2, ...\}$ and success probability p, i.e., $P(X_i = j) = p(1-p)^{j-1}$, then

$$Z = \sum_{1}^{N} X_i \sim geometric(p\pi).$$

Making use of this observation, it is possible (parallel to an exponential construction in Arnold, 1989), to define a future innovations geometric process as follows. For each $t = 0, 1, 2, 3, ...$ define i.i.d. $Bernoulli(p)$ random variables U_t. Then for each $t = 0, 1, 2, ...$ define a random variable N_t^+ as follows

$$N_t^+ = 1, \text{ if and only if } U_t = 1,$$

$$N_t^+ = i \text{ if and only if } U_t = U_{t+1} = \cdots = U_{t+i-2} = 0 \text{ and } U_{t+i-1} = 1.$$

These N_t^+'s will be $geometric(p)$ random variables with support $1, 2, ...$ They are identically distributed but are not independent. This process will be called the base-geometric process. The sample paths associated with this base process have simply describable form. A sample path consists of decreasing sequences of the form $k, k-1, k-2, ..., 2$ for various values of $k \geq 2$ separated by runs of 1's. See Fig. 3 for a typical sample path.

The future innovations geometric process is then defined by

$$X_t = \sum_{i=1}^{N_t^+} \varepsilon_{t+i-1}, \tag{10}$$

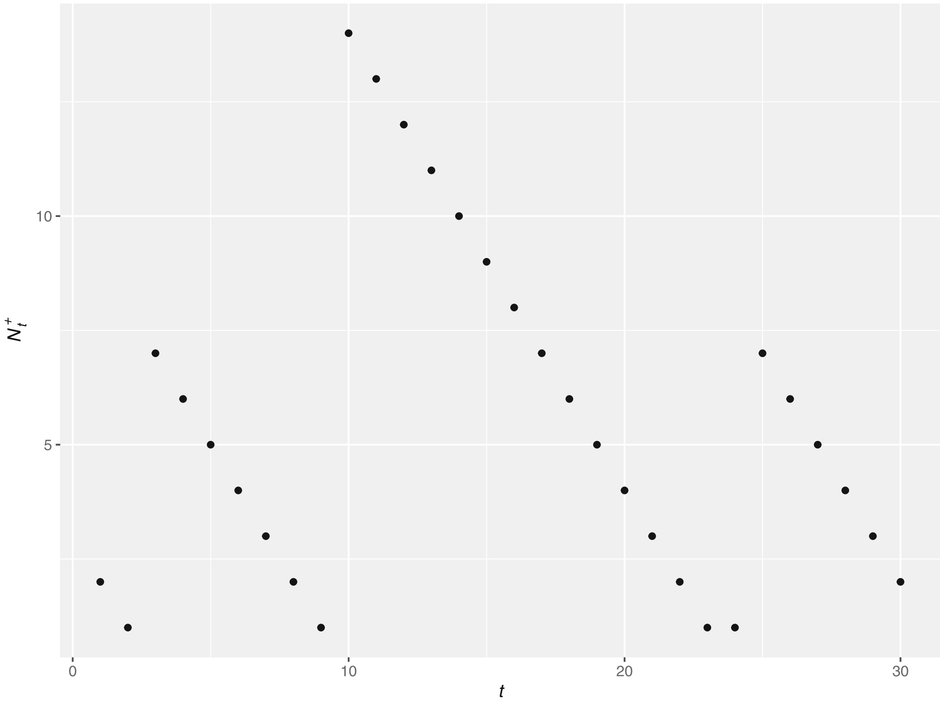

Figure 3: Type I sample path, N_t^+, for parameter $p = \frac{2}{7}$

where the ε_t's are i.i.d. geometric (π) variables, independent of N_t^+.

This is a stationary geometric process. It will be called a Type I process. The sample paths for this Type I process consist of blocks of decreasing integers. An example a Type I process is shown in Fig. 4.

A simple way to construct a stationary asymmetric discrete Laplace processes is then to take a difference of two independent Type I geometric processes. That is, let $\{X_t^{(1)}\}$ be a Type I geometric process with parameter p_1 and let $\{X_t^{(2)}\}$ be an independent Type I geometric process with parameter p_2 and define $\{Y_t\}$ by $Y_t = X_t^{(1)} - X_t^{(2)}$.

5 A Type II Asymmetric Discrete Laplace Process

For this construction we will utilize a Markov chain with a geometric stationary distribution. A convenient choice of such a chain is one with state

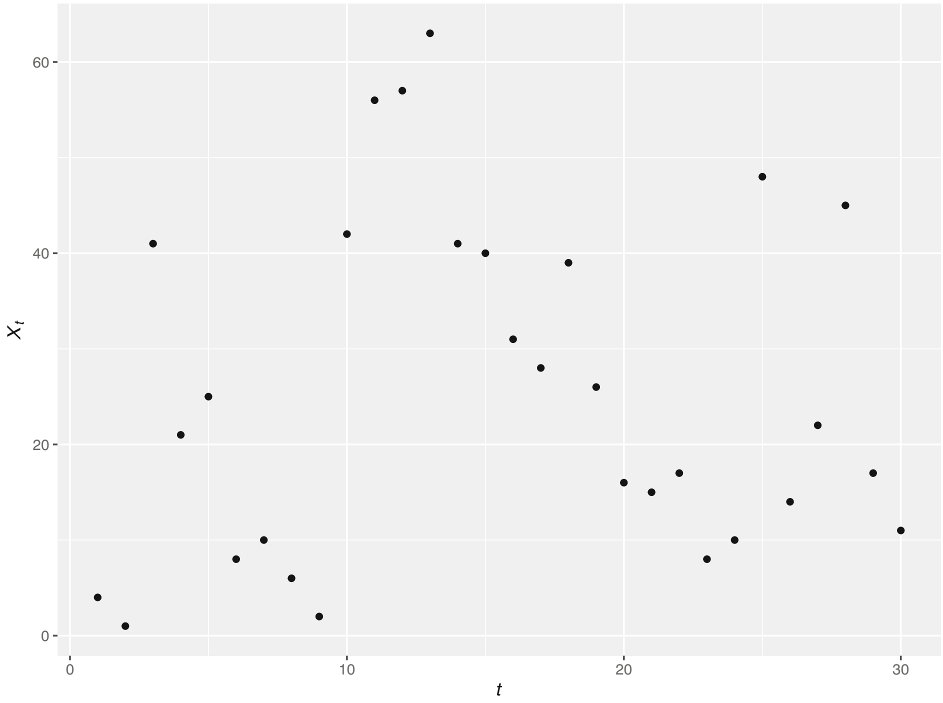

Figure 4: Type I sample path, X_t, for parameters $p = \frac{2}{7}$ and $\pi = \frac{3}{10}$, based on the N_t^+ realization in Fig. 3

space $\{1, 2, 3, 4, ...\}$ and the following transition matrix.

$$P = \begin{pmatrix} p & 1-p & 0 & 0 & 0 & ... \\ p & 0 & 1-p & 0 & 0 & ... \\ p & 0 & 0 & 1-p & 0 & ... \\ p & 0 & 0 & 0 & 1-p & ... \\ \vdots & \vdots & \vdots & ; & \vdots & ... \end{pmatrix} \tag{11}$$

If we define

$$\underline{p} = (p, p(1-p), p(1-p)^2, p(1-p)^3, p(1-p)^4, ...),$$

it is readily verified that

$$\underline{p} = \underline{p}P$$

confirming the fact that the Markov chain does have a *geometric*(p) stationary distribution.

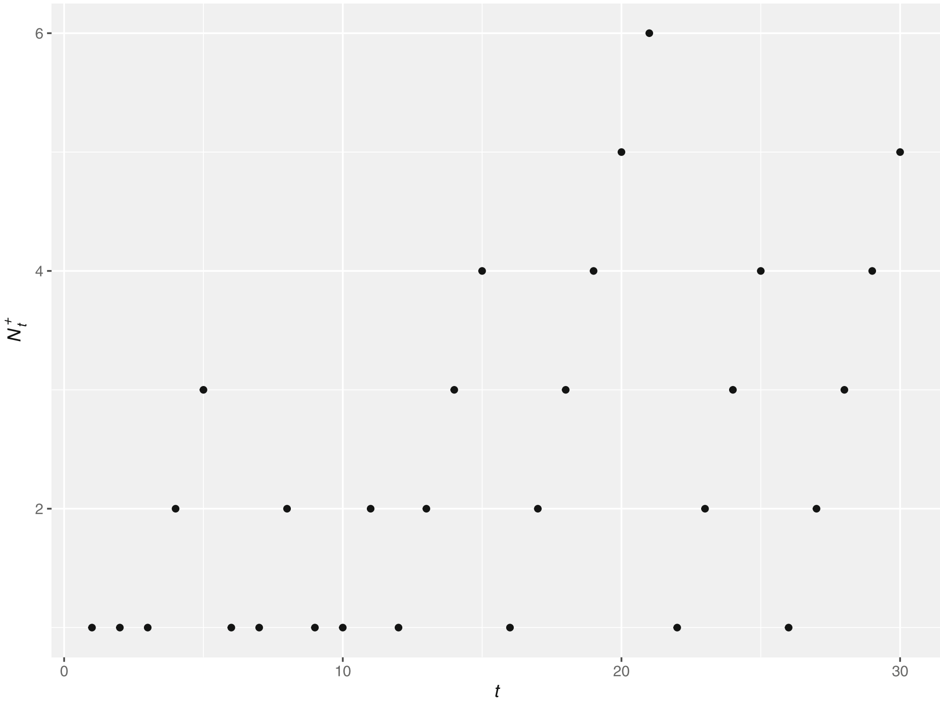

Figure 5: Type II sample path, N_t^+ for parameters $p = \frac{2}{5}$

A stationary sequence $\{Y_n\}$ with transition matrix P will be called a Type II base geometric process with parameter p. The sample paths of this geometric process (associated with the transition matrix (11) consist of increasing sequences of the form $2, 3, ..., k-1, k$ for various values of $k \geq 2$ separated by runs of 1's. See Fig. 5 for a typical sample path.

It will be noted that the Type I base process and the Type II base process are intimately related. One is just the time reversal of the other. Both are Markov chains.

The Type II future innovations geometric process is then defined by

$$X_t = \sum_{i=1}^{Y_t} \varepsilon_{t+i-1}, \tag{12}$$

where the ε_t's are i.i.d. geometric (π) variables, independent of Y_t's. It will be called a Type II geometric process. See Fig. 6 for a typical sample path.

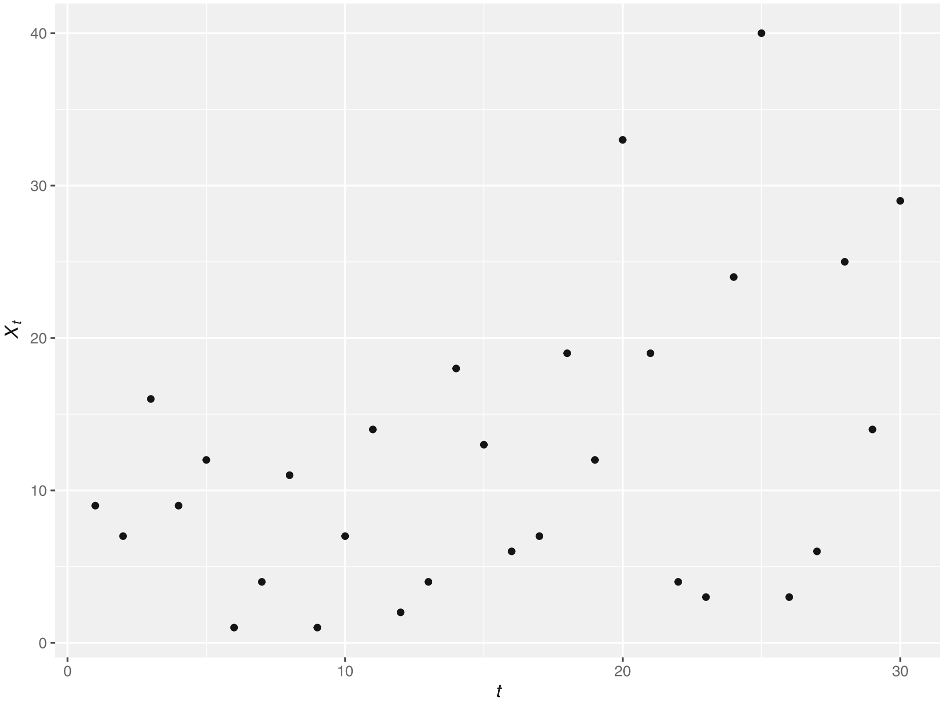

Figure 6: Type II sample path, X_t^+, for parameters $p = \frac{2}{5}$ and $\pi = \frac{1}{5}$, based on the N_t^+ realization in Fig. 5

A Type II stationary asymmetric discrete Laplace process is then constructed as a difference of two independent Type II geometric processes.. That is, let $\{X_n^{(1)*}\}$ be a Type II geometric process with parameter p_1 and let $\{X_n^{(2)*}\}$ be an independent Type II geometric process with parameter p_2 and define $\{Y_n\}$ by $Y_n = X_n^{(1)*} - X_n^{(2)*}$.

Note that any Markov chain with state space $\{1, 2, 3, ...\}$ and with a geometric stationary distribution can be used in such a construction to define geometric processes with a variety of sample path behaviors.

Note that, because of the closure of the class of ADL distributions under geometric compounding (following from Proposition 4.2 in Kozubowski and Inusah, 2006) we can also use the construction method used to build Type I and Type II geometric processes to directly construct ADL processes of Types I and II, by replacing the geometric variables, the ϵ_j's, in equation (10) by ADL variables.

6 A Generalized Asymmetric Discrete Laplace Process

For this we take two independent Type I or II geometric processes. That is, let $\{X_n^{(1)}\}$ be a Type I or II geometric process with parameter p_1 and let $\{X_n^{(2)}\}$ be an independent Type I or II geometric process with parameter p_2 and let $\{U_n\}$ be an independent sequence of i.i.d. $Bernoulli(\pi)$ random variables. Then define $\{Y_n\}$ by $Y_n = U_n X_n^{(1)} + (1-U_n)(-X_n^{(2)})$.

7 On Analogous Bivariate Processes

In the discussion of possible bivariate processes in this Section, we will make use of two very general bivariate discrete Laplace distributions introduced in Arnold and Arvanitis (2022) and described below.

A general bivariate discrete Laplace distribution of the first kind is defined as follows.

Begin with a set of 8 independent negative binomial random variables, $U_1, U_2, ..., U_8$, with $U_i \sim neg.bin.(\delta_j, p), \quad j = 1, 2, ..., 8$. Observe that all of the U_j's share a common value for p. Then define (X, Y) by

$$\begin{aligned} X &= (U_1 + U_5 + U_7) - (U_3 + U_6 + U_8), \\ Y &= (U_2 + U_6 + U_7) - (U_4 + U_5 + U_8), \end{aligned} \tag{13}$$

where it is assumed that the constraints, $\delta_1 + \delta_5 + \delta_7 = 1$, $\delta_3 + \delta_6 + \delta_8 = 1$, $\delta_2 + \delta_6 + \delta_7 = 1$, and $\delta_4 + \delta_5 + \delta_8 = 1$, have been imposed. This model will be called the bivariate discrete Laplace model of the first kind and if (X, Y) is as defined in Eq. 13 we will write $(X, Y) \sim \mathrm{BDL(I)}(\underline{\delta})$. Since there were four constraints on the δ_j's, this is a 5 parameter model. The marginal distributions are differences of independent $geometric(p)$ variables and thus have discrete Laplace densities.

This model, involving 8 independent component variables is the most general bivariate model whose symmetric discrete Laplace marginal variables are differences of sums of independent negative binomial variables, sharing the same value of the probability parameter p. The construction parallels that of the most general bivariate model with beta of the second kind marginals that are formed as ratios of independent sums of independent gamma variables with same scale parameter, introduced in Section 6 of Arnold and Ng (2011).

This approach results in a construction of a bivariate distribution with symmetric discrete Laplace marginals. It is not possible to use this kind of construction to yield asymmetric marginals. Asymmetric marginals can be encountered in the second bivariate construction, to be discussed next.

The second bivariate asymmetric discrete Laplace model that we will consider will utilize the closure under minimization property of the geometric distribution. For it we again begin with 8 independent random variables, $V_1, V_2, ..., V_8$ but this time we assume that they are geometrically distributed, thus $V_j \sim geo(\tau_j), \quad j = 1, 2, ..., 8$, where $\tau_j \in (0, 1), \quad j = 1, 2, ..., 8$, We then define

$$\begin{aligned} X &= min\{V_1, V_5, V_7\} - min\{V_3, V_6, V_8\}, \\ Y &= min\{V_2, V_6, V_7\} - min\{V_4, V_5, V_8\}. \end{aligned} \tag{14}$$

If (X, Y) has the structure shown in Eq. 14 then we will write $(X, Y) \sim$ BADL(II)($\underline{\tau}$) and say that it has a bivariate asymmetric discrete Laplace distribution of the second kind with parameter vector $\underline{\tau}$. Note that this second kind bivariate asymmetric discrete Laplace distributions has an 8 dimensional parameter space. The marginal distributions of the BADL(II) distribution are of the asymmetric discrete Laplace form. Thus:

$$X \sim ADL(1 - (1 - \tau_1)(1 - \tau_5)(1 - \tau_7), 1 - (1 - \tau_3)(1 - \tau_6)(1 - \tau_8)), \tag{15}$$
$$Y \sim ADL(1 - (1 - \tau_2)(1 - \tau_6)(1 - \tau_7), 1 - (1 - \tau_4)(1 - \tau_5)(1 - \tau_8)), \tag{16}$$

The marginal moments of the BADL(II) distribution are thus readily identified. However $cov(X, Y)$ is quite complicated and will usually be approximated by simulation using the definition Eq. 14.

See Arnold and Arvanitis (2021) for discussion of generalized versions of these bivariate discrete Laplace distributions. For example we may define a generalized version of the BADL(II) model as follows

$$\begin{aligned} X &= I_1[min\{(V_1, V_5, V_7\}] - (1 - I_1)[min\{V_3, V_6, V_8\}], \\ Y &= I_2[min\{V_2, V_6, V_7\}] - (1 - I_2)[min\{V_4, V_5, V_8)\}], \end{aligned} \tag{17}$$

where the I_j's are independent with $I_j \sim Bernoulli(\pi_j), \quad j = 1, 2.$ and the V_j's are independent of the I_j's with $V_j \sim geo(\tau_j), \quad j = 1, 2, ..., 8$. This is a 10 parameter model.

Means, variances and covariance of the coordinates of this random vector are not difficult to evaluate, or could be evaluated by simulation.

These bivariate ADL and generalized ADL models are discrete analogs of bivariate asymmetric Laplace and generalized asymmetric Laplace models discussed in Arnold and Arvanitis (2021). The geometric and negative binomial components in the present discrete models replace the exponential and gamma components in the earlier continuous models.

The full 8 or 10 parameter bivariate models will typically not be used. Inevitably, simplified sub-models will be found to be adequate in most applications. Such sub-models are obtained by eliminating various component variables in the full model, while being careful to retain enough component variables to preserve the ADL or GADL marginal distributions and usually including, as special cases, simplified models with independent marginals.

As an extreme example we might consider the following bivariate ADL model obtained by what is sometimes called trivariate reduction. It involves 3 independent geometric variables Y_0, Y_1, Y_2 with $Y_i \sim \textit{ geometric}(p_i), \quad i = 0, 1, 2$ and is defined by $X_1 = Y_1 - Y_0, X_2 = Y_2 - Y_0$. This model is recognizable as a special case of the bivariate model (16) in which only the variables V_1, V_2 and V_8 have been retained.

Simplification of the mutiparameter models can also be implemented by imposing linear constraints and or equality constraints on some of the parameters in the model. For example, certain symmetry conditions can be imposed by this approach. The big advantage of very simple sub-models is that it may become possible to interpret the effects that individual parameters have on the joint distribution (something that is usually not possible in the full model).

7.1 On Bivariate Discrete Laplace Processes In the discussion above, one method used to construct discrete asymmetric Laplace processes involved consideration of the difference between two independent geometric processes. To some extent, this approach can be used to construct analogous bivariate processes. For example we could begin with three independent geometric processes of the same type (i.e., Type I or Type II) denoted by $\{X_n^{(1)}\}$, $\{X_n^{(2)}\}$ and $\{X_n^{(3)}\}$, and define a bivariate discrete asymmetric Laplace process $\{(Y_n^{(1)}, Y_n^{(2)})\}$ by

$$Y_n^{(1)} = X_n^{(1)} - X_n^{(3)}$$
$$Y_n^{(2)} = X_n^{(2)} - X_n^{(3)}.$$

We have thus far been unable to construct a bivariate process utilizing general bivariate asymmetric Laplace distributions of Types I or II and we leave this as an intriguing open problem.

8 Discussion

The present paper is focussed on introducing an array of techniques for constructing asymmetric discrete Laplace processes. More detailed distributional properties of these processes will be discussed in a separate report together with consideration of feasible parameter estimation strategies whose effectiveness will be investigated via simulation.

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