Variations on the Univariate and Bivariate Asymmetric Laplace Themes

Barry C. Arnold and Matthew A. Arvanitis

Abstract Univariate symmetric and asymmetric Laplace (AL) models are reviewed together with certain related mixture models. Discrete versions of those models are then considered. Bivariate extensions of both the continuous and discrete models are discussed in some detail. Brief comments are provided on possible parameter estimation strategies in both univariate and bivariate cases. Discussion is provided dealing with certain related stochastic processes. Analogous gamma-difference models are introduced.

1 Introduction

Kozubowski and Podgórski (2000) introduced asymmetric Laplace models as ones having potential for dealing with data which exhibits heavier than normal tail behavior together with a lack of symmetry. Asymmetric Laplace models are becoming an increasingly useful choice for applications in quantile regression in which mean absolute error is preferred over mean squared error (Koenker & Bassett, 1978; Koenker et al., 2017). An example of this application is demonstrated by Geraci and Bottai (2007). In this work, Laplace models will be reviewed together with generalized versions, discrete analogs, bivariate and multivariate extensions and certain related stochastic processes. Some remarks are made regarding parameter estimation considerations for some of these models. The penultimate section of the work deals with an array of related stochastic processes. Section 8 includes discussion of related gamma difference models.

B. C. Arnold (✉)
University of California, Riverside, USA
e-mail: barry.arnold@ucr.edu

M. A. Arvanitis
USDA Forest Products Laboratory, Madison, USA
e-mail: matthew.arvanitis@usda.gov

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2 Univariate Laplace Models: Symmetric and Asymmetric

The classical Laplace distribution has a density of the form

$$f(x;\lambda) = (\lambda/2)e^{-\lambda|x|}I(-\infty < x < \infty), \tag{1}$$

where λ is a positive intensity parameter. It is well-known that a variable with this distribution can be represented as a sum or as a 1/2:1/2 mixture of a positive exponential variable and a negative exponential variable. We can thus write the Laplace variable X in either of the following two forms in terms of two independent identically distributed exponential(λ) random variables V_1 and V_2.

$$X = V_1 - V_2, \tag{2}$$

and

$$X = \begin{cases} V_1 \text{ with probability } 1/2 \\ -V_2 \text{ with probability } 1/2. \end{cases} \tag{3}$$

An asymmetric version of the Laplace model may be constructed as follows involving two independent exponential variables with possibly different intensity parameters. A random variable X is then said to have an asymmetric Laplace distribution if it can be represented in the form

$$X = V_1 - V_2, \tag{4}$$

where the V_i's are independent and $V_i \sim exp(\lambda_i),\quad i = 1, 2$. In this case we will write $X \sim AL(\lambda_1, \lambda_2)$.

Remark 1 It is an elementary exercise to prove that asymmetric Laplace variables defined in this way are infinitely divisible and, for brevity, thus excluded.

Just as in the classical Laplace case, a mixture representation of the AL model is available and may be described as follows.

Suppose that X has a distribution of an exponential (λ_1) variable with probability $[\lambda_2/(\lambda_1 + \lambda_2)]$, and has that of the negative of an exponential (λ_2) variable with probability $[\lambda_1/(\lambda_1 + \lambda_2)]$. If X is defined in this way, it can be verified that $X \sim AL(\lambda_1, \lambda_2)$. The density function of this asymmetric Laplace variable is given by

$$f_X(x) = \frac{\lambda_1\lambda_2}{\lambda_1 + \lambda_2}[e^{-\lambda_1 x}I(x \geq 0) + e^{\lambda_2 x}I(x < 0)]. \tag{5}$$

Note that this mixture model has a density that is continuous on the real line. An example of one such density is given in Fig. 1; the asymmetry is clearly apparent, and the right-tail wane is more rapid than that of the left. Expressions for $E(X)$ and $E(|X|)$ are included in Sect. 4.2 later. An alternative representation of the mixture model is

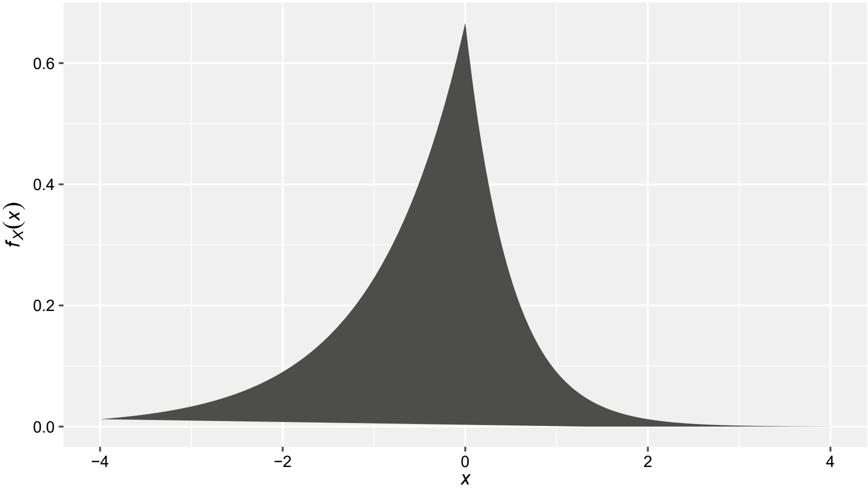

Fig. 1 Example of AL density with parameters $\lambda_1 = 2$ and $\lambda_2 = 1$

$$X = IV_1 + (1 - I)(-V_2), \tag{6}$$

where $V_i \sim exp(\lambda_i),\ i = 1, 2,$ and I is an independent Bernoulli random variable with $P(I = 1) = [\lambda_2/(\lambda_1 + \lambda_2)]$. This representation suggests introduction of a more general model with an additional parameter for more flexibility, as follows.

$$Y = IV_1 + (1 - I)(-V_2), \tag{7}$$

where $V_i \sim exp(\lambda_i),\ i = 1, 2,$ as before, but in this more general case assume that I is an independent Bernoulli random variable with $P(I = 1) = p \in (0, 1)$.

A random variable X that is of the form (7) will be said to have a generalized asymmetric Laplace distribution, and we will write $X \sim GAL(\lambda_1, \lambda_2, p)$. The corresponding density function of this generalized asymmetric Laplace variable is given by

$$f_X(x) = p\lambda_1 e^{-\lambda_1 x} I(x \geq 0) + (1 - p)\lambda_2 e^{\lambda_2 x} I(x < 0). \tag{8}$$

From this definition, and/or by perusing the graph of the density, it is apparent that unless $p = \frac{\lambda_2}{\lambda_1+\lambda_2}$, this GAL density will have a point of discontinuity at $x = 0$. An example of a GAL density is given in Fig. 2, where, unlike in the standard AL, the discontinuity is clearly apparent. Expressions for the first three moments of the GAL distribution may be found in Sect. 4.5 later.

Additional details on the construction, moments, and parameter estimation of the AL and GAL models can be found in Arnold and Arvanitis (2021).

In the next section, we discuss discrete Laplace models having both symmetric and asymmetric forms.

Fig. 2 Example of GAL density with parameters $\lambda_1 = 2$, $\lambda_2 = 1$, and $p = \frac{2}{3}$. The discontinuity is indicated by the vertical red dotted line at $x = 0$

3 Discrete Laplace Models, Symmetric and Asymmetric

A discrete analogue to the Laplace models (Inusah & Kozubowski, 2006) can be obtained by considering V_1 and V_2 with $V_i \sim geo(p_i)$, $i = 1, 2$, and defining

$$X = V_1 - V_2. \tag{9}$$

The resulting discrete density is thus given by

$$P(X = x) = \begin{cases} p_1 p_2 q_1^x (1 - q_1 q_2)^{-1}, x \geq 0, \\ p_1 p_2 q_2^{-x} (1 - q_1 q_2)^{-1}, x < 0. \end{cases}$$

An example of this discrete density is given in Fig. 3. Similar to the its continuous counterpart, the asymmetry is clearly apparent. This distribution, called the asymmetric discrete Laplace (ADL) distribution, is an obvious discrete parallel to the asymmetric Laplace model $Y = U_1 - U_2$ where the U_i's are independent with $U_i \sim exp(\lambda_i)$, $\ i = 1, 2$.

Remark 2 It is an elementary exercise to prove that asymmetric discrete Laplace variables are infinitely divisible.

We define a mixture alternative parallel to the GAL model as follows

$$Y = IV_1 + (1 - I)(-V_2), \tag{10}$$

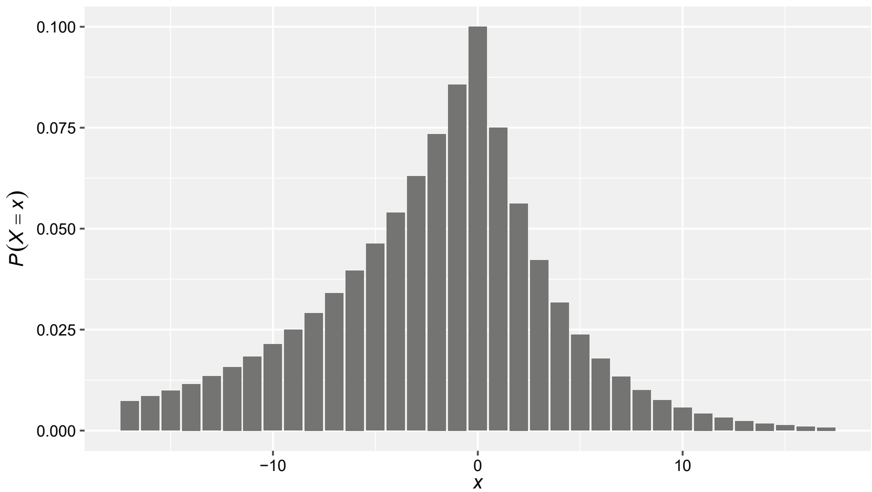

Fig. 3 Example of ADL density with parameters $p_1 = \frac{1}{4}$ and $p_2 = \frac{1}{7}$

where $V_i \sim geo(p_i), i = 1, 2,$ are independent and I is an independent Bernoulli random variable with $P(I = 1) = \pi$.

In this case Y is said to have a generalized asymmetric discrete Laplace (GADL) distribution, offering additional modeling flexibility. The discrete density of this GADL distribution is of the following form.

$$\begin{aligned} P(Y = y) &= \pi p_1 q_1^y, \quad y = 1, 2, \ldots, \\ &= \pi p_1 + (1-\pi)p_2, \quad y = 0, \\ &= (1-\pi)p_2 q_2^{-y}, \quad y = -1, -2, -3, \ldots, \end{aligned} \tag{11}$$

where p_1, p_2 and π are parameters ranging over the interval (0, 1).

From the characteristic function, or from the mixture representation (10), we find that for the GADL distribution

$$E(Y) = \frac{\pi(1-p_1)}{p_1} - \frac{(1-\pi)(1-p_2)}{p_2}$$

and

$$\begin{aligned} var(Y) = &\left\{ \frac{\pi(1-p_1)(2-p_1)}{p_1^2} + \frac{(1-\pi)(1-p_2)(2-p_2)}{p_2^2} \right\} \\ &- \left\{ \frac{\pi(1-p_1)}{p_1} - \frac{(1-\pi)(1-p_2)}{p_2} \right\}^2. \end{aligned}$$

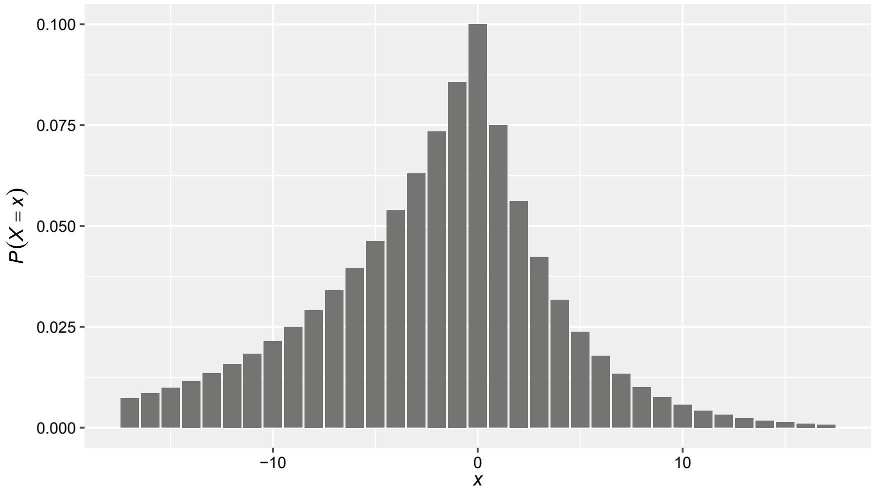

Fig. 4 Example of GDAL density with parameters $p_1 = \frac{1}{4}$ and $p_2 = \frac{1}{7}$, and $\pi = \frac{1}{2}$

In the special case of the GADL in which $X \sim ADL(p_1, p_2)$, the moments simplify to become

$$E(X) = (1 - p_1)/p_1 - (1 - p_2)/p_2$$

and

$$var(X) = (1 - p_1)/p_1^2 + (1 - p_2)/p_2^2.$$

Additional details on the ADL and GADL construction and moments can be found in Arnold and Arvanitis (2022) (Fig. 4).

4 Parameter Estimation for Univariate Laplace Models

For the Laplace, GAL, ADL, and GADL models, maximum likelihood, method of moments, and Bayesian parameter estimation procedures are described in this section.

4.1 Laplace: Maximum Likelihood

Suppose we have a sample $X_1, X_2, \ldots, X_n$ from an asymmetric Laplace distribution with density

$$f_X(x; \lambda_1, \lambda_2) = \frac{\lambda_1\lambda_2}{\lambda_1 + \lambda_2}[e^{-\lambda_1 x} I(x > 0) + e^{\lambda_2 x} I(x < 0)]$$

The sufficient statistics are of the form $U = \sum_1^n X_i I(X_i > 0)$, $V = -\sum_1^n X_i I(X_i < 0)$ and $W = \sum_1^n I(X_i > 0)$.

The mle's are given by solving the likelihood equations to get: $\widetilde{\lambda_1} = \frac{n}{U+\sqrt{UV}}$ and $\widetilde{\lambda_2} = \frac{n}{V+\sqrt{UV}}$.

4.2 Laplace: Method of Moments

Suppose we have a sample $X_1, X_2, \ldots, X_n$ from an asymmetric Laplace distribution represented in the form

$$X = V_1 - V_2,$$

where the V_i's are i.i.d. with $V_i \sim exp(\lambda_i)$, $i = 1, 2$. To estimate the parameters in the model, we will equate the first moment of X and the first absolute moment of X to the corresponding sample moments. Elementary computations yield

$$E(X) = \lambda_1^{-1} - \lambda_2^{-1}, \text{ and } E(|X|) = \frac{\lambda_1^2 + \lambda_2^2}{\lambda_1\lambda_2(\lambda_1 + \lambda_2)}.$$

Denote the corresponding sample moments by

$$M_1 = (1/n)\sum_1^n X_i, \text{ and } M_2 = (1/n)\sum_1^n |X_i|,$$

and set up and solve the equations $E(X) = M_1$ and $E(|X|) = M_2$ to get the following MOM estimates:

$$\widetilde{\lambda_1} = \left[\frac{M_2 + M_1 + \sqrt{M_2^2 - M_1^2}}{2}\right]^{-1}, \text{ and}$$

$$\widetilde{\lambda_2} = \left[\frac{M_2 - M_1 + \sqrt{M_2^2 - M_1^2}}{2}\right]^{-1}. \tag{12}$$

The authors note that, since the method of estimation is not the main theme of this work, we have excluded further details, such as analyzing the effectiveness of maximum likelihood versus method of moments estimation strategies, which may be found in future work.

4.3 Laplace: Bayesian

Suppose we have a sample $X_1, X_2, \ldots, X_n$ from an asymmetric Laplace distribution with density

$$f_X(x;\lambda_1,\lambda_2) = \frac{\lambda_1\lambda_2}{\lambda_1+\lambda_2}[e^{-\lambda_1 x}I(x>0) + e^{\lambda_2 x}I(x<0)].$$

Define $U = \sum_1^n X_i I(X_i > 0)$, $V = -\sum_1^n X_i I(X_i < 0)$, and $W = \sum_1^n I(X_i > 0)$. We wish to estimate the parameters from a Bayesian viewpoint. The likelihood for the sample is given by

$$L(\lambda_1,\lambda_2) = \left(\frac{\lambda_1\lambda_2}{\lambda_1+\lambda_2}\right)^n e^{-\lambda_1 u}e^{-\lambda_2 v};\ \lambda_1>0, \lambda_2>0.$$

If we take independent gamma priors for λ_1 and λ_2, say

$$\widetilde{\lambda_i} \sim \Gamma(\alpha_i, \delta_i), \quad i = 1, 2,$$

our posterior density will be of the form

$$f(\lambda_1,\lambda_2|\underline{X}=\underline{x}) \propto (\lambda_1+\lambda_2)^{-n}\lambda_1^{\alpha_1+n-1}\lambda_2^{\alpha_2+n-1}e^{-(u+\delta_1)\lambda_1}e^{-(v+\delta_2)\lambda_2}$$

From this joint posterior density for (λ_1, λ_2) the usual Bayes estimates of λ_1 and λ_2, namely $E(\lambda_1|\underline{X}=\underline{x})$ and $E(\lambda_2|\underline{X}=\underline{x})$, will be obtained by numerical integration.

4.4 GAL: Maximum Likelihood

Suppose we have a sample $X_1, X_2, \ldots, X_n$ from a generalized asymmetric Laplace distribution of the form

$$X = IY_1 + (1-I)(-Y_2),$$

where $I \sim Bernoulli(p)$ and the Y_i's are independent with $Y_i \sim exp(\lambda_i)$, $i = 1, 2$. Again let U, V, and W be the sufficient statistics. We can then easily solve the likelihood equations in this case to obtain

$$\widetilde{p} = \frac{W}{n},\ \widetilde{\lambda_1} = \frac{W}{U},\ \text{and}\ \widetilde{\lambda_2} = \frac{n-W}{V}. \tag{13}$$

4.5 GAL: Method of Moments

Suppose we have a sample $X_1, X_2, \ldots, X_n$ from a generalized asymmetric Laplace distribution of the form

$$X = IV_1 + (1 - I)(-V_2),$$

where $I \sim Bernoulli(p)$ and the V_i's are independent with $V_i \sim exp(\lambda_i), i = 1, 2$. We wish to estimate the parameters using the method of moments. For convenience, define $\delta_i = \lambda_i^{-1},\ \ i = 1, 2$. The first three moments are:

$$E(X) = p\delta_1 - (1 - p)\delta_2, \tag{14}$$
$$E(X^2) = 2p\delta_1^2 + 2(1 - p)\delta_2^2, \tag{15}$$
$$E(X^3) = 6p\delta_1^3 - 6(1 - p)\delta_2^3. \tag{16}$$

Denote the sample moments by $M_i = (1/n)\sum_{j=1}^{n} X_j^i, i = 1, 2, 3$.
The moment equations to solve are:

$$M_1 = p\delta_1 - (1 - p)\delta_2, \tag{17}$$
$$M_2 = 2p\delta_1^2 + 2(1 - p)\delta_2^2, \tag{18}$$
$$M_3 = 6p\delta_1^3 - 6(1 - p)\delta_2^3. \tag{19}$$

It can be shown that in order to satisfy the moment equations, $\hat{\delta}_1$ must be a root of a fifth degree polynomial

Eliminating complex and non-positive real roots yields all possible values for $\hat{\delta}_1$. For each, we can compute the other two estimates, if they exist, i.e.,

$$\hat{\delta}_2 = \frac{M_2 - 2M_1\hat{\delta}_1}{2(\hat{\delta}_1 - M_1)}, \tag{20}$$

and

$$\hat{p} = \frac{M_1 + \hat{\delta}_2}{\hat{\delta}_1 + \hat{\delta}_2}. \tag{21}$$

Any resulting values of $(\hat{\delta}_1, \hat{\delta}_2, p)$ which are in the parameter space are MOM estimates. Clearly, under these conditions, it may be the case that no such estimates exist or multiple estimates exist.

4.6 GAL: Bayesian Method

Suppose we have a sample $X_1, X_2, \ldots, X_n$ from a generalized asymmetric Laplace distribution of the form

$$X = IY_1 + (1 - I)(-Y_2),$$

where $I \sim Bernoulli(p)$ and the Y_i's are independent with $Y_i \sim exp(\lambda_i)$, $i = 1, 2$. Again let U, V, and W be the suficient statistics. We wish to estimate the parameters from a Bayesian viewpoint. In this case the likelihood will be

$$L(p, \lambda_1, \lambda_2) = p^w \lambda_1^w e^{-\lambda_1 u} (1 - p)^{n-w} \lambda_2^{n-w} e^{-\lambda_2 v}.$$

A conjugate prior with independent marginals is available in this case. Thus we can take

$$\widetilde{p} \sim Beta(\tau_1, \tau_2), \text{ and } \lambda_i \sim \Gamma(\alpha_i, \delta_i), \quad i = 1, 2.$$

The corresponding joint posterior density has independent marginals of the same form as in the prior, i.e.,

$$\widetilde{p}|\underline{X} = \underline{x} \sim Beta(\tau_1 + w, \tau_2 + n - w)$$

$$\lambda_1|\underline{X} = \underline{x} \sim \Gamma(\alpha_1 + w, \delta_1 + u)$$

and

$$\lambda_2|\underline{X} = \underline{x} \sim \Gamma(\alpha_2 + n - w, \delta_2 + v)$$

As estimates of the parameters we can take the posterior means, i.e.,

$$\hat{p}_{(B)} = \frac{\tau_1 + w}{\tau_1 + \tau_2 + n},$$

$$\hat{\lambda}_{1(B)} = \frac{\alpha_1 + w}{\delta_1 + u},$$

and

$$\hat{\lambda}_{2(B)} = \frac{\alpha_2 + n - w}{\delta_2 + v}.$$

If the hyperparameters of the prior are negligible, these estimates will reduce to agree with the mle's.

4.7 ADL: Maximum Likelihood

Suppose we have a sample $X_1, X_2, \ldots, X_n$ from an asymmetric discrete Laplace distribution defined in (9). We wish to estimate the parameters using maximum likelihood. Define

$$U = \sum_1^n X_i I(X_i \geq 0) \text{ and } V = -\sum_1^n X_i I(X_i < 0).$$

Taking partial derivatives of the log-likelihood and equating to 0, yields the following likelihood equations

$$\frac{n}{p_1} = \frac{n(1-p_2)}{p_1 + p_2 - p_1 p_2} + \frac{U}{1-p_1}, \tag{22}$$

$$\frac{n}{p_2} = \frac{n(1-p_1)}{p_1 + p_2 - p_1 p_2} + \frac{V}{1-p_2}. \tag{23}$$

In order to solve these equations it is convenient to rewrite them in terms of the means of the V_i's, thus we define $\mu_i = (1-p_i)/p_i, \quad i = 1, 2$. The associated likelihood equations then become

$$\mu_1 = \frac{\mu_1 \mu_2}{1 + \mu_1 + \mu_2} + \frac{U}{n}, \tag{24}$$

$$\mu_2 = \frac{\mu_1 \mu_2}{1 + \mu_1 + \mu_2} + \frac{V}{n}. \tag{25}$$

From these equations we deduce that

$$\mu_1 - \mu_2 = \frac{U}{n} - \frac{V}{n}. \tag{26}$$

Next, using (26), express μ_2 as a linear function of μ_1 and substitute this in equation (24). This upon rearranging is a quadratic equation in μ_1 which can be solved to yield the maximum likelihood estimate of μ_1, i.e.,

$$\widehat{\mu}_1 = \frac{U}{n} - \frac{1}{2} + \sqrt{\frac{1}{4} + \frac{UV}{n^2}}, \tag{27}$$

and then

$$\widehat{\mu}_2 = \widehat{\mu}_1 - \frac{U}{n} + \frac{V}{n.}. \tag{28}$$

4.8 ADL: Method of Moments

Suppose we have a sample $X_1, X_2, \ldots, X_n$ from an asymmetric discrete Laplace distribution represented in the form

$$X = V_1 - V_2,$$

where the V_i's are i.i.d. with $V_i \sim geo(p_i),\ \ i = 1, 2.$

We will equate the first two moments of X to the corresponding sample moments. Elementary computations yield

$$E(X) = \frac{1-p_1}{p_1} - \frac{1-p_2}{p_2}$$

and

$$E(X^2) = \frac{1-p_1}{p_1^2} + \frac{1-p_2}{p_2^2} + (E(X))^2$$

Denote the first two sample moments, based on a sample of size n, by $M_1 = (1/n)\sum_{i=1}^n X_i$ and $M_2 = (1/n)\sum_{i=1}^n X_i^2$, and define $S^2 = M_2 - M_1^2$ and set up the equations $M_1 = E(X)$ and $S^2 = var(X)$ which may be solved to obtain method of moments estimates of the p_i's. Here, as in the maximum likelihood case, it is convenient to rewrite the equations in terms of the means of the V_i's (i.e., $\mu_i = (1-p_i)/p_i,\ \ i = 1, 2$). The moment equations are of the form

$$M_1 = \mu_1 - \mu_2, \tag{29}$$

$$S^2 = \mu_1(1+\mu_1) + \mu_2(1+\mu_2). \tag{30}$$

From Eq. (29) we can write $\mu_2 = \mu_1 - M_1$ and substitute this into (32). This then can be rearranged into a quadratic function of μ_1 which is readily solved. In this way we obtain method of moments estimates of the following form.

$$\widetilde{\mu}_1 = \frac{1}{2}\left[M_1 - 1 + \sqrt{1 - M_1^2 + 2M_1 S^2}\right], \tag{31}$$

and

$$\widetilde{\mu}_2 = \widetilde{\mu}_1 - M_1. \tag{32}$$

4.9 ADL: Bayesian Method

Suppose we have a sample $X_1, X_2, \ldots, X_n$ from an asymmetric discrete Laplace distribution with density

$$f_X(x; p_1, p_2) = \left(\frac{p_1 p_2}{p_1 + p_2 - p_1 p_2}\right)(1 - p_1)^{xI(x \geq 0)}(1 - p_2)^{-xI(x<0)}$$

Define
$U = \sum_1^n X_i I(X_i > 0)$, $V = -\sum_1^n X_i I(X_i < 0)$ and $W = \sum_1^n I(X_i > 0)$
We wish to estimate the parameters from a Bayesian viewpoint.
The likelihood for the sample is given by

$$L(p_1, p_2) = \left(\frac{p_1 p_2}{p_1 + p_2 - p_1 p_2}\right)^n (1 - p_1)^u (1 - p_2)^v.$$

If we take independent beta priors for p_1 and p_2, a natural choice for these parameters, i.e.,

$$\widetilde{p}_i \sim Beta(\alpha_i, \beta_i), \quad i = 1, 2,$$

then the posterior will be of the form

$$f(p_1, p_2 | \underline{X} = \underline{x}) \propto \left(\frac{p_1^{n+\alpha_1} p_2^{n+\alpha_2}}{(p_1 + p_2 - p_1 p_2)^n}\right)(1 - p_1)^{u+\beta_1}(1 - p_2)^{v+\beta_2}.$$

From this joint posterior density the usual Bayes estimates of p_1 and p_2, namely $E(p_i | \underline{X} = \underline{x}), \quad i = 1, 2$, will be obtained by numerical integration.

4.10 GADL: Maximum Likelihood

Suppose instead we have a sample $X_1, X_2, \ldots, X_n$ from a generalized asymmetric discrete Laplace distribution of the form

$$X = IW_1 + (1 - I)(-W_2),$$

where $I \sim Bernoulli(\pi)$ and the W_i's are independent with $W_i \sim geo(p_i), i = 1, 2$.

We wish to estimate the parameters using maximum likelihood. Here too, we will define

$$U = \textstyle\sum_1^n X_i I(X_i \geq 0) \text{ and } V = -\sum_1^n X_i I(X_i < 0),$$

and also define $N_0 = \sum_1^n I(X_i = 0)$, $N_1 = \sum_1^n I(X_i > 0)$ and $N_2 = \sum_1^n I(X_i < 0)$

Taking partial derivatives of the log-likelihood and equating to 0, yields the following likelihood equations

$$\frac{N_1}{\pi} = \frac{N_2}{1-\pi} - \frac{N_0(p_1 - p_2)}{\pi p_1 + (1-\pi)p_2}, \tag{33}$$

$$\frac{N_1}{p_1} = \frac{U}{1-p_1} - \frac{N_0 \pi}{\pi p_1 + (1-\pi)p_2}, \tag{34}$$

$$\frac{N_2}{p_2} - \frac{V}{1 = p_2} - \frac{N_0(1-\pi)}{\pi p_1 + (1-\pi)p_2}. \tag{35}$$

Note these equations are particularly easy to solve if $N_0 = 0$. In other cases an iterative solution may obtained. Note that if p_1 and p_2 are known, then Eq. (33) is equivalent to a quadratic equation in π. If p_1 and π are known, then Eq. (34) is equivalent to a linear equation in p_2. And, finally, if p_2 and π are known, then Eq. (35) is equivalent to a linear equation in p_1.

4.11 GADL: Method of Moments

Suppose we have a sample $X_1, X_2, \ldots, X_n$ from a generalized asymmetric discrete Laplace distribution of the form

$$X = IW_1 + (1-I)(-W_2)$$

where $I \sim Bernoulli(\pi)$ and the W_i's are independent with $W_i \sim geo(p_i), i = 1, 2$.

We wish to estimate the parameters using the method of moments. The first three moments about zero of X are:

$$E(X) = \pi \frac{1-p_1}{p_1} - (1-\pi)\frac{1-p_2}{p_2}$$

$$E(X^2) = \pi \left\{ 2\left(\frac{1-p_1}{p_1}\right)^2 + \frac{1-p_1}{p_1} \right\} + (1-\pi)\left\{ 2\left(\frac{1-p_2}{p_2}\right)^2 + \frac{1-p_2}{p_2} \right\}$$

$$E(X^3) = \pi \left\{ 6\left(\frac{1-p_1}{p_1}\right)^3 + 6\left(\frac{1-p_1}{p_1}\right)^2 + \left(\frac{1-p_1}{p_1}\right) \right\}$$
$$-(1-\pi)\left\{ 6\left(\frac{1-p_2}{p_2}\right)^3 + 6\left(\frac{1-p_2}{p_2}\right)^2 + \left(\frac{1-p_2}{p_2}\right) \right\}$$

If we denote the first three sample moments, based on a sample of size n, by $M_1 = (1/n)\sum_{i=1}^n X_i$, $M_2 = (1/n)\sum_{i=1}^n X_i^2$, and $M_3 = (1/n)\sum_{i=1}^n X_i^3$, and set up the equations $M_j = E(X^j), \quad j = 1, 2, 3$, then our method of moments estimates will be the solution to these three equations.

Note that, if p_1 and p_2 are known then the equation $M_3 = E(X^3)$ is a linear equation in π. Also, if p_2 and π are known, then the equation $M_2 = E(X^2)$ is equivalent to a quadratic equation in p_1, and finally, if π and p_1 are known then the equation $M_1 = E(X)$ is equivalent to a linear equation in p_2. Using these observations, an iterative scheme for identifying the method of moments estimates is readily set up.

4.12 GADL: Bayesian Method

Suppose we have a sample $X_1, X_2, \ldots, X_n$ from a generalized asymmetric discrete Laplace distribution of the form

$$X = IY_1 + (1 - I)(-Y_2),$$

where $I \sim Bernoulli(\pi)$ and the Y_i's are independent with $Y_i \sim geo(p_i), i = 1, 2$.

Recall that we defined

$$U = \sum_1^n X_i I(X_i \geq 0) \text{ and } V = -\sum_1^n X_i I(X_i < 0),$$

and also defined $N_0 = \sum_1^n I(X_i = 0)$, $N_1 = \sum_1^n I(X_i > 0)$ and $N_2 = \sum_1^n I(X_i < 0)$

We wish to estimate the parameters from a Bayesian viewpoint.

In this case the likelihood will be

$$L(\pi, p_1, p_2) = \pi^{n_1} p_1^{n_1}(1 - p_1)^u (1 - \pi)^{n_2} p_2^{n_2}(1 - p_2)^v [\pi p_1 + (1 - \pi)p_2]^{n_0}.$$

A plausible prior with independent marginals will be of the form

$$\widetilde{\pi} \sim Beta(\tau_1, \tau_2), \text{ and } \widetilde{p_i} \sim Beta(\alpha_i, \beta_i), \quad i = 1, 2.$$

The corresponding joint posterior density will be

$$\begin{aligned} f(\pi, p_1, p_2 | \underline{X} = \underline{x}) \propto \pi^{n_1+\tau_1-1} & p_1^{n_1+\alpha_1-1}(1 - p_1)^{u+\beta_1-1} \\ & \times (1 - \pi)^{n_2+\tau_2-1} p_2^{n_2+\alpha_2-1}(1 - p_2)^{v+\beta_2-1} \\ & \times [\pi p_1 + (1 - \pi)p_2]^{n_0}. \end{aligned} \tag{36}$$

If $n_0 = 0$ then the posterior density will have independent beta distributed marginals. If $n_0 > 0$ then a posterior density that is a mixture of distributions with independent marginals will be encountered.

5 Bivariate Models

Arnold and Arvanitis (2021) and Arnold and Arvanitis (2022) also provide constructions of eight bivariate models with AL, GAL, ADL, and GADL marginals. These models are summarized in this section.

1. Bivariate Asymmetric Laplace of the First Kind (BAL(I)): Consider 8 independent gamma variables $U_1, U_2, \ldots, U_8$ with $U_j \sim \Gamma(\delta_j, 1), \quad j = 1, 2, \ldots, 8$. We then define (X, Y) by

$$
\begin{aligned}
X &= \lambda_{11}^{-1}(U_1 + U_5 + U_7) - \lambda_{12}^{-1}(U_3 + U_6 + U_8), \\
Y &= \lambda_{21}^{-1}(U_2 + U_6 + U_7) - \lambda_{22}^{-1}(U_4 + U_5 + U_8),
\end{aligned} \tag{37}
$$

   where it is assumed that the constraints, $\delta_1 + \delta_5 + \delta_7 = 1$, $\delta_3 + \delta_6 + \delta_8 = 1$, $\delta_2 + \delta_6 + \delta_7 = 1$, and $\delta_4 + \delta_5 + \delta_8 = 1$, have been imposed. Then $(X, Y) \sim BAL(1)(\lambda_{11}, \lambda_{12}, \lambda_{21}, \lambda_{22}, \underline{\delta})$ with marginals given by

$$
X \sim AL(\lambda_{11}, \lambda_{12}), \quad Y \sim AL(\lambda_{21}, \lambda_{22}). \tag{38}
$$

   An example of a BAL(I) distribution is depicted in Fig. 5.
2. Bivariate Asymmetric Laplace of the Second Kind (BAL(II)): Consider 8 independent random variables, $V_1, V_2, \ldots, V_8$ but this time we assume that they are exponentially distributed, thus $V_j \sim exp(\lambda_j), \quad j = 1, 2, \ldots, 8$. We then define

$$
\begin{aligned}
X &= min\{V_1, V_5, V_7\} - min\{V_3, V_6, V_8\}, \\
Y &= min\{V_2, V_6, V_7\} - min\{V_4, V_5, V_8\}.
\end{aligned} \tag{39}
$$

   Then $(X, Y) \sim BAL(II)(\underline{\lambda})$, with marginals given by

$$
X \sim AL(\lambda_1 + \lambda_5 + \lambda_7, \lambda_3 + \lambda_6 + \lambda_8), \tag{40}
$$

$$
Y \sim AL(\lambda_2 + \lambda_6 + \lambda_7, \lambda_4 + \lambda_5 + \lambda_8), \tag{41}
$$

3. Generalized Bivariate Asymmetric Laplace of the First Kind (GBAL(I)): Similar to the BAL(I), the GBAL(I) is given by

$$
\begin{aligned}
X &= I_1\lambda_{11}^{-1}(U_1 + U_5 + U_7) - (1 - I_1)\lambda_{12}^{-1}(U_3 + U_6 + U_8), \\
Y &= I_2\lambda_{21}^{-1}(U_2 + U_6 + U_7) - (1 - I_2)\lambda_{22}^{-1}(U_4 + U_5 + U_8),
\end{aligned} \tag{42}
$$

   with the same constraints as the BAL(I) and where the I_j's are independent with $I_j \sim Bernoulli(p_j), \quad j = 1, 2$.
4. Generalized Bivariate Asymmetric Laplace of the Second Kind (GBAL(II)): Similar to the BAL(II), the GBAL(II) is given by

$$X = I_1[min\{(U_1, U_5, U_7\}] - (1 - I_1)[min\{U_3, U_6, U_8\}],$$
$$Y = I_2[min\{(U_2, U_6, U_7\}] - (1 - I_2)[min\{U_4, U_5, U_8)\}], \tag{43}$$

the λ_j's are positive parameters, and where the I_j's are independent with $I_j \sim Bernoulli(p_j),\quad j = 1, 2.$

5. Bivariate Discrete Laplace of the First Kind (BDL(I)): Consider a set of 8 independent random variables $U_1, U_2, \ldots, U_8$ with $U_i \sim neg.bin.(\delta_j, p),\quad j = 1, 2, \ldots, 8$. Note that all of the U_j's share a common value for p. Then, define (X, Y) by

$$X = (U_1 + U_5 + U_7) - (U_3 + U_6 + U_8),$$
$$Y = (U_2 + U_6 + U_7) - (U_4 + U_5 + U_8), \tag{44}$$

where it is assumed that the constraints, $\delta_1 + \delta_5 + \delta_7 = 1$, $\delta_3 + \delta_6 + \delta_8 = 1$, $\delta_2 + \delta_6 + \delta_7 = 1$, and $\delta_4 + \delta_5 + \delta_8 = 1$, have been imposed. We write $(X, Y) \sim BDL(1)(\underline{\delta})$. Since there were four constraints on the δ_j's, this is a 5 parameter model. The marginal distributions are differences of independent $geometric(p)$ variables and thus have discrete (symmetric) Laplace densities.

6. Bivariate Asymmetric Discrete Laplace Distribution of the Second Kind (BADL(II)): Consider 8 independent random variables, $V_1, V_2, \ldots, V_8$ but this time we assume that they are geometrically distributed, thus $V_j \sim geo(\tau_j),\quad j = 1, 2, ..., 8$, where $\tau_j \in (0, 1),\quad j = 1, 2, \ldots, 8$, Then define

$$X = min\{V_1, V_5, V_7\} - min\{V_3, V_6, V_8\},$$
$$Y = min\{V_2, V_6, V_7\} - min\{V_4, V_5, V_8\}, \tag{45}$$

using a construction parallel to that used in the construction of the BAL(II) model. If (X, Y) has the structure shown in (45) then we will write $(X, Y) \sim BADL(II)(\underline{\tau})$ and say that it has a bivariate asymmetric discrete Laplace distribution of the second kind with parameter vector $\underline{\tau}$. Note that the second kind bivariate asymmetric discrete Laplace distributions has an 8 dimensional parameter space. The marginal distributions of the BADL(II) distribution are of the asymmetric discrete Laplace form. Thus the marginals are given by

$$X \sim ADL(1 - (1 - \tau_1)(1 - \tau_5)(1 - \tau_7), 1 - (1 - \tau_3)(1 - \tau_6)(1 - \tau_8)), \tag{46}$$
$$Y \sim ADL(1 - (1 - \tau_2)(1 - \tau_6)(1 - \tau_7), 1 - (1 - \tau_4)(1 - \tau_5)(1 - \tau_8)), \tag{47}$$

7. Generalized Bivariate Discrete Laplace Distribution of the First Kind (GBDL(I)): Setting $U_1, U_2, \ldots, U_8$ identically to those for the BDL(I), set

$$X = I_1(U_1 + U_5 + U_7) - (1 - I_1)(U_3 + U_6 + U_8),$$
$$Y = I_2(U_2 + U_6 + U_7) - (1 - I_2)(U_4 + U_5 + U_8), \tag{48}$$

where it is assumed that the constraints, $\delta_1 + \delta_5 + \delta_7 = 1$, $\delta_3 + \delta_6 + \delta_8 = 1$, $\delta_2 + \delta_6 + \delta_7 = 1$, and $\delta_4 + \delta_5 + \delta_8 = 1$, have been imposed, and where the I_j's are independent with $I_j \sim Bernoulli(\pi_j),\quad j = 1, 2$, and the U_j's are independent with $U_j \sim neg.bin.(\delta_j, p),\quad j = 1, 2, 3..., 8$. We write $(X, Y) \sim GBDL(1)(\pi_1, \pi_2, \underline{\delta})$. Since there were four constraints on the δ_j's, this is a 7 parameter model.

8. Generalized Bivariate Asymmetric Discrete Laplace Distribution of the Second Kind (GBADL(II)): Setting $V_1, V_2, \ldots, V_8$ identically to those for the BADL(I), set

$$\begin{aligned} X &= I_1[min\{(V_1, V_5, V_7\}] - (1 - I_1)[min\{V_3, V_6, V_8\}], \\ Y &= I_2[min\{V_2, V_6, V_7\}] - (1 - I_2)[min\{V_4, V_5, V_8)\}], \end{aligned} \tag{49}$$

where the I_j's are independent with $I_j \sim Bernoulli(\pi_j),\quad j = 1, 2.$ and the V_j's are independent with $V_j \sim geo(\tau_j),\quad j = 1, 2, \ldots, 8$. This is a 10 parameter model.

Each of these models is large and flexible and will tend to be used to construct smaller models with fewer parameters for specific applications. To do so, for Type I distributions, we apply the convention that $\delta_j = 0 \implies U_j \equiv 0$, and, for Type II distributions, we apply the conventions that $\lambda_j = 0 \implies V_j \equiv \infty$ or $\tau_j = 0 \implies V_j \equiv 0$, for all j.

Remark 3 Even more general distributions than (48) and (49) can be constructed in which the I_i's are dependent indicator random variables.

Remark 4 Higher dimensional versions of all the bivariate models discussed in this paper are readily described. Since, for example, the completely general trivariate version of the asymmetric discrete Laplace model of Type II will involve 26 independent geometric components each with its own τ parameter, only submodels including just a limited number of components will be tractable and useful for modeling purposes. One very simple trivariate submodel model that might find application is the following.

$$\begin{aligned} X &= \min\{V_1, V_7\} - V_{19}, \\ Y &= \min\{V_2, v_7\} - V_{19}, \\ Z &= V_3 - V_{19}. \end{aligned}$$

which involves only 5 of the 26 components in the general model, but still has a non-trivial dependence structure.

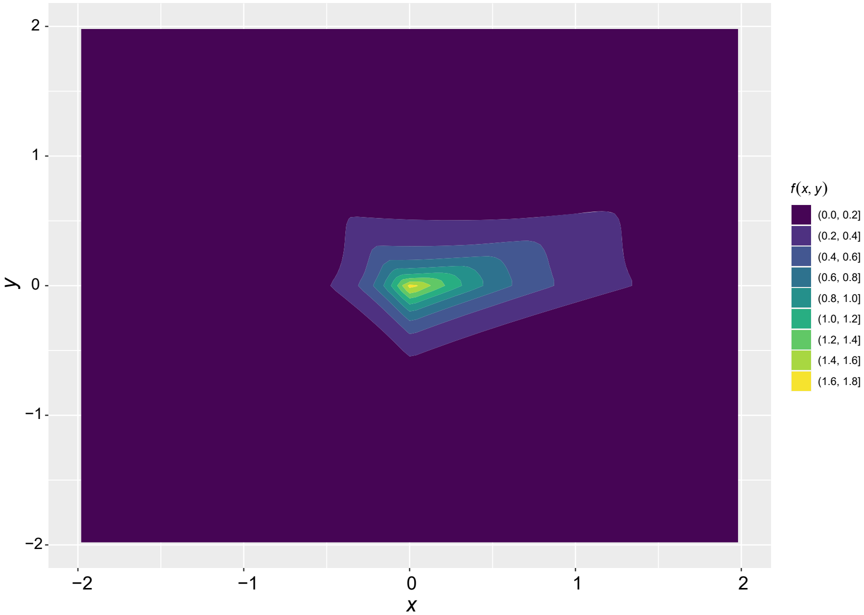

Fig. 5 Density plot for the $BAL(I)\left(\lambda_{11}=1, \lambda_{12}=3, \lambda_{21}=2, \lambda_{22}=4, \underline{\delta}=\left(\frac{1}{2}, 0, \frac{1}{2}, 1, 0, \frac{1}{2}, 0, \frac{1}{2}\right)\right)$. Notice that the density exhibits stretching away from the origin in the top two quadrants as a result of the parameters δ_6 and δ_8 being positive

6 Asymmetric Laplace Processes

A spectrum of asymmetric Laplace processes will be described in this Section. A future project will involve a more detailed development of the distribution theory associated with these processes. In some cases, an analogous discrete process can be defined using geometric distributed components rather than exponential ones.

In constructing the asymmetric Laplace processes in this section we will make use of certain well-known properties of the exponential distribution. If $X_1 \sim exp(\lambda_1)$ and $X_2 \sim exp(\lambda_2)$ are independent then

$$Y = \min\{X_1, X_2\} \sim \exp(\lambda_1 + \lambda_2)$$

If $X_i,\ i = 1, 2, \ldots$ are i.i.d. $exp(\lambda)$ variables and N independent of the X_i's has a geometric distribution with support $\{1, 2, \ldots\}$ and success probability p, i.e., $P(X_i = j) = p(1-p)^{j-1}$, then

$$Z = \sum_{i=1}^{N} X_i \sim \exp(p\lambda),$$

i.e., a geometric sum of i.i.d. exponential variables has an exponential distribution with a modified intensity parameter. This naturally motivates consideration of the possibility that a similar property will be exhibited by the classical symmetric and the asymmetric Laplace models. That this is true was first observed by Kozubowski (1994). See Sect. 6.3 for pertinent details.

The asymmetric process models to be discussed below can be expected to provide alternatives to commonly used assumptions of the availability of i.i.d. symmetric observations. The new processes will continue to have identically distributed observations, but will allow for asymmetry and some degree of dependence. A particular arena in which the i.i.d. assumption is routinely invoked is the use of control charts in monitoring possible changes in production quality.

6.1 A Type I Asymmetric Laplace Process

Suppose that $U_i,\ i = 1, 2, \ldots$ are i.i.d. $\exp(\lambda_1)$ variables, and also that $V_i,\ i = 1, 2, \ldots$ are i.i.d. $\exp(\lambda_2)$ variables and that the two sequences are independent. Using these sequences of variables we define a process $\{X_n,\ n = 1, 2, \ldots\}$ as follows

$$X_n = Y_n - Z_n$$

where, $Y_1 = U_1$ and $Z_1 = V_1$ and for $n = 2, 3, \ldots$ we define

$$Y_n = 2\min\{Y_{n-1}, U_n\}, \tag{50}$$

$$Z_n = 2\min\{Z_{n-1}, V_n\}. \tag{51}$$

From properties of the exponential distribution it is evident that, for each n, we have $X_n \sim AL(\lambda_1, \lambda_2)$. The process $\{X_n\}$ is a stationary asymmetric Laplace process (which we will call a Type I AL process). It is not a Markov process. However each of the processes $\{Y_n\}$ and $\{Z_n\}$ are first order Markov chains.

6.2 A Type II Asymmetric Laplace Process

In this case, we use the fact that if $X_i,\ i = 1, 2, \ldots$ are i.i.d. $\exp(\lambda)$ variables, and N independent of the X_i's has a geometric distribution with support $\{1, 2, \ldots\}$ and success probability p, i.e., $P(X_i = j) = p(1-p)^{j-1}$, then

$$Z = \sum_{i=1}^{N} X_i \sim \exp(p\lambda).$$

Making use of this observation Arnold (1989) defined what he called a future innovations exponential process as follows. For each $t = 0, \pm 1, \pm 2, \ldots$ define i.i.d. $Bernoulli(p)$ random variables U_t. Then for each $t = 0, 1, 2, \ldots$ define a random variable N_t^+ as follows

$$N_t^+ = 1, \text{ if and only if } U_t = 1,$$
$$N_t^+ = i \text{ if and only if } U_t = U_{t+1} = \cdots = U_{t+i-2} = 0. U_{t+i-1} = 1.$$

These N_t^+'s will be $geometric(p)$ random variables with support 1, 2, ... They are identically distributed but are not independent. The future innovations exponential process is then defined by

$$X_t = p \sum_{i=1}^{N_t^+} \varepsilon_{t+i-1}, \tag{52}$$

where the ε_t's are i.i.d. exponential variables, independent of N_t^+.

This is a stationary non-Markovian exponential process. It will be called a Type II process. A simple way to construct an asymmetric Laplace processes is then to take a difference of two independent Type II exponential processes. That is, let $\{X_t^{(1)}\}$ be a Type II exponential process with intensity parameter λ_1 and let $\{X_t^{(2)}\}$ be an independent Type II exponential process with intensity parameter λ_2 and define $\{Y_t\}$ by $Y_t = X_t^{(1)} - X_t^{(2)}$. Note that, in this construction, the two Bernoulli parameters (the$p$'s,) involved in the construction can be equal or unequal as desired.

6.3 A Type II(A) Asymmetric Laplace Process

Instead of considering differences of independent exponential processes, a more attractive construction will involve geometric sums of i.i.d. asymmetric Laplace variables. It turns out that geometric sums of asymmetric Laplace variables are again of the asymmetric Laplace form with modified intensity parameters. This can be verified as follows.

If $X_i,\ i = 1, 2, \ldots$ are i.i.d. $AL(\lambda_1, \lambda_2)$ variables and N independent of the X_i's has a geometric distribution with support $\{1, 2, \ldots\}$ and success probability p, i.e., $P(X_i = j) = p(1-p)^{j-1}$, then

$$Z = \sum_{i=1}^{N} X_i \sim AL(\delta_1, \delta_2),$$

where $\delta_1 = h_1(\underline{\lambda})$ and $\delta_2 = h_2(\underline{\lambda})$ are solutions to the following equations

$$\frac{1}{p\lambda_1} - \frac{1}{p\lambda_2} = \frac{1}{\delta_1} - \frac{1}{\delta_2} \tag{53}$$

$$p\lambda_1\lambda_2 = \delta_1\delta_2. \tag{54}$$

The value of δ_1 that satisfies these equations is a solution to a particular quadratic equation and using it and (54) one can obtain the corresponding value of δ_2.

It turns out that an alternative parameterization, introduced by Kozubowski (1994), and used by him and his coauthors in the study of asymmetric Laplace variables and processes, is more convenient to use in the context of geometric sums of AL variables.

The Kozubowski parameters are

$$\mu = \frac{1}{\lambda_1} - \frac{1}{\lambda_2}, \tag{55}$$

$$\sigma^2 = \frac{1}{\lambda_1\lambda_2}. \tag{56}$$

Using these parameters, we can rewrite the geometric sum of AL variables result in the following much simpler form:

If $X_i,\ i = 1, 2, \ldots$ are i.i.d. $AL(\mu, \sigma^2)$ variables and N independent of the X_i's has a geometric distribution with support $\{1, 2, \ldots\}$ and success probability p, i.e., $P(X_i = j) = p(1-p)^{j-1}$, then

$$Z = \sum_{i=1}^{N} X_i \sim AL(\mu/p, \sigma^2/p).$$

We can thus define a future innovations asymmetric Laplace process as follows (using the Kozubowski parameters).

For each $t = 0, \pm 1, \pm 2, \ldots$ define i.i.d. $Bernoulli(p)$ random variables U_t. Then for each $t = 0, 1, 2, \ldots$ define a random variable N_t^+ as follows

$$N_t^+ = 1, \text{ if and only if } U_t = 1,$$
$$N_t = i \text{ if and only if } U_t = U_{t+1} = \cdots = U_{t+i-2} = 0. U_{t+i-1} = 1.$$

These N_t^+'s will be $geometric(p)$ random variables with support $1, 2, \ldots$. They are identically distributed but are not independent. The future innovations asymmetric Laplace process is then defined by

$$X_t = \sum_{i=1}^{N_t^+} \varepsilon_{t+i-1}, \tag{57}$$

where the ε_t's are i.i.d. asymmetric Laplace $(p\mu, p\sigma^2)$ variables, independent of N_t^+. The process so constructed will have $X_t \sim AL(\mu, \sigma^2)$ for every t.

This is a stationary non-Markovian asymmetric Laplace process. It will be called a Type II(A) process

6.4 Types III and III(A) Asymmetric Laplace Processes

We begin by recalling the past innovation exponential process defined in Anderson and Arnold (1993). It is defined in terms of the same *Bernoulli* random variables, the U_t''s, and the same exponential variables, the ε_t's as in Subsect. 5.2. For any t we define a geometric random variable N_t^- by

$$N_t^- = 1, \text{ if and only if } U_t = 1,$$
$$N_t^- = i \text{ if and only if } U_t = U_{t-1} = \cdots = U_{t-i+2} = 0.U_{t-i-1} = 1.$$

These N_t^-'s will be $geometric(p)$ random variables with support $1, 2, \ldots$. They are identically distributed but are not independent. The past innovations exponential process is then defined by

$$X_t = p \sum_{i=1}^{N_t^-} \varepsilon_{t-i-1}, \tag{58}$$

where the ε_t's are i.i.d. exponential variables, independent of N_t^-.

This is a stationary Markovian exponential process. It will be called a Type III process. It can be alternatively written in an autoregressive form as follows:

$$X_t = (1 - U_t)X_{t-1} + p\varepsilon_t. \tag{59}$$

Without going into details, it is clear that an asymmetric Laplace process, to be called a Type III process, can be built as the difference of two independent Type III exponential processes.

To obtain the Type III(A) asymmetric Laplace model , we merely assume that, in the definition of the past innovation exponential process, we define the ε_t's to be independent asymmetric Laplace$(\mu/p, \sigma^2/p)$ variables, independent of the N_t^-'s. The process so constructed will have $X_t \sim AL(\mu, \sigma^2)$ for every t.

6.5 Type IV(A) Asymmetric Laplace Processes

Using an autoregressive type formulation, we may define other variants of the asymmetric Laplace process.

For example, the model that we will call a Type IV(A) model was introduced and studied by Jayakumar et al. (2012). It can be expressed in the form.

$$X_t = pX_{t-1} + \varepsilon_t. \tag{60}$$

In order for this process to have an asymmetric Laplace(μ, σ^2) distribution for each X_t, the innovation variables, the ε's must be defined as follows

$$\begin{aligned} \varepsilon_t &= 0 \text{ with probability } \pi_1, \\ &= E_{t1} \text{ with probability } \pi_2, \\ &= -E_{t2} \text{ with probability } \pi_3, \end{aligned}$$

where E_{t1} and E_{t2} are independent exponential variables with intensities k/σ and σ/k respectively where

$$k = \frac{2}{\frac{\mu}{\sigma} + \sqrt{4 + \left(\frac{\mu}{\sigma}\right)^2}}$$

and

$$\pi_1 = p^2, \quad \pi_2 = (1-p)\frac{1+pk^2}{1+k^2} \quad \text{and} \quad \pi_3 = (1-p)\frac{p+k^2}{1+k^2}.$$

7 About Generalized Asymmetric Laplace Processes

At this point, it is inevitable that one might consider the possibility of constructing GAL processes using methods analogous to those used for describing AL processes. Unfortunately, in the generalized setting, we no longer have the property of closure under geometric summation. At this time, the construction of generalized asymmetric Laplace processes remains an open (though intriguing) problem.

8 Univariate and Bivariate Gamma-Difference Models

In this Section we will briefly discuss gamma-difference models which can be viewed as extensions of the asymmetric Laplace models discussed in this paper.

If W_1 and W_2 are independent gamma distributed random variables with $W_j \sim \Gamma(\delta_j, 1)$ $j = 1, 2$, then the difference $Z = W_1 - W_2$ has what is known as a gamma-difference distribution with parameters δ_1 and δ_2, and we write $Z \sim GD(\delta_1, \delta_2)$. Klar (2015) provides details, possible applications and historical perspective on this model. Many early appearances of the model dealt with the symmetric case in which $\delta_1 = \delta_2$.

If $Z \sim GD(\delta_1, \delta_2)$ then its moment generating function is of the form

$$M_Z(t) = (1-t)^{-\delta_1}(1+t)^{-\delta_2}, \quad |t| < 1. \tag{61}$$

The moments of Z could be obtained from this moment generating function, or they may more easily be obtained by expanding $(W_1 - W_2)^k$, where k is a positive integer, and using available expressions for gamma moments. The mean, variance ad skewness of Z are thus, respectively:

$$\delta_1 - \delta_2, \qquad \delta_1 + \delta_2, \text{ and } \frac{2(\delta_1 - \delta_2)}{(\delta_1 + \delta_2)^{3/2}}. \tag{62}$$

The distribution of Z is asymmetric unless $\delta_1 = \delta_2$. Klar provided an expression for the density of Z that involves Whittaker-W functions.

Of course, if δ_1 and δ_2 are positive integers, successive integration by parts can be used to evaluate the density. The representation $Z = W_1 - W_2$ not only allows for ready computation of the moments of Z, but also permits straightforward simulation of realizations from its distribution.

A simple bivariate versions of the GD distribution can be developed using the "variables in common" methodology. For it, we can begin with 3 independent gamma variables U_1, U_2 and U_3 with $U_j \sim \Gamma(\delta_j, 1),\ \ j = 1, 2, 3$. Then define $X = U_1 - U_3$ and $Y = U_2 - U_3$.

Instead, we may develop a more flexible bivariate model in a manner analogous to that used in the development of the Arnold-Ng bivariate beta(2) model. We thus begin with 8 independent gamma variables $U_1, U_2, \ldots, U_8$ with $U_j \sim \Gamma(\delta_j, 1),\ \ j = 1, 2, \ldots, 8$.. We then define (X, Y) by

$$\begin{aligned} X &= (U_1 + U_5 + U_7) - (U_3 + U_6 + U_8), \\ Y &= (U_2 + U_6 + U_7) - (U_4 + U_5 + U_8). \end{aligned} \tag{63}$$

If (X, Y) is as defined by (63) then we write $(X, Y) \sim BGD(\underline{\delta})$, to be read as a bivariate gamma-difference distribution with parameter vector $\underline{\delta}$. Clearly (X, Y) has gamma-difference marginals. The BGD distribution is identifiable since each U_j plays a different role in the construction. Moreover, the BGD model represents the most general model in which both X and Y are linear combinations of independent gamma variables with all coefficients equal to 1 or -1. Of course sub-models of (63), in which certain δ_j's are set equal to 0, with the usual convention that the corresponding U_j's are equal to 0 with probability 1, may frequently be found adequate in application scenarios.

9 Conclusion

In this work, certain asymmetric Laplace models and their discrete counterparts are discussed. Standard parameter estimation techniques are exhibited for the models. In addition, bivariate variants of these families of distributions are discussed. We finished with a brief discussion on higher-dimensional models, related stochastic processes and gamma-difference distributions, viewed as extensions of the asymmetric Laplace model.

References

Anderson, D. N., & Arnold, B. C. (1993). Linnik distributions and processes. *Journal of Applied Probability, 30*(2), 330–340.

Arnold, B., & Arvanitis, M. (2021). Asymmetric univariate and bivariate Laplace and generalized Laplace distributions. *Journal of the Iranian Statistical Society, Behboodian Special Issue, 20*(1), 61–81.

Arnold, B. C. (1989). A logistic process constructed using geometric minimization. *Statistics & Probability Letters, 7*(3), 253–257.

Arnold, B. C., & Arvanitis, M. A. (2022). Univariate and bivariate discrete Laplace and generalized discrete Laplace distributions. *Gujarat Journal of Statistics and Data Science, 38*(2), 7–18.

Geraci, M., & Bottai, M. (2007). Quantile regression for longitudinal data using the asymmetric Laplace distribution. *Biostatistics, 8*(1), 140–154.

Inusah, S., & Kozubowski, T. J. (2006). A discrete analogue of the Laplace distribution. *Journal of statistical planning and inference, 136*(3), 1090–1102.

Jayakumar, K., Kuttykrishnan, A., & Kozubowski, T. (2012). A first order autoregressive asymmetric Laplace process. *Journal of Probability and Statistical Science, 10*(1), 1–14.

Klar, B. (2015). A note on gamma difference distributions. *Journal of Statistical Computation and Simulation, 85*(18), 3708–3715.

Koenker, R., Chernozhukov, V., He, X., & Peng, L. (2017). Handbook of quantile regression.

Koenker, R., & Bassett Jr, G. (1978). Regression quantiles. *Econometrica: Journal of the Econometric Society,* 33–50.

Kozubowski, T. J. (1994). Representation and properties of geometric stable laws. *Approximation, Probability, and Related Fields,* 321–337.

Kozubowski, T. J., & Podgórski, K. (2000). A multivariate and asymmetric generalization of Laplace distribution. *Computational Statistics, 15*, 531–540.

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Editors
Somesh Kumar
Department of Mathematics
Indian Institute of Technology Kharagpur
Kharagpur, West Bengal, India

Barry C. Arnold
Department of Statistics
University of California
Riverside, CA, USA

Kunio Shimizu
Center for Training Professors in Statistics
The Institute of Statistical Mathematics
Tachikawa, Tokyo, Japan

Arnab Kumar Laha
Operations and Decision Sciences
Indian Institute of Management Ahmedabad
Ahmedabad, Gujarat, India

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